Algebraic and Path-Based Conditions for Local Network Identifiability

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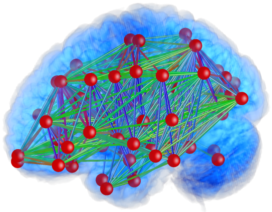
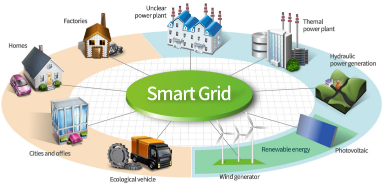
MTNS 2022

UCLouvain

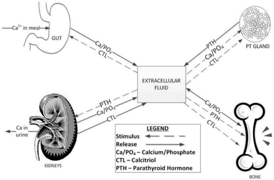
Neuroscience [Chiêm et al. 2021]



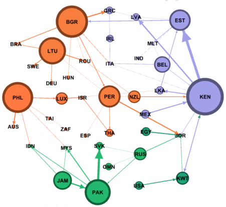
Power grids



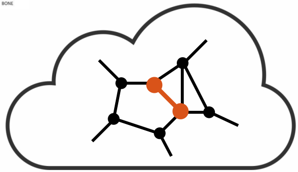
Physiological models [Christie et al. 2014]



Stock market [Shahzad et al. 2018]



Excitations →



→ Measurements

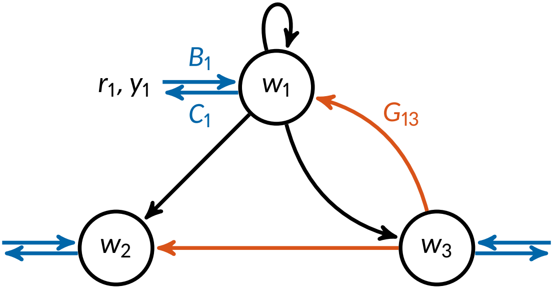
Model

$$\text{state} \longleftarrow w = G\,w + B\,r \longrightarrow \text{excitation}$$

$$\text{measure} \longleftarrow y = C\,w$$

From given exc/meas, can we recover the unknown transfer fcts?

i.e. From r at B and y at C, can we recover the unknown G_{ij}?

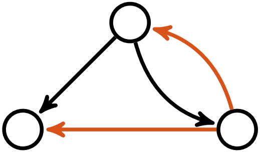


Assumptions: Network topology is known Otherwise: [van Waarde et al. 2019]

Not all nodes are excited/measured

Global transfer matrix $\underbrace{C\,(I - G)^{-1}\,B}_{T(G)}$ is known

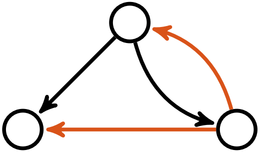
Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

Identifiability



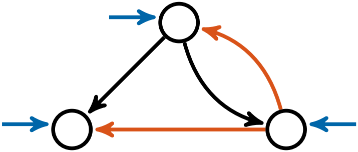
Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,B}_{\text{known}} = C\,(I - G)^{-1}\,B$ admit a unique solution G ?

Identifiability is a property of the *graph topology*.

It does *not* depend on the problem parameters.

Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{C\,T\,\not{B}}_{\text{known}} = C\,(I - G)^{-1}\,\not{B}$ admit a unique solution G ?

All nodes excitated: i.e. $B = I$

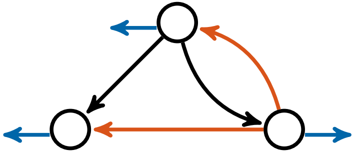
Necessary and sufficient path-based condition

[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

Identifiability



Network *identifiable* if the unknown transfer fcts can be recovered.

Does $\underbrace{\not{C}\,T\,B}_{\text{known}} = \not{C}\,(I - G)^{-1}\,B$ admit a unique solution G ?

All nodes excitated: i.e. $B = I$ — Necessary and sufficient path-based condition

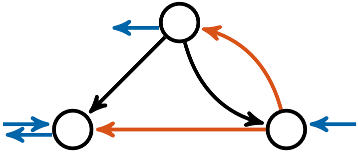
[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

[Cheng, Shi, Van den Hof 2019]

All nodes measured: i.e. $C = I$ — Dual results

Identifiability



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[Hendrickx, Gevers, Bazanella 2017]

Algo allocating measurements in the graph

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All nodes measured: i.e. $C = I$

Dual results

General case: Need to linearize

From the definition of identifiability...

