Remarks on the strong maximum principle

Haïm Brezis(1),(2) and Augusto C. Ponce(1),(2)

1 Introduction

The strong maximum principle asserts that if $u$ is smooth, $u \geq 0$ and $-\Delta u \geq 0$ in a connected domain $\Omega \subset \mathbb{R}^N$, then either $u \equiv 0$ or $u > 0$ in $\Omega$. The same conclusion holds when $-\Delta$ is replaced by $-\Delta + a(x)$ with $a \in L^p(\Omega)$, $p > \frac{N}{2}$ (this is a consequence of Harnack’s inequality; see e.g. Stampacchia [1], and also Trudinger [1], Corollary 5.3). Another formulation of the same fact says that if $u(x_0) = 0$ for some point $x_0 \in \Omega$, then $u \equiv 0$ in $\Omega$. A similar conclusion fails, however, when $a \not\in L^p(\Omega)$, for any $p > \frac{N}{2}$. For instance, $u(x) = |x|^2$ satisfies $-\Delta u + a(x)u = 0$ in $B_1$ with $a = \frac{2N}{|x|^2} \not\in L^{N/2}(\Omega)$.

If $u$ vanishes on a larger set, one may still hope to conclude, under some weaker condition on $a$, that $u \equiv 0$ in $\Omega$. Such a result was obtained by Bénilan-Brezis [1, Appendix D] (with a contribution by R. Jensen) in the case where $a \in L^1(\Omega)$ and supp $u$ is a compact subset of $\Omega$. Their maximum principle has been further extended by Ancona [1], who proved Theorem 1 below.

We recall that a function $v : \Omega \to \mathbb{R}$ is quasicontinuous if there exists a sequence of open subsets $(\omega_n)$ of $\Omega$ such that $v|_{\Omega \setminus \omega_n}$ is continuous $\forall n \geq 1$ and cap $\omega_n \to 0$ as $n \to \infty$, where cap $\omega_n$ denotes the $H^1$-capacity of $\omega_n$.

Theorem 1 (Ancona [1]) Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set. Let $u \in L^1(\Omega)$, $u \geq 0$ a.e. in $\Omega$, be such that $\Delta u$ is a Radon measure on $\Omega$. Then there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in $\Omega$.

Let $a \in L^1(\Omega)$, $a \geq 0$ a.e. in $\Omega$. If

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

in the following sense

$$\int_E \Delta u \leq \int_E au \quad \text{for every Borel set } E \subset \Omega,$$

and if $\tilde{u} = 0$ on a set of positive $H^1$-capacity in $\Omega$, then $u = 0$ a.e. in $\Omega$.

The proof given by Ancona is purely based on Potential Theory, while ours is more direct in the spirit of PDE’s. We also discuss carefully the meaning of the condition $-\Delta u + au \geq 0$ in $\Omega$.

The next two corollaries follow immediately from the theorem above:

Corollary 2 Let $u$ and $a$ be as in Theorem 1, and suppose (1) is satisfied.

If $u = 0$ on a subset of $\Omega$ with positive measure, then $u = 0$ a.e. in $\Omega$.

If $u$ is continuous in $\Omega$ and $u = 0$ on a subset of $\Omega$ with positive $H^1$-capacity, then $u \equiv 0$ in $\Omega$.
Corollary 3 Let $u$ and $a$ be as in Theorem 1. Suppose that $\Delta u \in L^1(\Omega)$.
If
$$-\Delta u + au \geq 0 \quad a.e. \text{ in } \Omega$$
and $u = 0$ on a subset of $\Omega$ with positive measure, then $u = 0$ a.e. in $\Omega$.

The next corollary follows from Theorem 1 and Remark 3:

Corollary 4 Let $u$ and $a$ be as in Theorem 1. Suppose that $au \in L^1_{loc}(\Omega)$.
If
$$-\Delta u + au \geq 0 \quad \text{in } D'(\Omega),$$
i.e.
$$\int u\Delta \varphi \leq \int au\varphi \quad \forall \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \text{ in } \Omega,$$and $u = 0$ on a subset of $\Omega$ with positive measure, then $u = 0$ a.e. in $\Omega$.

