Local network identifiability: Path conditions

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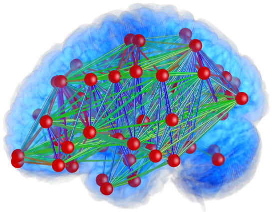
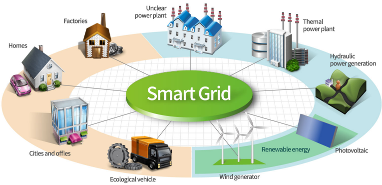
Benelux meeting 2023

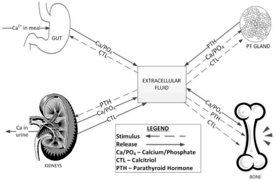
Neuroscience [Chiêm et al. 2021]



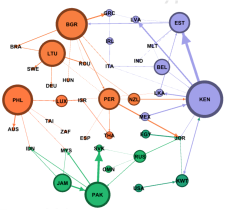
Power grids

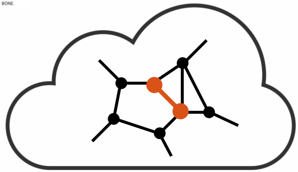


Physiological models [Christie et al. 2014]



Stock market [Shahzad et al. 2018]



Excitations →  → Measurements

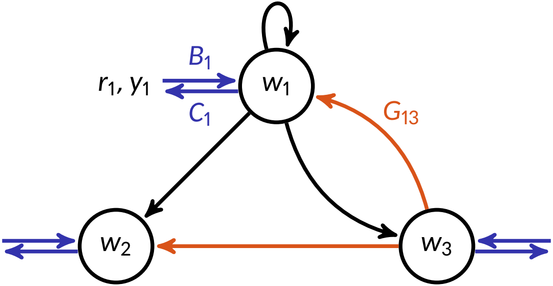
Model

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given exc/meas, can we recover the unknown transfer fcts?

i.e. From r at B and y at C, can we recover the unknown G_{ij}?



Assumptions: Network topology is known

Not all nodes are excited/measured

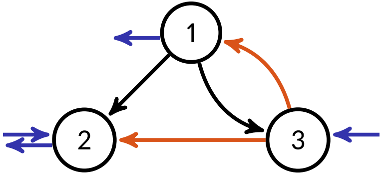
Global transfer matrix $C\,\underbrace{(I - G)^{-1}}_{T(G)}\,B$ is known

Example

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given exc/meas, can we recover the unknown transfer fcts?



Assumption: Global transfer matrix $C\,\underbrace{(I - G)^{-1}}_{T(G)}\,B$ is known

$$G = \begin{bmatrix} 0 & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

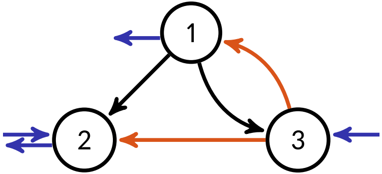
$$T \triangleq (I - G)^{-1} = \frac{1}{1 - G_{13}G_{31}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix}$$

Example

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given exc/meas, can we recover the unknown transfer fcts?



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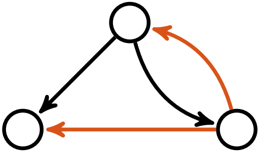
$$\mathbf{C}\,T\,\mathbf{B} = \frac{1}{1 - G_{13}G_{31}} \begin{bmatrix} 1 & 0 & G_{13} \\ G_{21} + G_{31}G_{23} & 1 & G_{23} + G_{13}G_{21} \\ G_{31} & 0 & 1 \end{bmatrix}$$

Outline

1. Context
2. Local identifiability
    - Algebraic condition
3. Decoupled identifiability
    - Path-based conditions
4. Future perspectives

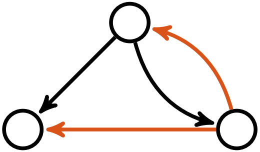
Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

Identifiability is generic



Network *identifiable* if the unknown transfer fcts can be recovered.

$$\text{Does } \underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B \text{ admit a unique solution } G\text{ ?}$$

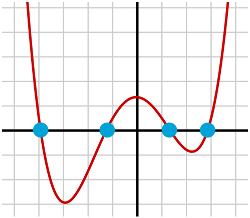
Identifiability is a property of the **graph topology.**

It does almost **not** depend on parameters G_{ij}.