Definition: Network identifiable at G if for all $\tilde{G}$:

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

Network *generically* identifiable if it holds at *almost*[1] *all* G.

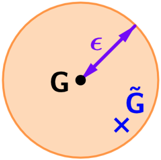
[1] except possibly on a lower-dimensional set

… we introduce *local* identifiability

Definition: Network *locally* identifiable at G if for all $\tilde{G}$ *on an ϵ—ball:*

$$C\,T(\tilde{G})\,B = C\,T(G)\,B \Rightarrow \tilde{G} = G$$

Network *generically locally* identifiable if it holds at *almost all* G.



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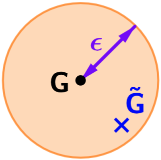
- Necessary for generic identifiability
- No counter-example to sufficiency known



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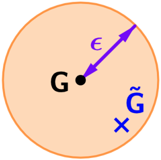


Theorem 1

G generically locally identif $\Leftrightarrow$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\,\Delta$

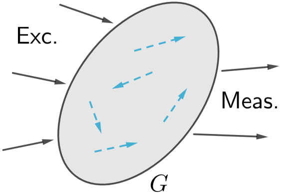
almost everywhere

Outline

1. Definition of decoupled identifiability (necessary)
   Allows a novel approach based on a larger graph
2. Ingredients for our path-based condition
3. Path-based necessary condition and a sufficient one

From Theorem 1 ...

Generic ...	Network G	
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	



... we introduce decoupled identifiability

Generic ...	Network G	
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	
	$\Downarrow$	
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
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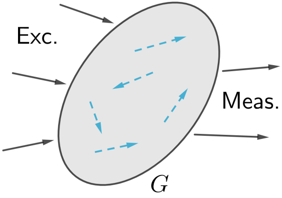
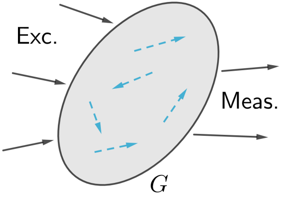
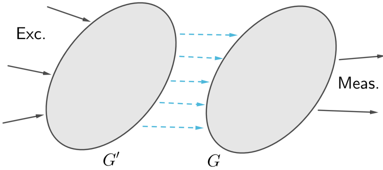
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	



... and the decoupled network

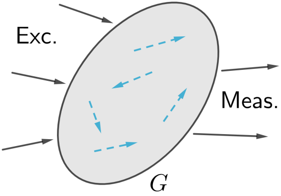
Generic ...	Network G	Decoupled network
identifiable	$C\tilde{T}B = CTB \Rightarrow \tilde{G} = G$	$C T \Delta T' B = 0 \Rightarrow \Delta = 0$
	$\Downarrow$	
local identif	$C T \Delta T B = 0 \Rightarrow \Delta = 0$	
	$\Downarrow$	
decoupled-identif	$C T \Delta T' B = 0 \Rightarrow \Delta = 0$	





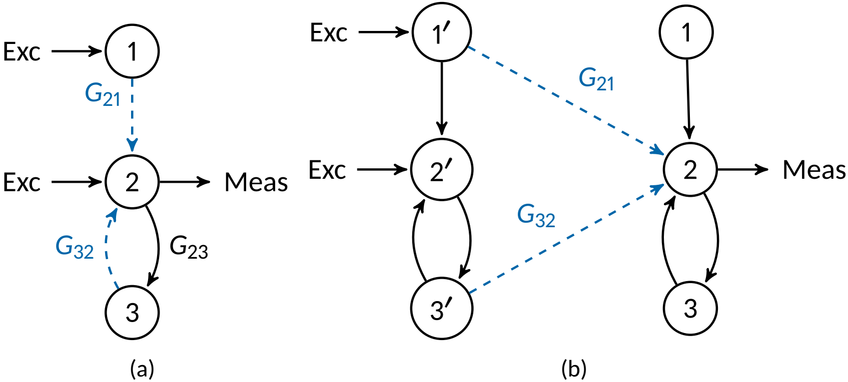
Necessary... and sufficient?

