Combinatorial Characterization for Global Identifiability of Separable Networks with Partial Excitation and Measurement

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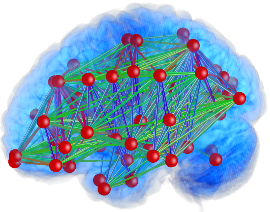
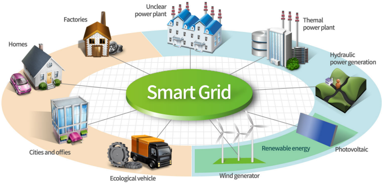
UCLouvain

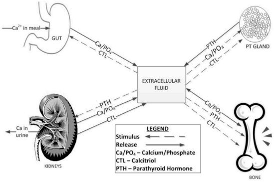
Neuroscience [Chiêm et al. 2021]



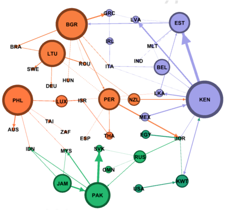
Power grids



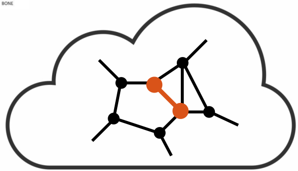
Physiological models [Christie et al. 2014]



Stock market [Shahzad et al. 2018]



Excitations →



→ Measurements

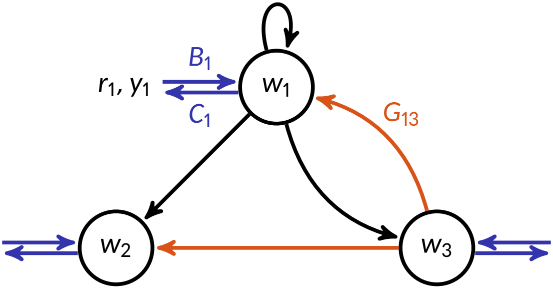
Model

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given excitations r at B and measurements y at C,

can we recover the unknown transfer fcts G_{ij} ?



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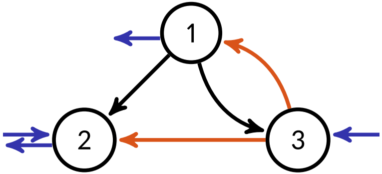

Framework: Not all nodes are excited/measured

Handle known transfer functions

No use of noise. But reformulation at [Shi, Cheng, Van den Hof 2019]

Single frequency – scalar case

Assumptions: Network topology is known

Global transfer matrix $\underbrace{C\,(I - G)^{-1}\,B}_{T(G)}$ is known

Example

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given excitations r at B and measurements y at C,
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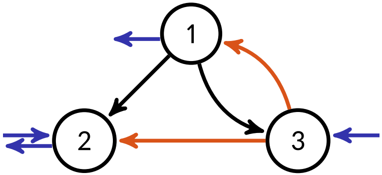

$$G = \begin{bmatrix} 0 & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T \triangleq (I - G)^{-1} = \frac{1}{1 - G_{13}G_{31}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix}$$

Example

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

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From given excitations r at B and measurements y at C,
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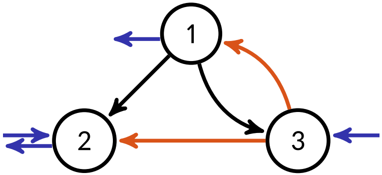

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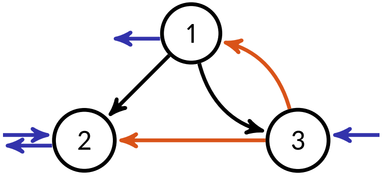
$$\mathbf{C}\,T\,\mathbf{B} = \frac{1}{1 - G_{13}G_{31}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix}$$

Example



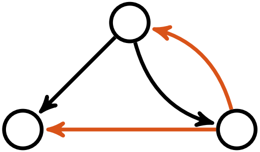
The question is:

From $\quad \mathbf{C}\,T\,\mathbf{B} = \dfrac{1}{1 - G_{13}G_{31}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix},$

can we uniquely recover G_{13} and G_{23}?