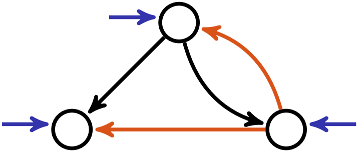
Identifiability is a generic property. It holds:

- either for almost all parameters G_{ij};
- or for no parameters G_{ij}.

Example: the nonzeroness of a polynomial is generic



State of the Art



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,\not{B}}_{\text{known}} = C\,(I - G)^{-1}\,\not{B}$ admit a unique solution G ?

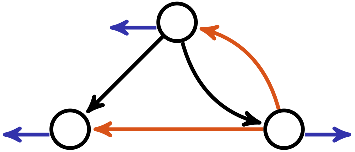
All nodes excitated: i.e. $B = I$

Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

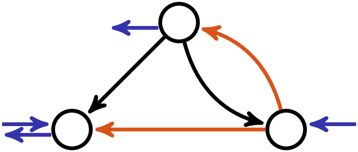
[Cheng, Shi, Van den Hof 2019]



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{\not{C}\; T\; B}_{\text{known}} = \not{C}\,(I - G)^{-1}\, B$ admit a unique solution G ?

All nodes excitated: i.e. $B = I$	Necessary and sufficient path-based condition [Hendrickx, Gevers, Bazanella 2017]
	Algo allocating measurements in the graph [Cheng, Shi, Van den Hof 2019]
All nodes measured: i.e. $C = I$	Dual results



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

All nodes excitated: Necessary and sufficient path-based condition

i.e. $B = I$

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

All nodes measured: Dual results

i.e. $C = I$

General case: Need to linearize

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From the definition of global identifiability…

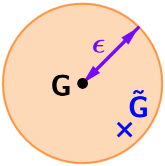
Definition: Network globally identifiable at G if for all $\tilde{G}$:

$$C\, T(\tilde{G})\, B = C\, T(G)\, B \Rightarrow \tilde{G} = G$$

… we introduce *local* identifiability

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

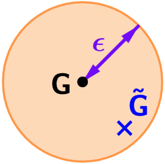
$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$



Local identifiability is necessary

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

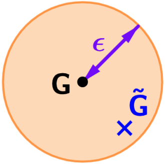


- Necessary for global identifiability
- No counter-example to sufficiency known

Algebraic condition

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$



- Necessary for global identifiability
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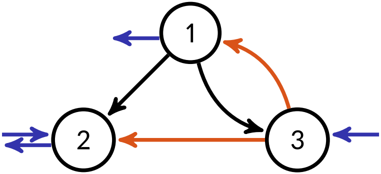
Theorem 1 (CDC 2020)

G generically locally identif $\iff$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\ \Delta$

almost everywhere

Example



$$G = \begin{bmatrix} 0 & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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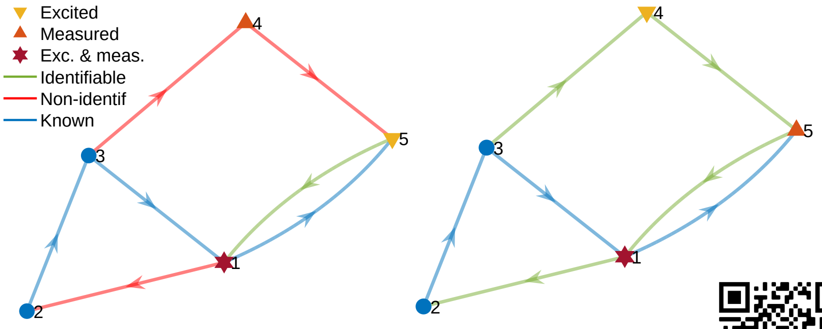
almost everywhere