Remark 1 In view of Corollary 4 above, it would seem natural to replace condition (1) in Theorem 1 by
$$\int u\Delta \varphi \leq \int au\varphi \quad \forall \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \text{ in } \Omega,$$(2)which makes sense even if $au \notin L^1_{loc}(\Omega)$ (note that $au\varphi \geq 0$ a.e., so that the right-hand side is always well-defined, possibly taking the value $+\infty$). However, the strong maximum principle is no longer true in general. See Remark 4.

There are several interesting questions related to Theorem 1:

Open problem 1. In the statement of Theorem 1, suppose in addition that $\text{supp } u \subset \Omega$ is a compact set. Can one replace the assumption $a \in L^1_{loc}$ by a weaker condition, for example $a^{1/2} \in L^1_{loc}$ (or $a^{1/2} \in L^p_{loc}$ for some $p > 1$), and still conclude that $u = 0$ a.e. in $\Omega$?

Note that one cannot hope to go below $L^{1/2}$. For instance the $C^2$-function $u$ given by
$$u(x) = \begin{cases} (1 - |x|^2)^4 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases},$$satisfies $-\Delta u + au \geq 0$ for some function $a(x)$ such that $a(x) \sim \frac{1}{(1 - |x|)^2}$ for $|x| \lesssim 1$. Here $a^\alpha \in L^1$ $\forall \alpha < 1/2$, but $a^{1/2} \notin L^1$.

Here is another one:

Open problem 2. Assume $u \in C^0$, $u \geq 0$, and $a \in L^q_{loc}$ for some $q \geq 1$, $a \geq 0$ a.e., satisfy (1). Suppose that $u = 0$ on a set $E$ with $\text{cap}_{1,2q}(E) > 0$, where $\text{cap}_{1,2q}$ refers to the capacity associated with the Sobolev space $W^{1,2q}$. Can one conclude that $u \equiv 0$?

Theorem 1 above shows that the answer is positive when $q = 1$. It is also true when $q > \frac{N}{2}$ by the strong maximum principle mentioned above (note that if $q > \frac{N}{2}$ and $x_0$ is any point, then $\text{cap}_{1,2q}(\{x_0\}) > 0$).
2 Some comments about condition (1)

Since in the statement of Theorem 1 it may happen that $au \not\in L^1_{\text{loc}}(\Omega)$, and so $au$ is not necessarily a distribution, one should be careful in order to give a precise meaning to the inequality

$$\Delta u \leq au \quad \text{in } \Omega.$$ 

More generally, let $\mu$ be a Radon measure on $\Omega$ and $f$ a measurable function, $f \geq 0$ a.e. in $\Omega$. Here are two possible definitions for the inequality $\mu \leq f$ in $\Omega$:

**Definition 1** We shall write $\mu \leq_1 f$ in $\Omega$ if

$$\int_E d\mu \leq \int_E f \quad \text{for every Borel set } E \subset \Omega.$$ 

**Definition 2** We shall write $\mu \leq_2 f$ in $\Omega$ if

$$\int \varphi d\mu \leq \int f \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega.$$ 

In the first definition, we view $f$ as the nonnegative measure $f \, dx$, while in the second one $f$ is treated as if it were a distribution.

**Remark 2** If $\mu \leq_1 f$ in $\Omega$, then $\mu \leq_2 f$ in $\Omega$. However, the converse is not true in general. See Remark 4 below.

**Remark 3** If we assume in addition that $f \in L^1_{\text{loc}}(\Omega)$, then $\mu \leq_1 f$ in $\Omega$ if, and only if, $\mu \leq_2 f$ in $\Omega$.