Generic ...	Network G	
identifiable	$CTB = C\tilde{T}B \Rightarrow G = \tilde{G}$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex known
local identif	$CT\Delta TB = 0 \Rightarrow \Delta = 0$	
	$\Downarrow \quad \Uparrow ?$	No counter-ex known
decoupled-identif	$CT\Delta T'B = 0 \Rightarrow \Delta = 0$	(10^6 systematic trials)



github.com/alegat/identifiable

Basic example



(a): Unknowns in dashed blue

(b): Decoupled network: unknowns in the middle

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Ingredient 1: assignations

Theorem 1 can be formulated as a determinant, which can be expressed as the sum over all *assignations:*

$$\sigma : \text{unknown edges} \longrightarrow (\text{excitation}, \text{measurement})$$

Reminder:

Theorem 1

G generically locally identif $\iff$ $\boxed{CT\Delta TB = 0 \Rightarrow \Delta = 0}$ $\forall\, \Delta$ almost everywhere

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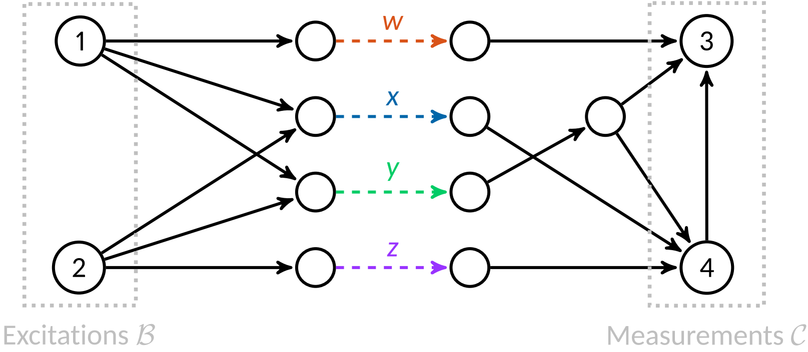
Example:



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Theorem 1 can be formulated as a determinant, which can be expressed as the sum over all *assignations:*

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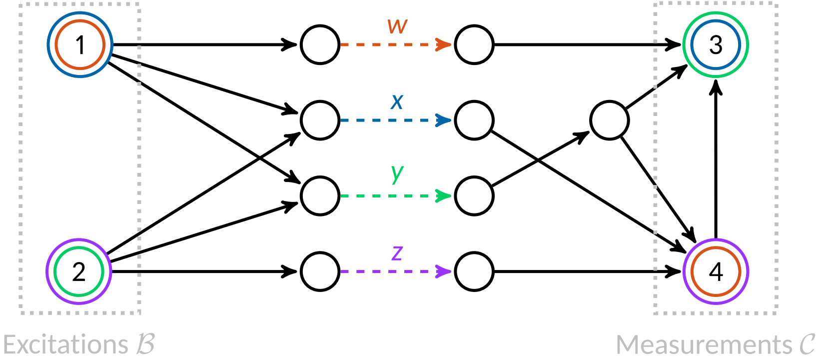
Example: $w \to (1,4),\ x \to (1,3),\ y \to (2,3),\ z \to (2,4)$

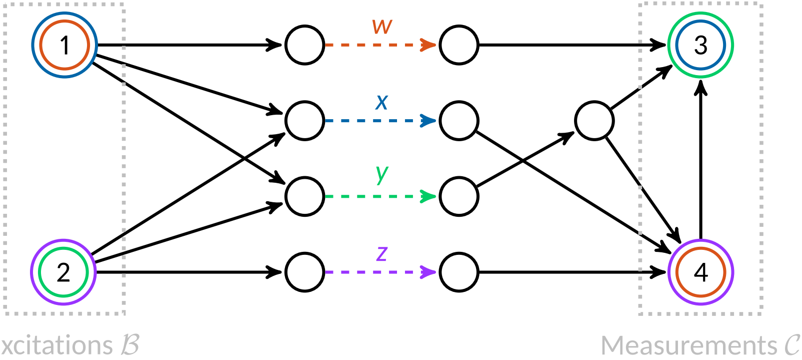


Ingredient 2: connected assignations

$$\sigma : \text{unknown edges} \longrightarrow (\text{excitation}, \text{measurement})$$

σ is *connected* if for each unknown edge, there is a path from its assigned exc. to its assigned meas., including the unknown edge.