Outline

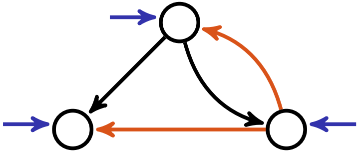
Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

State of the Art



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,\not{B}}_{\text{known}} = C\,(I - G)^{-1}\,\not{B}$ admit a unique solution G ?

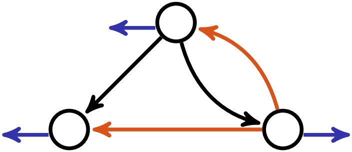
All nodes excitated: i.e. $B = I$

Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]



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Necessary and sufficient path-based condition

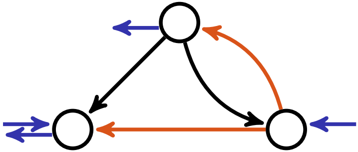
[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

All nodes measured: i.e. $C = I$

Dual results



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

All nodes excited: i.e. $B = I$ — Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

All nodes measured: i.e. $C = I$ — Dual results

General case: Our approach: we linearize

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1. Introduction
2. Model
3. Identifiability
4. Algebraic characterization
    - Local identifiability
    - Decoupled identifiability
    - Separable networks
5. Combinatorial characterization
6. Conclusion

From the definition of global identifiability…

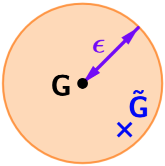
Definition: Network globally identifiable at G if for all $\tilde{G}$:

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

… we introduce *local* identifiability

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

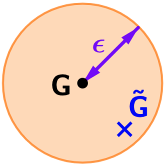
$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$



Local identifiability is necessary

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an* ϵ*—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$



- Necessary for global identifiability
- No counter-example to sufficiency known

Local identifiability is generic

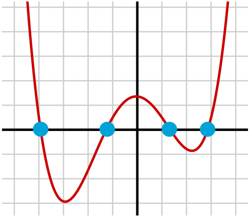
Proposition 1 (CDC 2020)

Local identifiability is a generic property. It holds:

- either for almost all parameters G_{ij};
- or for no parameters G_{ij}.

Unformally, it does almost only depend on the graph topology.

Example: the nonzeroness of a polynomial is generic

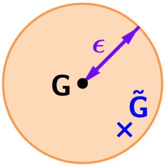


When all nodes are excited/measured, **global** identifiability is also a generic property.

Algebraic characterization

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$



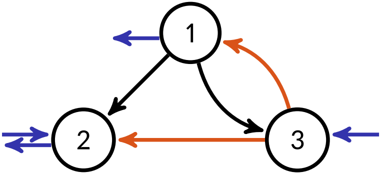
- Necessary for global identifiability
- No counter-example to sufficiency known

Theorem 1 (CDC 2020)

G generically locally identif	$\Leftrightarrow$	$CT\Delta TB = 0 \Rightarrow \Delta = 0$ $\forall\,\Delta$, generically	$\Leftrightarrow$	K is full-rank generically

where $K \triangleq (B^\top T^\top \otimes CT) I_{G\Delta}$ $\quad$ $I_{G\Delta}$ selects the columns corresponding to unknown transfer functions

Example

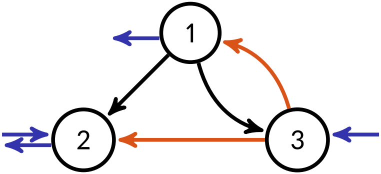


$$G = \begin{bmatrix} 0 & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T \triangleq (I - G)^{-1} = \frac{1}{\underbrace{1 - G_{13}G_{31}}_{\triangleq D}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix}$$

Theorem 1 (CDC 2020)

G generically locally identif $\iff$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\, \Delta$, generically $\iff$ K is full-rank generically



$$K \triangleq (B^\top T^\top \otimes CT) I_{G^\Delta} = \frac{1}{D^2}$$

	G_{13}	G_{23}
$2 \rightarrow 1$	0	0
$2 \rightarrow 2$	0	0
$3 \rightarrow 1$	1	0
$3 \rightarrow 2$	$G_{21} + G_{23}G_{31}$	1