**Remark 4** Theorem 1 above is no longer true in general (even for the case where $\Delta u \in L^1(\Omega)$) if we replace (1) by

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

i.e. if

$$\int u \Delta \varphi \leq \int au \varphi \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega.$$ 

In fact, let $N \geq 2$. Take $v \in L^1(\mathbb{R}^N)$, $v \geq 0$ a.e. in $\mathbb{R}^N$, such that $\text{supp } v \subset B_1$, $\Delta v \in L^1(\mathbb{R}^N)$, but $v$ is unbounded (this is possible since $N \geq 2$). In particular, there exists $b \in L^1(\mathbb{R}^N)$, $b \geq 0$ a.e. in $\mathbb{R}^N$, such that $bv \not\in L^1(\mathbb{R}^N)$.

Let $(x_j) \subset B_1$ be a dense sequence in $B_1$ and, for each $j \geq 1$, let

$$\gamma_j := \min \left\{ \frac{1}{j}, \frac{1 - |x_j|}{2} \right\}.$$
We define
\[ u(x) := \sum_{j=1}^{\infty} \frac{1}{2j\gamma_j N - 2} v\left(\frac{x-x_j}{\gamma_j}\right), \]
\[ a(x) := \sum_{j=1}^{\infty} \frac{1}{2j\gamma_j N} b\left(\frac{x-x_j}{\gamma_j}\right). \]

Then
\[ u \in L^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N, \]
\[ \Delta u \in L^1(\mathbb{R}^N), \]
\[ a \in L^1(\mathbb{R}^N), \quad a \geq 0 \text{ a.e. in } \mathbb{R}^N, \]
and
\[ \int u \Delta \varphi \leq \int au \varphi \quad \forall \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \text{ in } \Omega, \]
(note that the integral in the right-hand side is either 0 or +\infty), but \( \text{supp } u \subset \overline{B}_1 \) and \( u \not\equiv 0 \) in \( \mathbb{R}^N \). On the other hand, in view of Theorem 1, the inequality \( \Delta u \leq au \) is not satisfied.

From now on, we shall always consider the inequality \( \Delta u \leq au \) in the sense of Definition 1. In particular we shall omit the subscript 1 in the symbol \( \leq_1 \).

### 3 Proof of the quasicontinuity statement of Theorem 1

Before proving the first part of Theorem 1 (see Lemma 1 below), we make the following remark:

**Remark 5** If \( v \in H^1_{\text{loc}}(\Omega) \), then there exists \( \tilde{v} : \Omega \to \mathbb{R} \) quasicontinuous such that \( v = \tilde{v} \) a.e. in \( \Omega \) (see e.g. Lewy-Stampacchia [1]). In addition, \( \tilde{v} \) is well-defined modulo polar subsets of \( \Omega \), i.e. if \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are two quasicontinuous functions such that \( \tilde{v}_1 = v = \tilde{v}_2 \) a.e. in \( \Omega \), then there exists a polar set \( P \subset \Omega \) such that \( \tilde{v}_1(x) = \tilde{v}_2(x) \quad \forall x \in \Omega \setminus P \) (see Deny [1]).

**Notation.** Given \( k > 0 \), we denote by \( T_k : \mathbb{R} \to \mathbb{R} \) the truncation function
\[
T_k(s) := \begin{cases} 
  k & \text{if } s \geq k, \\
  s & \text{if } -k < s < k, \\
  -k & \text{if } s \leq -k. 
\end{cases}
\]

The existence of a quasicontinuous function \( \tilde{u} : \Omega \to \mathbb{R} \) such that \( u = \tilde{u} \) a.e. in \( \Omega \) as in the statement of Theorem 1 is a consequence of Lemma 1 below (see Ancona [1]):
Lemma 1 Let \( \Omega \subset \mathbb{R}^N \) be an open set. Assume \( u \in L^1(\Omega) \) is such that \( \Delta u \) is a Radon measure on \( \Omega \). Then

\[
T_k(u) \in H^1_{\text{loc}}(\Omega) \quad \forall k > 0,
\]

and, for each open subset \( A \subset \subset \Omega \), there exists \( C_A > 0 \) so that

\[
\int_A |\nabla T_k(u)|^2 \leq k \left( \int_\Omega |\Delta u| + C_A \int_\Omega |u| \right) \quad \forall k > 0.
\]

Moreover, there exists \( \tilde{u} : \Omega \to \mathbb{R} \) quasicontinuous such that \( u = \tilde{u} \) a.e. in \( \Omega \).