Example: $w \rightarrow (1, 4),\ x \rightarrow (1, 3),\ y \rightarrow (2, 3),\ z \rightarrow (2, 4)$



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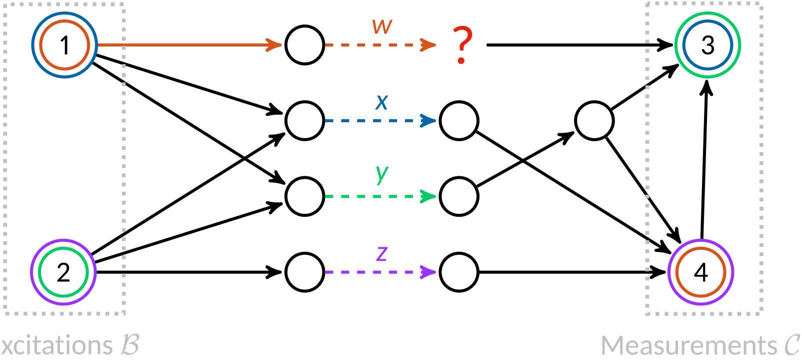
Example: ~~$w \rightarrow (1, 4)$~~, $x \rightarrow (1, 3)$, $y \rightarrow (2, 3)$, $z \rightarrow (2, 4)$ KO



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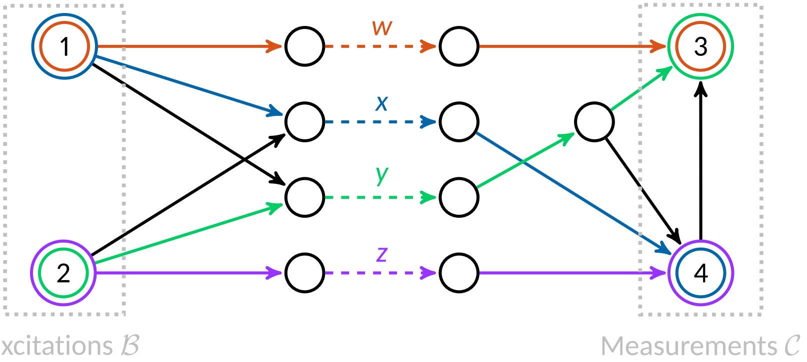
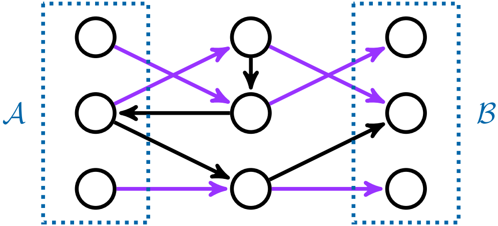


Ingredient 3: vertex-disjoint paths

Theorem 1 can be formulated in terms of *generic rank* of T.

> The generic rank of a matrix T between two sets $\mathcal{A}$ and $\mathcal{B}$ equals the max number of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$.



3 v-d paths

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1. Definition of decoupled identifiability (necessary)
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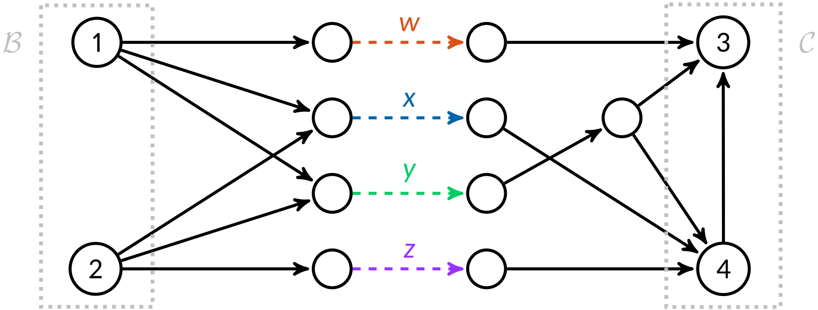
Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

- For each excitation b,
  - (a) $|\mathcal{C}|$ unknown edges are assigned to b
  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measurements $\mathcal{C}$.
- For each measurement, dual of (a) and (b) with $\mathcal{B}$.



