Graph-Theoretical Condition for Local Network Identifiability

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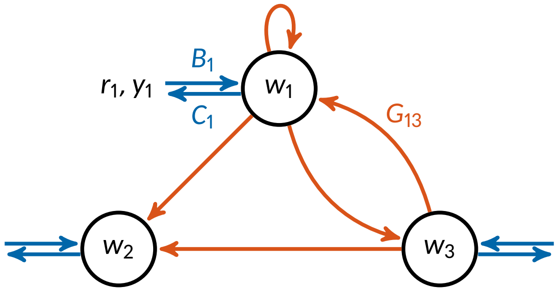
Model

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From the given exc/meas, which transfer functions can be recovered?

i.e. From r at B and y at C, which G_{ij} can be recovered?



Assumption: Network topology is known, **partial** exc/meas

Terminology: TFs that can be recovered are *identifiable*

From the definition of identifiability...

Known:	exc. & measures B, C, data $T = (I - G)^{-1}$, topology
To find:	transfer functions G

Does $CTB = C(I-G)^{-1}B$ admit a unique solution G?

Definition: G generically identifiable if

$$CTB = C\tilde{T}B \Rightarrow G = \tilde{G}$$

... we introduce *local* identifiability

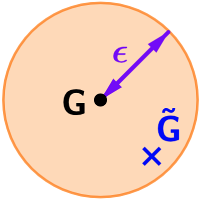
Known: exc. & measures B, C, data $T = (I - G)^{-1}$, topology

To find: transfer functions G

Does $CTB = C(I - G)^{-1}B$ admit a unique solution G?

Definition: G generically *locally* identifiable if *on an ϵ—ball,*

$$CTB = C\tilde{T}B \Rightarrow G = \tilde{G}$$



- Necessary for generic identifiability
- No counter-example to sufficiency found yet

Theorem 1 (Legat; Hendrickx 2020)

G generically locally identif $\Leftrightarrow$ $\boxed{C T \Delta T B = 0 \Rightarrow \Delta = 0}$ $\forall\ \Delta$

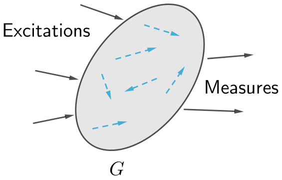
almost everywhere

Contribution

1. Definition of decoupled identifiability (necessary)
2. Allows a novel approach based on a larger graph
3. Graph-theoretical necessary condition and a sufficient one

From Theorem 1 ...

Generic ...	Network G	
identifiable	$CTB = CT'B \Rightarrow G = G'$	
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	



... we introduce decoupled identifiability

Generic ...	Network G	
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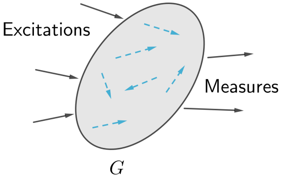
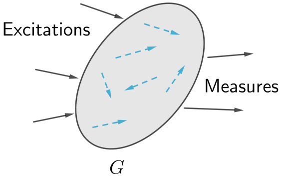
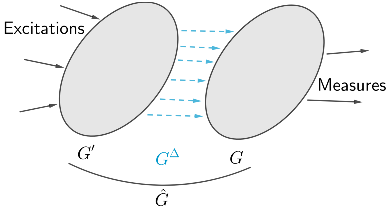
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	



... and the decoupled network

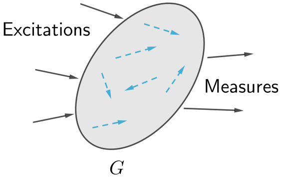
Generic …	Network G	Decoupled network $\hat{G}$
identifiable	$CTB = CT'B \Rightarrow G = G'$	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
	$\Downarrow$	
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	



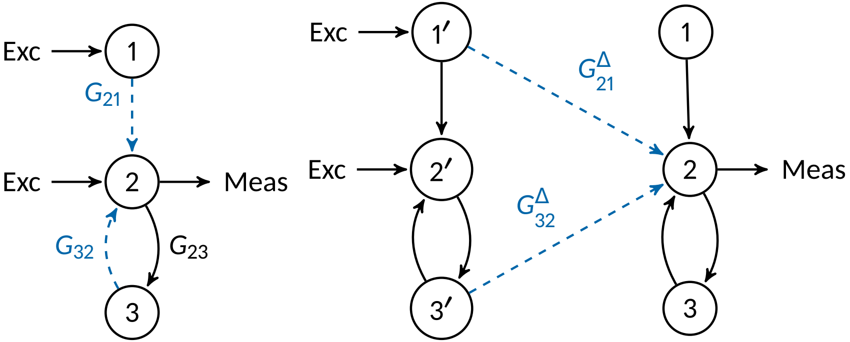


Necessary... and sufficient?