Theorem 1 (CDC 2020)

G generically locally identif $\Leftrightarrow$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\, \Delta$, generically $\Leftrightarrow$ K is full-rank generically

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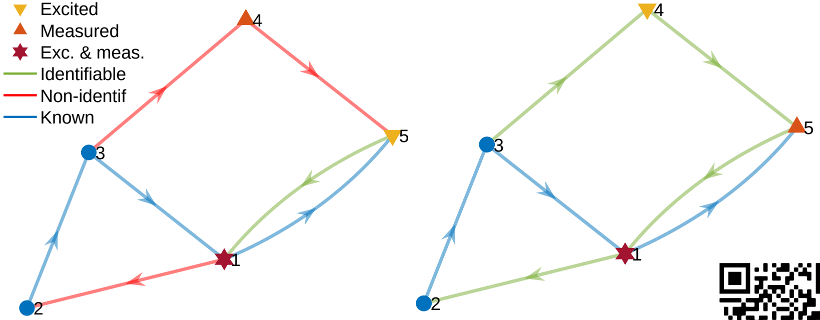
$\longrightarrow$ Probability-1 algorithm: randomized, proba 0 of inaccuracy



(a) (b)

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What we have so far

Global identifiability

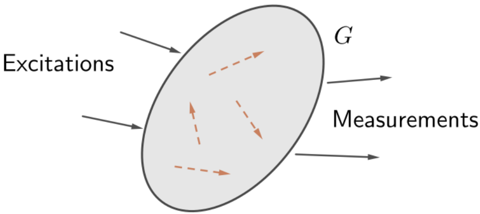
$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

→ Graph interpretation?



Interpretation on a larger network

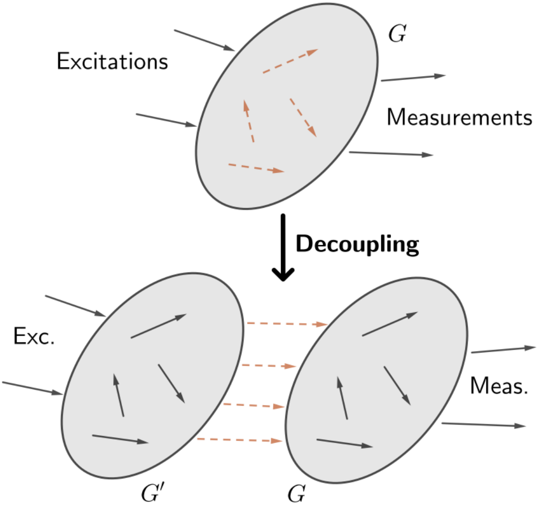
Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$$\Downarrow$$

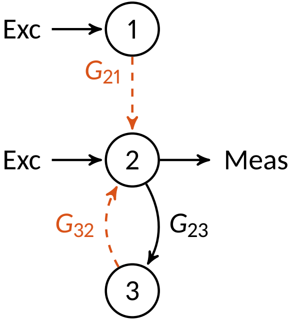
Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$



Global identifiability

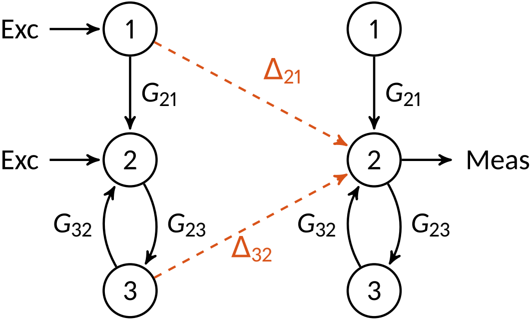
$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$



$\Downarrow$

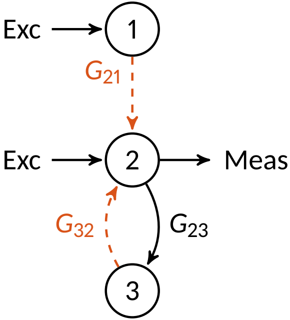
Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$