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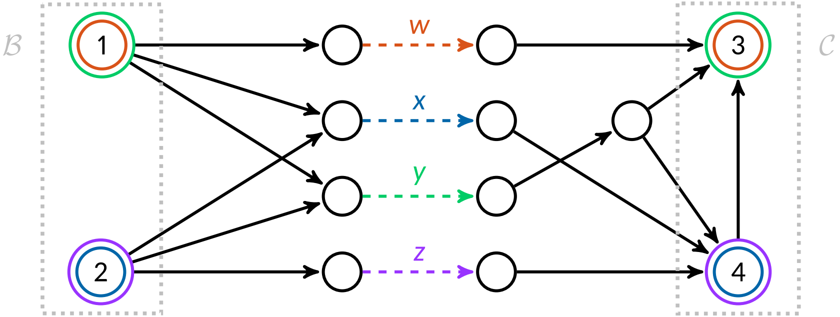

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected ($\longrightarrow$ OK) assignation σ such that:

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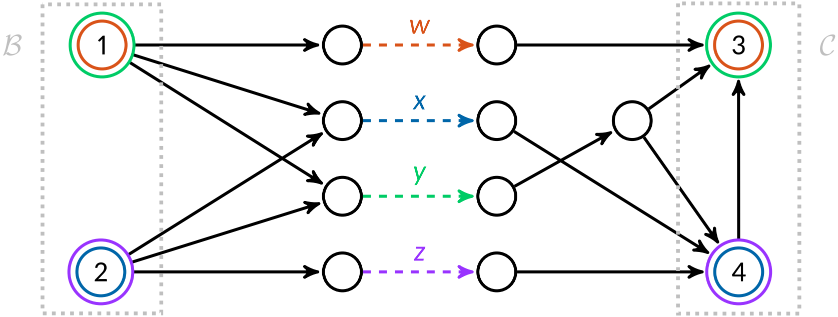

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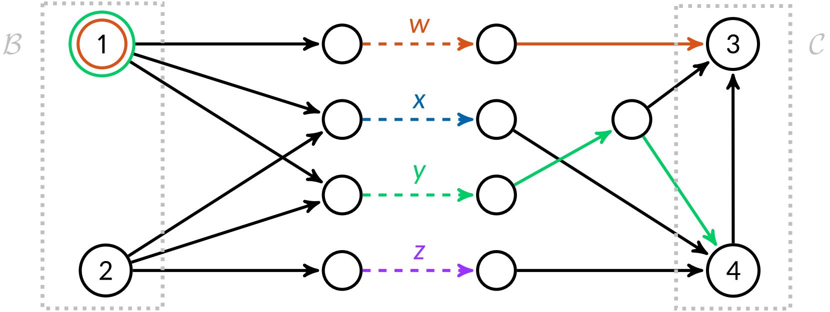

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected (⟶ OK) assignation σ such that:

- For each excitation b,
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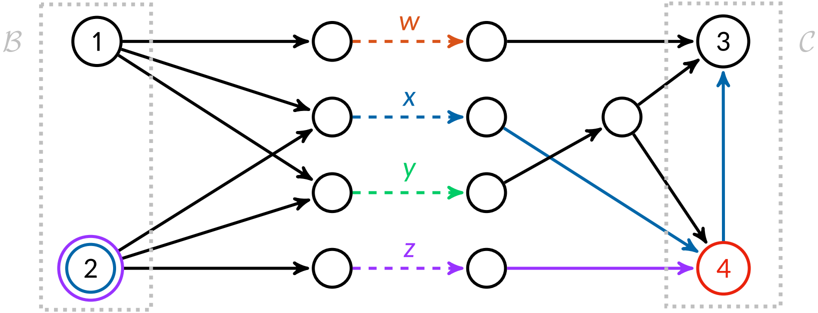
  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measurements $\mathcal{C}$ ⟶ KO
- For each measurement, dual of (a) and (b) with $\mathcal{B}$.