$\longrightarrow$ Probability-1 algorithm: randomized, proba 0 of inaccuracy



(a) (b)

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What we have so far

Global identifiability

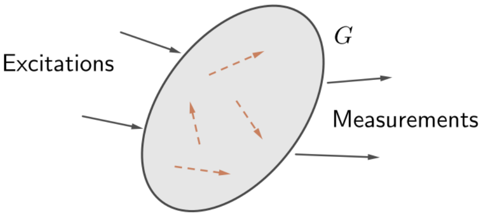
$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$$\Downarrow$$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

→ Graph interpretation?



Interpretation on a larger network

Global identifiability

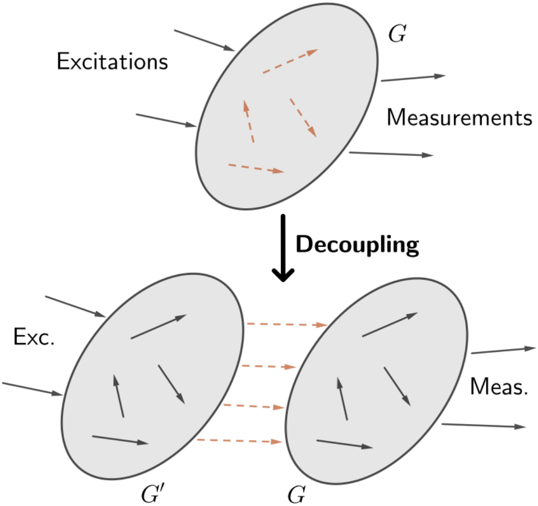
$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

Left graph = right graph?



Decoupled identifiability

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow$

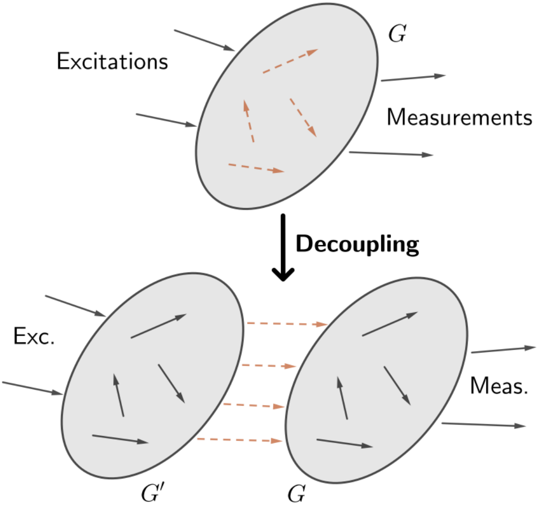
Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

$\Downarrow$

Decoupled identifiability

$CT\Delta T'B = 0 \Rightarrow \Delta = 0$



Necessary... and sufficient?