Global identifiability

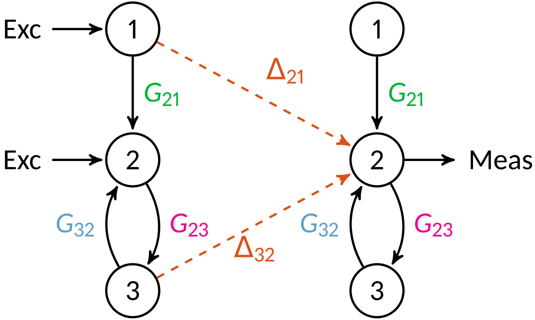
$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$



$\Downarrow$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$



Same transfer functions
in the two copies

Decoupled identifiability

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$

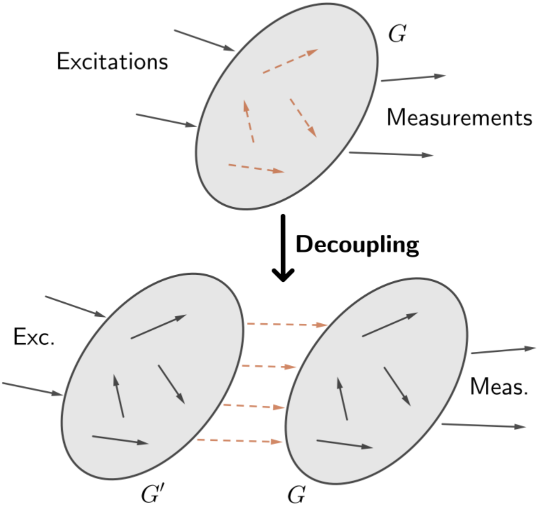
Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

$\Downarrow$

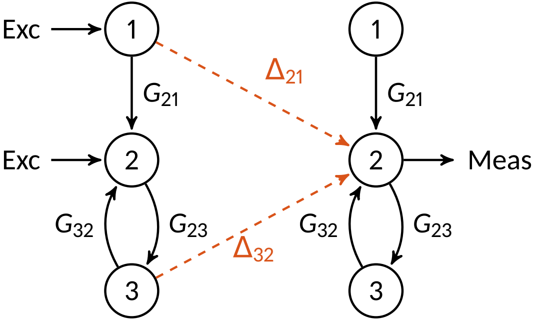
Decoupled identifiability

$CT\Delta T'B = 0 \Rightarrow \Delta = 0$



Local identifiability

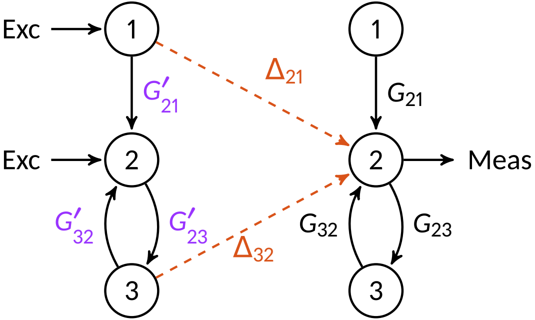
$C T \Delta T B = 0 \Rightarrow \Delta = 0$



$\Downarrow$

Decoupled identifiability

$C T \Delta T' B = 0 \Rightarrow \Delta = 0$



Decoupled network

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$

Local identifiability

$C T \Delta T B = 0 \Rightarrow \Delta = 0$

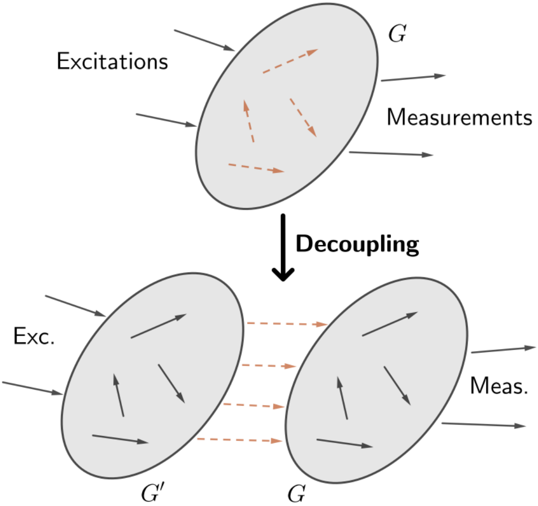
$\Downarrow$

Decoupled identifiability

$C T \Delta T' B = 0 \Rightarrow \Delta = 0$

$\Updownarrow$

Global identifiability of the decoupled network



Necessary... and sufficient?