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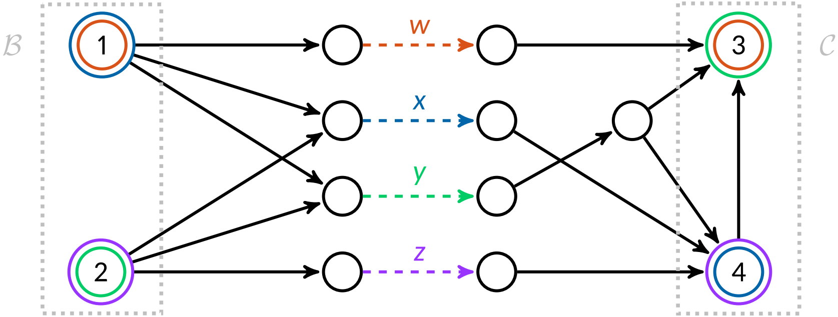
  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measurements $\mathcal{C}$ $\longrightarrow$ OK
- For each measurement, dual of (a) and (b) with $\mathcal{B}$ $\longrightarrow$ OK



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  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measurements $\mathcal{C}$
- For each measurement, dual of (a) and (b) with $\mathcal{B}$

This condition is also *necessary* for generic *(local) identifiability* since:

Generic identif $\Rightarrow$ Generic local identif $\Rightarrow$ Generic decoupled-identif

A necessary condition and a sufficient one

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

- For each excitation b,
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- For each measurement, dual of (a) and (b) with $\mathcal{B}$

If there is only one such assignation, then this condition is also sufficient for generic decoupled identifiability.

Theorem 2

If a network is generically decoupled-identifiable, then there is at least one connected assignation σ such that:

- For each excitation b,
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  - (b) there are $|\mathcal{C}|$ vertex-disjoint paths between the edges assigned to b and the measurements $\mathcal{C}$
- For each measurement, dual of (a) and (b) with $\mathcal{B}$

If there is only one such assignation, then this condition is also sufficient for generic decoupled identifiability.

- σ is *not necessarily bijective:* two unknown edges can be assigned to the same (excitation, measurement) pair
- The vertex-disjoint paths of condition (b) do not necessarily match the assigned measurements.

⟶ There *could be a stronger version* of Theorem 2.

Take-home message

- Introduced generic *decoupled*-identifiability,
  - Necessary for generic (local) identifiability
  - New: larger graph which decouples excitations and measures
- Derived a *path-based necessary condition* which also applies to generic (local) identifiability
- Whether the sufficient condition extends as well remains an open question
- There *could be a stronger version* of our theorem, extending previous results under full excitation/measurement
- **Further work:** when not all edges are identifiable, obtain a path-based condition for the recovery of some edges

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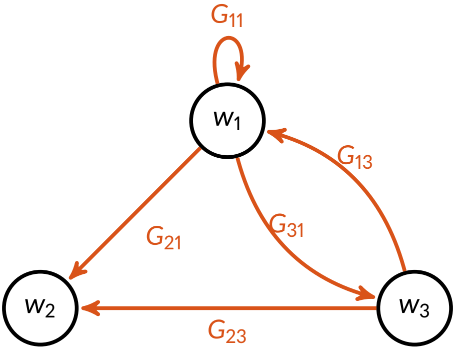
Back-up slides

Model

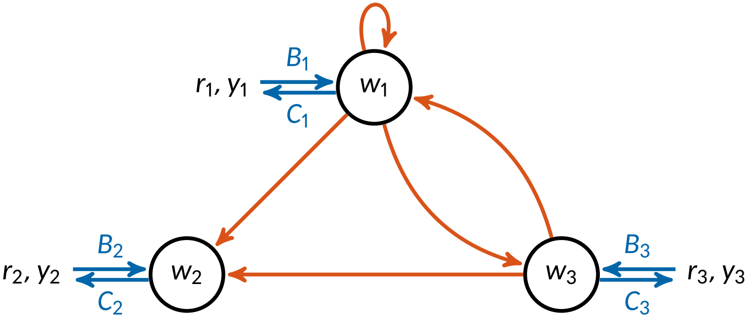
$$\text{state} \longleftarrow w_i(t) = \sum G_{ij}(q)\, w_j(t)$$