Generic ...	Network G	
identifiable	$CTB = CT'B \Rightarrow G = G'$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex so far
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex so far
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	



Basic example



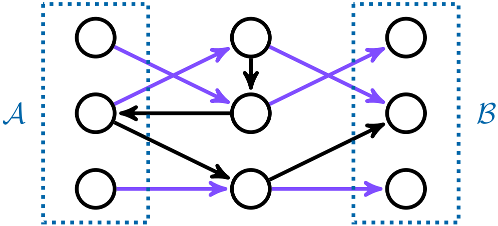
(a) (b)

(a): Unknowns in dashed blue

(b): Decoupled network: unknowns in the middle

Ingredients for our graph-theoretical condition

- Theorem 1 (Legat, Hendrickx 2020), which characterizes identifiability in terms of the rank of a matrix
- A square matrix is full-rank iFF its determinant is nonzero
- The determinant can be expressed as the sum over all row-column permutations by the Leibniz formula
- The generic rank of a matrix between two sets $\mathcal{A}$ and $\mathcal{B}$ equals the maximum number of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$



Theorem

If a network is generically decoupled-identifiable, then there is at least one assignation σ such that:

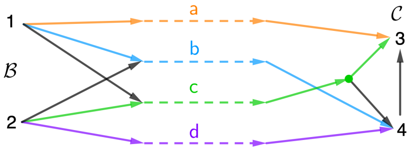
(a) $|\mathcal{C}|$ (= 2 here) unknown edges are assigned to each excitation

(b) $|\mathcal{B}|$ (= 2 here) unknown edges are assigned to each measure

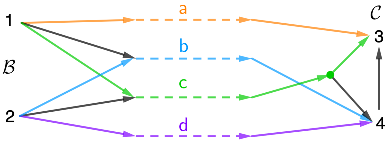
(c) σ is connected (e.g. edge a must be assigned to (1,3))

(d) for each excitation b, there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measures $\mathcal{C}$.

(e) for each measure c, there are $|\mathcal{B}|$ vertex-disjoint paths between the edges assigned to c and the measures $\mathcal{B}$.



σ OK



σ KO

A necessary condition and a sufficient one

Theorem

If a network is generically decoupled-identifiable, then there is at least one assignation σ such that:

(a) $|\mathcal{C}|$ unknown edges are assigned to each excitation

(b) $|\mathcal{B}|$ unknown edges are assigned to each measure

(c) σ is connected

(d) for each excitation b, there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measures $\mathcal{C}$.

(e) for each measure c, there are $|\mathcal{B}|$ vertex-disjoint paths between the edges assigned to c and the measures $\mathcal{B}$.

If there is only one such assignation, then this condition is also sufficient for generic decoupled identifiability.

Discussion

- The assignation σ of our theorem is *not necessarily bijective:* two edges assigned to the same excitation can be assigned to the same measure
- The assignations of condition (a) do not necessarily match the vertex-disjoint paths of condition (e).
- The assignations of condition (b) do not necessarily match the vertex-disjoint paths of condition (d).

⟶ There *could be a stronger version* of our theorem

- Our necessary condition is also necessary for (generic) (local) identifiability
- No counter-example to sufficiency found so far. The *possible equivalence* between generic: identif, local identif and decoupled-identif *remains an open question.*

Take-home message

- Introduced generic *decoupled-*identifiability,
  - Necessary for (generic) (local) identifiability
  - New: larger graph which decouples excitations and measures
- Derived a *graph-theoretical necessary condition* which applies to (generic) (local) identifiability
- Whether the sufficient condition extends as well remains an open question
- There *could be a stronger version* of our theorem, extending previous results of full excitation/measurement
- **Further work:** when not all edges are identifiable, obtain a graph-theoretical condition for the recovery of some edges