Proof. We shall split the proof of Lemma 1 into two steps:

Step 1. Proof of (3) and (4).

Proof. We first extend \( u \) to the whole \( \mathbb{R}^N \) so that \( u \equiv 0 \) outside \( \Omega \). Let \( \rho \in C^\infty_0(B_1) \) be a radial, nonnegative, mollifier. Set

\[
u_{\epsilon}(x) := \rho_{\epsilon} \ast u(x) = \int_\Omega \rho_{\epsilon}(x - y)u(y) \, dy \quad \forall x \in \Omega.
\]

For \( k > 0 \) fixed, we have \( T_k(u_{\epsilon}) \in H^1(\Omega) \) and

\[
\nabla T_k(u_{\epsilon}) = \nabla u_{\epsilon} \chi_{\{|u_{\epsilon}| < k\}},
\]

where \( \chi_{\{|u_{\epsilon}| < k\}} \) denotes the characteristic function of the set \( \{|u_{\epsilon}| < k\} \).

Given an open set \( A \subset \subset \Omega \), let \( \varphi \in C^\infty_0(\Omega) \) be such that \( 0 \leq \varphi \leq 1 \) in \( \Omega \) and \( \varphi \equiv 1 \) on \( A \). On the one hand, using (5) and integrating by parts, we have

\[
\int \nabla T_k(u_{\epsilon})^2 \varphi = \int \nabla T_k(u_{\epsilon}) \cdot (\nabla u_{\epsilon}) \varphi
\]

\[
= - \int T_k(u_{\epsilon})(\Delta u_{\epsilon}) \varphi - \int T_k(u_{\epsilon}) \nabla u_{\epsilon} \cdot \nabla \varphi. \tag{6}
\]

On the other hand,

\[
\int T_k(u_{\epsilon}) \nabla u_{\epsilon} \cdot \nabla \varphi = - \int u_{\epsilon} \nabla T_k(u_{\epsilon}) \cdot \nabla \varphi - \int u_{\epsilon} T_k(u_{\epsilon}) \Delta \varphi
\]

\[
= - \int T_k(u_{\epsilon}) \nabla T_k(u_{\epsilon}) \cdot \nabla \varphi - \int u_{\epsilon} T_k(u_{\epsilon}) \Delta \varphi
\]

\[
= -1/2 \int \nabla [T_k(u_{\epsilon})]^2 \cdot \nabla \varphi - \int u_{\epsilon} T_k(u_{\epsilon}) \Delta \varphi
\]

\[
= 1/2 \int [T_k(u_{\epsilon})]^2 \Delta \varphi - \int u_{\epsilon} T_k(u_{\epsilon}) \Delta \varphi
\]

\[
= - \int T_k(u_{\epsilon}) \left( u_{\epsilon} - 1/2 T_k(u_{\epsilon}) \right) \Delta \varphi
\]

\[
\geq -k \int |u_{\epsilon}| |\Delta \varphi|.
\]

\[
\tag{7}
\]

5
It follows from (6) and (7) that
\[ \int_A |\nabla T_k(u_\varepsilon)|^2 \leq \int |\nabla T_k(u_\varepsilon)|^2 \varphi \leq k \left( \int_{\text{supp } \varphi} |\Delta u_\varepsilon| + \| \Delta \varphi \|_{L^\infty} \int_{\text{supp } \varphi} |u_\varepsilon| \right). \]

In particular, for every $0 < \varepsilon < \text{dist (supp } \varphi, \partial \Omega)$,
\[ \int_A |\nabla T_k(u_\varepsilon)|^2 \leq k \left( \int_{\Omega} |\Delta u| + \| \Delta \varphi \|_{L^\infty} \int_{\Omega} |u| \right). \]

Letting $\varepsilon \downarrow 0$, we conclude that $T_k(u) \in H^1(A)$ and (4) holds with $C_A = \| \Delta \varphi \|_{L^\infty}$.