q is the shift operator, i.e. $q^{-1}w(t) = w(t-1)$



state ⟵ $w_i(t) = \sum G_{ij}(q)\, w_j(t) + B_i\, r_i(t)$ ⟶ excitation

measure ⟵ $y_i(t) = C_i\, w_i(t)$ $\qquad B_i, C_i \in \{0, 1\}$

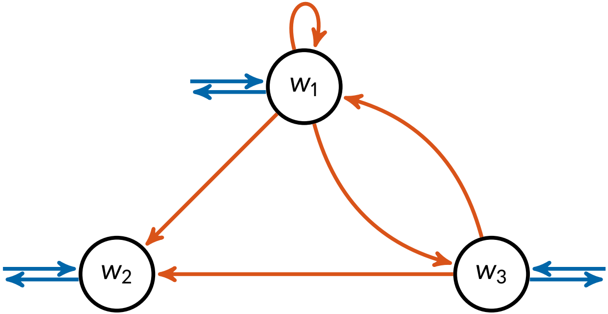


state ⟵ $w = G\,w + B\,r$ ⟶ excitation

measure ⟵ $y = C\,w$

Which nodes to excite/measure to recover the transfer functions?

i.e. how to choose B, C to accurately recover G?

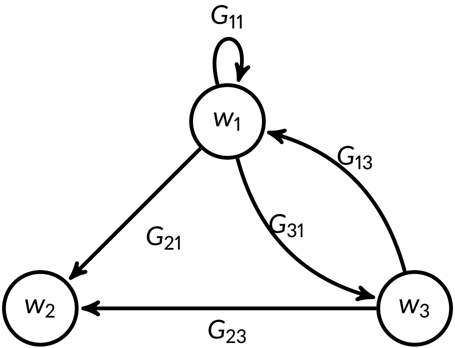


Transfer functions that can be recovered are *identifiable*

Network topology is defined by the nonzero entries of G, and is assumed to be known (often the case).

$$G = \begin{bmatrix} G_{11} & 0 & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & 0 \end{bmatrix}$$

Theorem: *Identifiability* is a generic property of *network topology*: it only depends* on the structure of G, but not on its parameters G_{ij}.



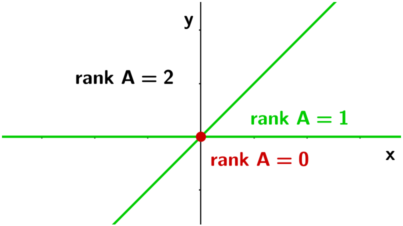
Genericity

- A generic property holds everywhere except possibly on a *lower-dimensional set.*
- A lower-dimensional set has *Lebesgue-measure zero* → 0-probability of falling in this set when sampling randomly

Example: The matrix

$$A = \begin{bmatrix} x & 0 \\ 0 & x - y \end{bmatrix}$$

has generic rank 2. Its rank drops on $\{x = 0\} \cup \{x = y\}$.



Identifiability is generic – example

Global input-output transfer function:

$$CTB \triangleq C(I-G)^{-1}B = \begin{pmatrix} G_{42}G_{21} + G_{43}G_{31} & G_{42} & G_{43} & 1 & 0 \\ G_{52}G_{21} + G_{53}G_{31} & G_{52} & G_{53} & 0 & 1 \end{pmatrix}$$

$$\Rightarrow G_{42}, G_{43}, G_{52}, G_{53} \text{ identif, and } \begin{pmatrix} G_{42} & G_{43} \\ G_{52} & G_{53} \end{pmatrix} \begin{pmatrix} G_{21} \\ G_{31} \end{pmatrix} = \begin{pmatrix} T_{41} \\ T_{51} \end{pmatrix}$$

$\Rightarrow G_{21}, G_{31}$ identifiable except when $G_{42}G_{53} + G_{43}G_{52} = 0$.