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$ $\Uparrow$?

Local identifiability

$C T \Delta T B = 0 \Rightarrow \Delta = 0$

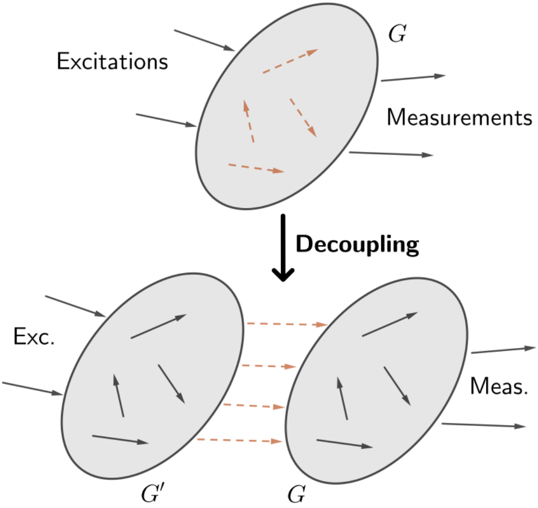
$\Downarrow$ $\Uparrow$?

Decoupled identifiability

$C T \Delta T' B = 0 \Rightarrow \Delta = 0$

No counter-example known

`github.com/alegat/identifiable`



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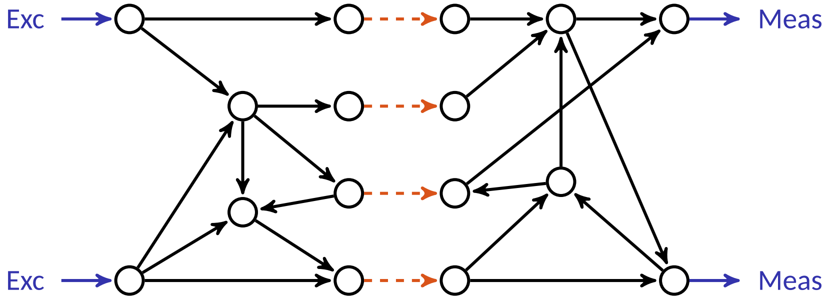
Separable networks

Generalization of the decoupled network:

Excited and measured subgraphs can now have different topologies



Proposition 2

On separable networks, local identifiability $\Leftrightarrow$ global identifiability

$\Rightarrow$ Global identifiability can be studied via rank K

Other network topologies

The decoupled network is a particular case of separable network, where the excited and measured subgraph have same topology

$\Rightarrow$ Conditions derived for separable networks apply to decoupled identifiability

Reminder

Global Identifiability $\Rightarrow$ Local Identifiability $\Rightarrow$ Decoupled Identif

$\Rightarrow$ *Necessary* conditions derived for separable networks apply to global identifiability of *any* network topology

Summary

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$$\Downarrow$$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

$$\Downarrow$$

Decoupled identifiability

$CT\Delta T'B = 0 \Rightarrow \Delta = 0$

$$\Updownarrow$$

Global identifiability of the decoupled network	$\Leftarrow$	Separable networks (for which global $\Leftrightarrow$ local)