**Step 2.** Under the assumptions of the lemma, there exists a function $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in $\Omega$.

**Proof.** By (3) and Remark 5, for each $k > 0$ there exists $\tilde{T_k(u)} : \Omega \to \mathbb{R}$ quasicontinuous such that $T_k(u) = \tilde{T_k(u)}$ a.e. in $\Omega$.

Let $v_k := \frac{1}{k} T_k(u)$, so that
\[ v_k \to 0 \quad \text{in } L^q(\Omega) \quad \forall q \in [1, \infty) \]
and, by (4),
\[ \int_A |\nabla v_k|^2 \to 0 \quad \forall A \subset \subset \Omega. \]

In particular, $v_k \to 0$ in $H^1_{\text{loc}}(\Omega)$, which implies there exists a polar set $P \subset \Omega$ such that
\[ \tilde{v}_k(x) = \frac{1}{k} \tilde{T_k(u)}(x) \to 0 \quad \forall x \in \Omega \setminus P. \]

We conclude that
\[ \text{cap} \left[ \left| \tilde{T_k(u)} \right| > \frac{k}{2} \right] = \text{cap} \left[ |\tilde{v}_k| > \frac{1}{2} \right] \to 0. \tag{8} \]

Set
\[ w(x) := \begin{cases} \sup_{k \in \mathbb{N}} \left\{ T_k(u)(x) \right\} & \text{if } \sup_{k \in \mathbb{N}} \left| T_k(u)(x) \right| < \infty, \\ 0 & \text{otherwise}, \end{cases} \tag{9} \]
so that $w = u$ a.e. in $\Omega$. By (8) and the quasicontinuity of the functions $\tilde{T_k(u)}$, it is easy to see that $w$ is quasicontinuous in $\Omega$. This concludes the proof of the lemma.
4 A variant of Kato’s inequality when $\Delta u$ is a Radon measure

We start with the following (see Ancona [1])

**Lemma 2** Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$, $u \geq 0$ a.e. in $\Omega$, is such that $\Delta u$ is a Radon measure on $\Omega$. Then,

$$\Delta T_k(u) \text{ is a Radon measure } \forall k > 0.$$ 

Moreover, for any $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. in $\Omega$, we have

$$\Delta T_k(u) - a T_k(u) \leq (\Delta u - au)^+ \text{ in } D'(\Omega). \quad (10)$$

**Proof.** We shall use the same notation as in the proof of Lemma 1. By the standard $L^1$-version of Kato’s inequality (see Kato [1]) we have (note that $T_k|_{\mathbb{R}_+}$ is concave)

$$\Delta T_k(u_\varepsilon) \leq t_k(u_\varepsilon) \Delta u_\varepsilon \text{ in } \Omega \quad \forall \varepsilon > 0,$$ \quad (11)

where the function $t_k: \mathbb{R}_+ \to \mathbb{R}$ is given by

$$t_k(s) := \begin{cases} 
1 \text{ if } 0 \leq s \leq k, \\
0 \text{ if } s > k.
\end{cases}$$

Since $T_k(s) \geq t_k(s) s$ $\forall s \geq 0$ and $a \geq 0$ a.e. in $\Omega$, it follows from (11) that

$$\Delta T_k(u_\varepsilon) - a T_k(u_\varepsilon) \leq t_k(u_\varepsilon) (\Delta u_\varepsilon - au_\varepsilon) \leq (\Delta u_\varepsilon - au_\varepsilon)^+ \text{ in } D'(\Omega).$$