Global identifiability

$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$

$\Downarrow \quad \Uparrow?$

Local identifiability

$CT\Delta TB = 0 \Rightarrow \Delta = 0$

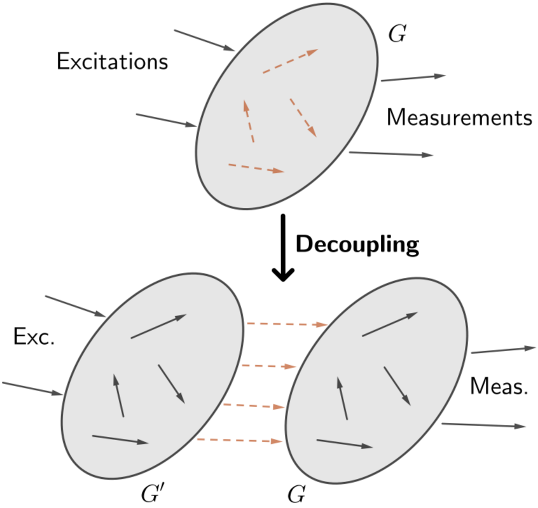
$\Downarrow \quad \Uparrow?$

Decoupled identifiability

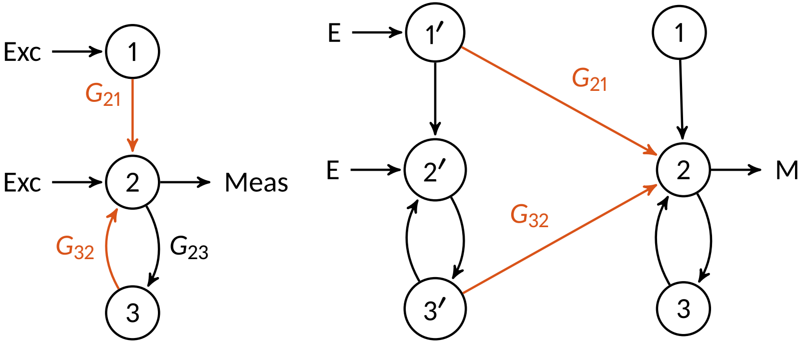
$CT\Delta T'B = 0 \Rightarrow \Delta = 0$

No counter-example known

`github.com/alegat/identifiable`



Basic example



(a) (b)

(b) is the decoupled network of (a)

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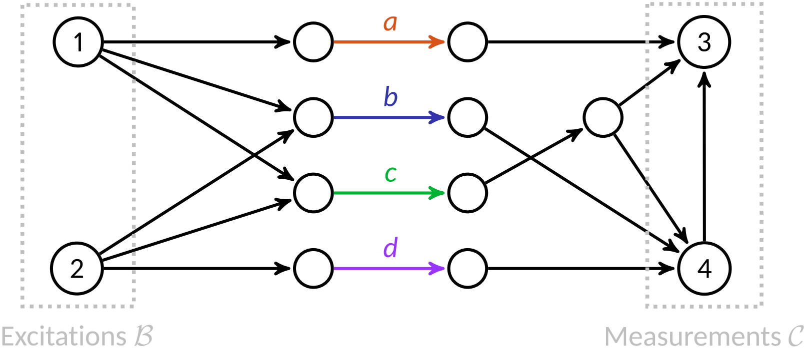
Theorem 1 - CDC 2020

G generically locally identif $\Leftrightarrow$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\ \Delta$

almost everywhere

Example:



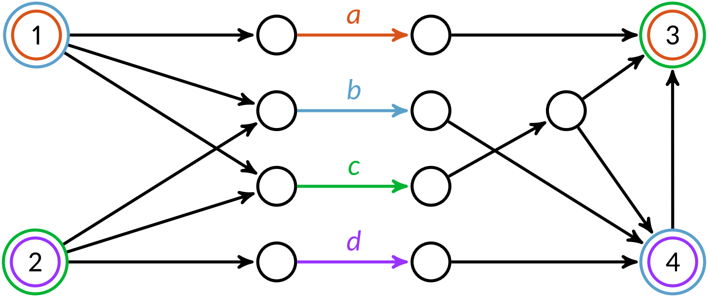
Theorem 2 (weak condition) – CDC 2021

Decoupled identif $\Rightarrow$ $\exists$ a connected bijective assignation

$\exists$ *only one* connected bijective assignation $\Rightarrow$ Decoupled identif

Example of assignation:

$$a \rightarrow (1, 4),\ b \rightarrow (1, 3),\ c \rightarrow (2, 3),\ d \rightarrow (2, 4)$$



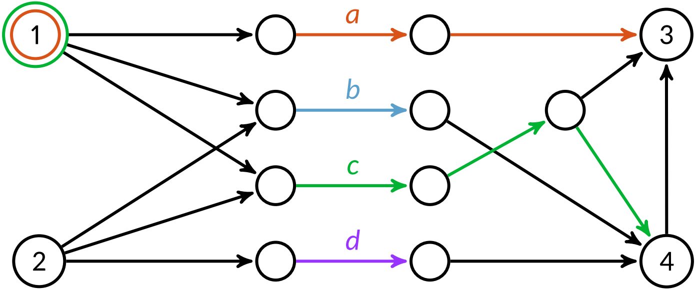
Theorem 3 (strong condition) – CDC 2021

Decoupled identif $\Rightarrow$ $\exists$ a connected assignation s.t:

- for each excitation *i*, there are 2 vertex-disjoint paths between the edges assigned to *i* and the measurements
- dual condition for the measurements

If $\exists$ only one such $\Rightarrow$ Decoupled identif

Example: Excitation 1 : OK



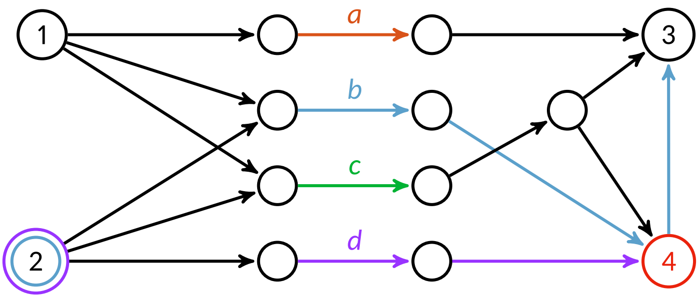
Theorem 3 (strong condition)

Decoupled identif $\Rightarrow$ $\exists$ a connected assignation s.t:

- for each excitation *i*, there are 2 vertex-disjoint paths between the edges assigned to *i* and the measurements
- dual condition for the measurements

If $\exists$ only one such $\Rightarrow$ Decoupled identif

Example: Excitation 2 : KO



Theorem 2 (weak condition)

Decoupled identif $\Rightarrow$ $\exists$ a connected *bijective* assignation

$\exists$ *only one* connected *bijective* assignation $\Rightarrow$ Decoupled identif

Theorem 3 (strong condition)

Decoupled identif $\Rightarrow$ $\exists$ a connected *vertex-disjoint* assignation

$\exists$ *only one* connected *v-disjoint* assignation $\Rightarrow$ Decoupled identif

$\longrightarrow$ There *could be a stronger condition* combining Theorems 2 & 3, extending previous results under full excitation/measurement

[Hendrickx, Gevers, Bazanella 2017]

Take-home message

- Introduced *decoupled* identifiability
  - Necessary for local and global identifiability
  - New: larger graph which decouples excitations and measurements
- Derived *path-based necessary conditions* which also apply to local and global identifiability
- Whether the sufficient conditions extend as well remains an open question
- There *could be a stronger version* of our conditions, extending previous results under full excitation/measurement
- **Further work:** when not all edges are identifiable, obtain a path-based condition for the recovery of some edges

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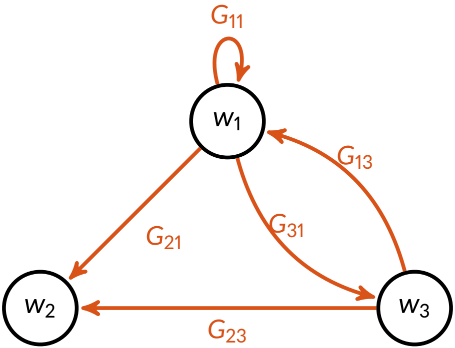
Back-up slides

Model

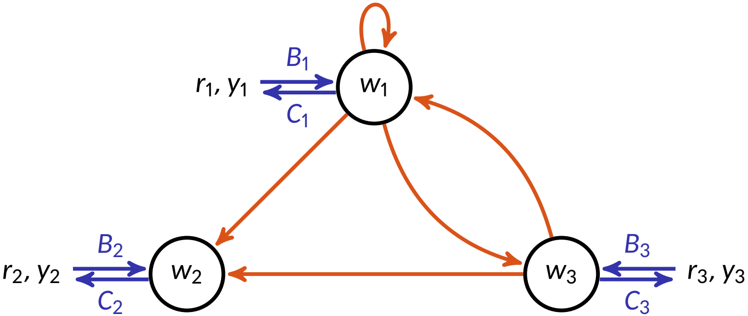
state $\longleftarrow$ $w_i(t) = \sum G_{ij}(q)\, w_j(t)$