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Framework

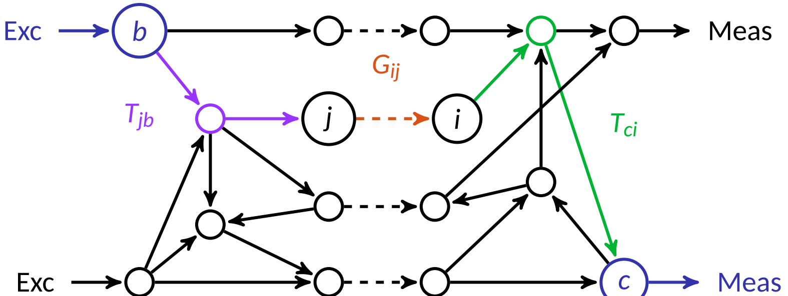
- *Global* identifiability on *separable* networks
  - $\Rightarrow$ Applies on decoupled identifiability of all networks
- Identifiability of the *whole* network
- We assume: # unknowns = # excitations × # measurements
  - $\Rightarrow$ Matrix K is now square, and: K full-rank $\Leftrightarrow \det K \neq 0$

Theorem 2

A separable network is globally identifiable generically $\iff$ $\det K \neq 0$ generically

$$K = (B^\top T^\top \otimes CT) I_{G^\Delta} =$$

	$\cdots$	unknown G_{ij}	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
exc b $\rightarrow$ meas c	$\cdots$	$T_{jb} T_{ci}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$



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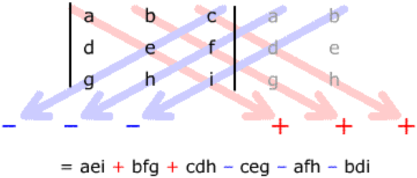
$K = (B^\top T^\top \otimes CT) I_{G^\Delta} =$

	$\cdots$	unknown G_{ij}	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
exc b $\longrightarrow$ meas c	$\cdots$	$T_{jb} T_{ci}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

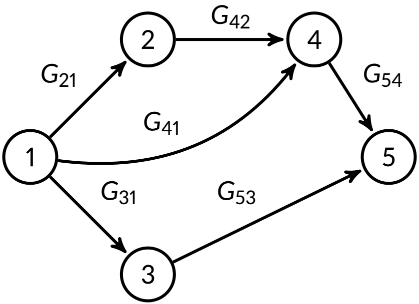
$$\det K = \sum_{\text{assignations } \sigma} \text{sign}(\sigma) \prod_{G_{ij}} T_{jb} T_{ci} \qquad \text{(Leibniz)}$$



Lemma 1

The entries of $T(G) = (I - G)^{-1}$ are

$$T_{ji} = \sum_{\substack{\text{all paths} \\ i \to j}} G_{j*} \ \dots \ G_{*i}.$$



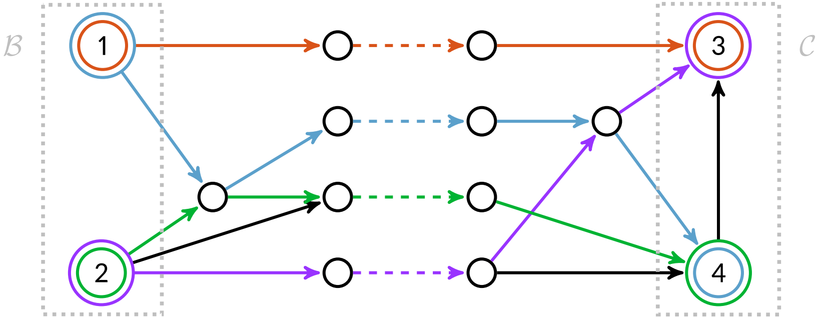
$$T_{51} = G_{54}G_{42}G_{21} + G_{54}G_{41} + G_{53}G_{31}$$

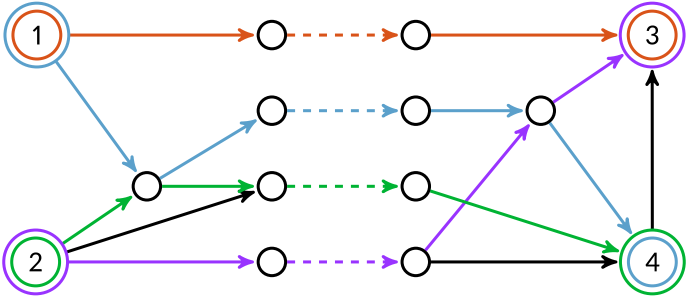
Take a collection of paths that link an excitation and a measurement to each unknown edge, bijectively.

A *monomial* μ is the product of all transfer functions of these paths

(where edges taken n times by the paths are to the power n in μ)

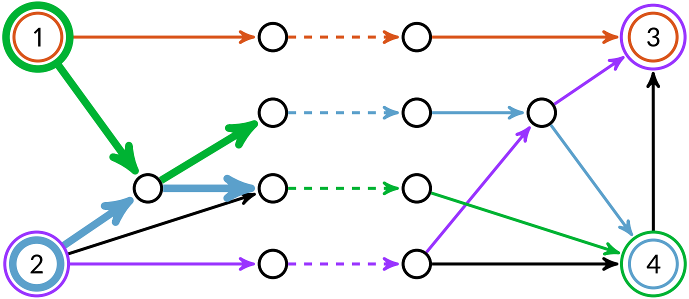
$$\mu \triangleq G_{..} \ldots G_{..} \cdot G_{..} \ldots G_{..} \cdot G_{..} \ldots G_{..} \cdot G_{..} \ldots G_{..}$$





Two different collections of paths using the same transfer functions

⇒ Same monomial μ, but different sign in expression of $\det K$



After algebra, we obtain

$$\det K = \sum_{\mu \in M} r(\mu)\, \mu$$

where

$$\mu = \prod_{\text{all unknown edges } \alpha} G_{\text{measurement},*} \dots G_{*,\alpha} \cdot G_{\alpha,*} \dots G_{*,\text{excitation}}$$

and the repetition $r(\mu)$ accounts for the sign of term μ in the expression of $\det K$.

Theorem 3

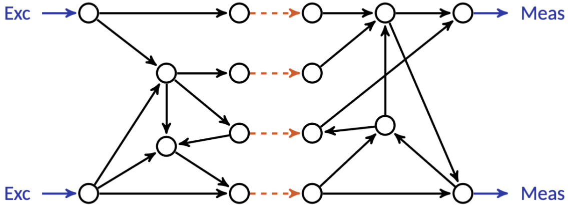
A separable network is generically globally identifiable $\Leftrightarrow$ there is at least one monomial $\mu \in M$ such that its repetition $r(\mu) \neq 0$

Future perspectives

- Path-based condition from our combinatorial characterization
- Algorithm for the synthesis problem
- Gap between local and global identifiability
- Gap between decoupled and local identifiability
- Path-based condition for single edge identifiability

Take-home message

- For identifiability: Global ⇒ Local ⇒ Decoupled
- Introduced *Separable networks*, for which Global ⇔ Local



- Necessary conditions derived on separable networks apply on networks of any topology, through the decoupled network
- Combinatorial *necessary and sufficient characterization*
- Further work:
  - Path-based condition from our combinatorial characterization
  - Algorithm for the synthesis problem

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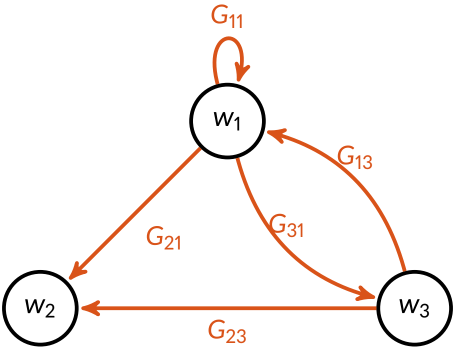
Back-up slides