q is the shift operator, i.e. $q^{-1}w(t) = w(t-1)$



state ⟵ $w_i(t) = \sum G_{ij}(q)\, w_j(t) + B_i\, r_i(t)$ ⟶ excitation

measure ⟵ $y_i(t) = C_i\, w_i(t)$ $\qquad B_i, C_i \in \{0, 1\}$

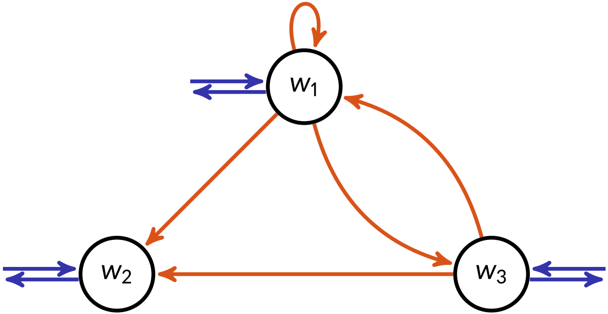


state ⟵ $w = G\,w + B\,r$ ⟶ excitation

measure ⟵ $y = C\,w$

Which nodes to excite/measure to recover the transfer functions?

i.e. how to choose B, C to accurately recover G?

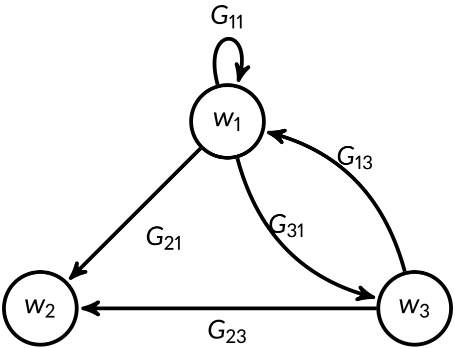


Transfer functions that can be recovered are *identifiable*

Network topology is defined by the nonzero entries of G, and is assumed to be known (often the case).

$$G = \begin{bmatrix} G_{11} & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix}$$

Theorem: *Identifiability* is a generic property of *network topology*: it only depends* on the structure of G, but not on its parameters G_{ij}.



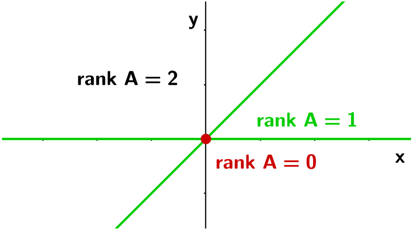
Genericity

- A generic property holds everywhere except possibly on a *lower-dimensional set*.
- A lower-dimensional set has *Lebesgue-measure zero*
  → 0-probability of falling in this set when sampling randomly

Example: The matrix

$$A = \begin{bmatrix} x & 0 \\ 0 & x - y \end{bmatrix}$$

has generic rank 2. Its rank drops on $\{x = 0\} \cup \{x = y\}$.



Identifiability is generic – example

Global input-output transfer function:

$$CTB \triangleq C(I-G)^{-1}B = \begin{pmatrix} G_{42}G_{21} + G_{43}G_{31} & G_{42} & G_{43} & 1 & 0 \\ G_{52}G_{21} + G_{53}G_{31} & G_{52} & G_{53} & 0 & 1 \end{pmatrix}$$

$$\Rightarrow G_{42}, G_{43}, G_{52}, G_{53} \text{ identif, and } \begin{pmatrix} G_{42} & G_{43} \\ G_{52} & G_{53} \end{pmatrix} \begin{pmatrix} G_{21} \\ G_{31} \end{pmatrix} = \begin{pmatrix} T_{41} \\ T_{51} \end{pmatrix}$$

$\Rightarrow G_{21}, G_{31}$ identifiable except when $G_{42}G_{53} + G_{43}G_{52} = 0$.