Model

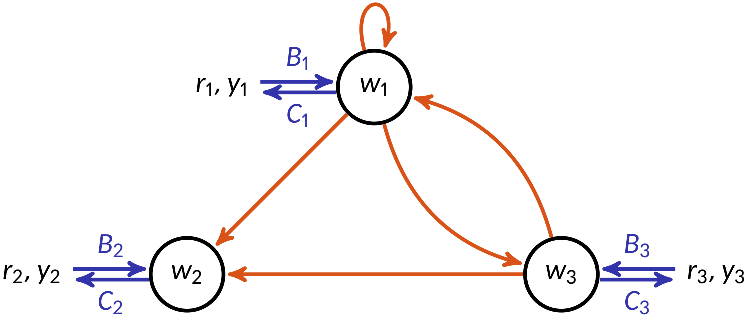
$$\text{state} \longleftarrow w_i(t) = \sum G_{ij}(q)\, w_j(t)$$

q is the shift operator, i.e. $q^{-1}w(t) = w(t-1)$



$$\text{state} \longleftarrow w_i(t) = \sum G_{ij}(q)\, w_j(t) + B_i\, r_i(t) \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y_i(t) = C_i\, w_i(t) \qquad\qquad B_i, C_i \in \{0, 1\}$$

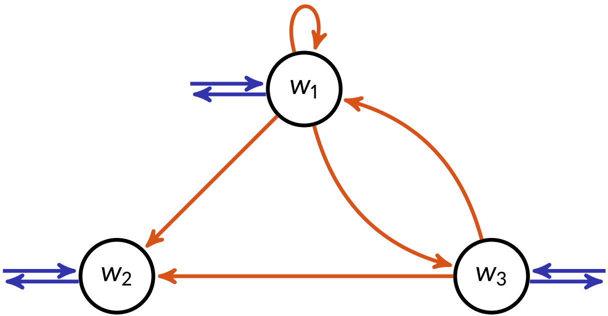


$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

Which nodes to excite/measure to recover the transfer functions?

i.e. how to choose B, C to accurately recover G?

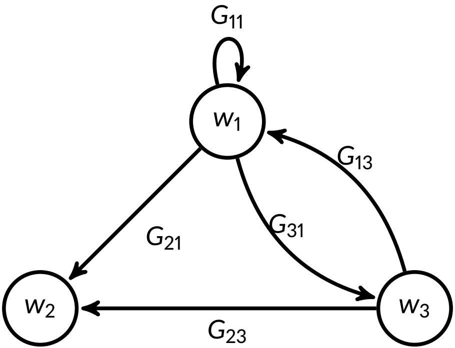


Transfer functions that can be recovered are *identifiable*

Network topology is defined by the nonzero entries of G, and is assumed to be known (often the case).

$$G = \begin{bmatrix} G_{11} & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix}$$

Theorem: *Identifiability* is a generic property of *network topology*: it only depends* on the structure of G, but not on its parameters G_{ij}.



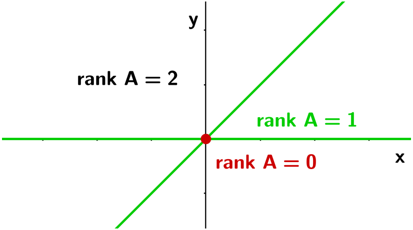
Genericity

- A generic property holds everywhere except possibly on a *lower-dimensional set.*
- A lower-dimensional set has *Lebesgue-measure zero* $\longrightarrow$ 0-probability of falling in this set when sampling randomly

Example: The matrix

$$A = \begin{bmatrix} x & 0 \\ 0 & x - y \end{bmatrix}$$

has generic rank 2. Its rank drops on $\{x = 0\} \cup \{x = y\}$.



Identifiability is generic – example

Global input-output transfer function:

$$CTB \triangleq C(I-G)^{-1}B = \begin{pmatrix} G_{42}G_{21} + G_{43}G_{31} & G_{42} & G_{43} & 1 & 0 \\ G_{52}G_{21} + G_{53}G_{31} & G_{52} & G_{53} & 0 & 1 \end{pmatrix}$$

$\Rightarrow G_{42}, G_{43}, G_{52}, G_{53}$ identif, and $\begin{pmatrix} G_{42} & G_{43} \\ G_{52} & G_{53} \end{pmatrix} \begin{pmatrix} G_{21} \\ G_{31} \end{pmatrix} = \begin{pmatrix} T_{41} \\ T_{51} \end{pmatrix}$

$\Rightarrow G_{21}, G_{31}$ identifiable except when $G_{42}G_{53} + G_{43}G_{52} = 0$.