In other words, we have

$$\int T_k(u_\varepsilon) \Delta \varphi - a T_k(u_\varepsilon) \varphi \leq \int (\Delta u_\varepsilon - au_\varepsilon)^+ \varphi \quad \forall \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \text{ in } \Omega. \quad (12)$$

For $\lambda > 0$, let $\Omega_\lambda := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \lambda \}$. Thus, if $0 < \varepsilon < \lambda$, we get

$$\Delta u_\varepsilon - au_\varepsilon = (\Delta u - au)_\varepsilon + (au)_\varepsilon - au_\varepsilon$$

$$\leq \rho_\varepsilon \ast (\Delta u - au)^+ + |(au)_\varepsilon - au| + |au - au_\varepsilon| \text{ in } \Omega_\lambda.$$ 

Therefore, for any $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$ in $\Omega$, and $0 < \varepsilon < \text{dist}(\text{supp } \varphi, \partial \Omega)$, we may write

$$\int (\Delta u_\varepsilon - au_\varepsilon)^+ \varphi \leq \int \rho_\varepsilon \ast (\Delta u - au)^+ \varphi +$$

$$+ \| \varphi \|_{L^\infty} \left\{ \|(au)_\varepsilon - au\|_{L^1} + \| a \|_{L^\infty} \| u - u_\varepsilon \|_{L^1} \right\} \quad (13)$$

$$= \int (\rho_\varepsilon \ast \varphi)(\Delta u - au)^+ + o(1).$$

Since $\rho_\varepsilon \ast \varphi \to \varphi$ uniformly in $\Omega$ and $(\Delta u - au)^+$ is a Radon measure in $\Omega$, by letting $\varepsilon \downarrow 0$ in (12) and (13), we conclude that

$$\int T_k(u) \Delta \varphi - a T_k(u) \varphi \leq \int (\Delta u - au)^+ \varphi \quad \forall \varphi \in C^\infty_0(\Omega), \varphi \geq 0 \text{ in } \Omega,$$ 

so that $T_k(u)$ is a Radon measure (take for instance $a \equiv 0$) and (10) holds.
Lemma 3 Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$, $u \geq 0$ a.e. in $\Omega$, is such that $\Delta u$ is a Radon measure on $\Omega$. Let $a \in L^1(\Omega)$, $a \geq 0$ a.e. in $\Omega$. If

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

in the following sense

$$\int_E \Delta u \leq \int_E au \quad \text{for every Borel set } E \subset \Omega,$$

then

$$-\Delta T_k(u) + aT_k(u) \geq 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \forall k > 0.$$ (15)

Proof. By the preceding lemma applied with $a_i := T_i(a)$, where $i$ is a positive integer, we know that

$$\Delta T_k(u) - a_iT_k(u) \leq (\Delta u - a_iu)^+ \quad \text{in } \mathcal{D}'(\Omega).$$ (16)

On the other hand, from (14) we get

$$\int_E (\Delta u - a_iu) \leq \int_E (a - a_i)u \quad \text{for every Borel set } E \subset \Omega.$$ (17)

Since $(a - a_i)u \geq 0$ a.e. in $\Omega$, (17) implies that

$$0 \leq \int_E (\Delta u - a_iu)^+ \leq \int_E (a - a_i)u \quad \text{for every Borel set } E \subset \Omega.$$ (18)

Hence, $(\Delta u - a_iu)^+$ is a nonnegative measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, we have

$$(\Delta u - a_iu)^+ \in L^1(\Omega) \quad \forall i = 1, 2, \ldots.$$ (19)

We now return to (18) to conclude that

$$0 \leq (\Delta u - a_iu)^+ \leq (a - a_i)u \quad \text{a.e. in } \Omega.$$ (20)

In particular,

$$(\Delta u - a_iu)^+ \downarrow 0 \quad \text{a.e. in } \Omega \text{ as } i \uparrow \infty.$$ (21)

It follows from (19) and (20) that

$$(\Delta u - a_iu)^+ \to 0 \quad \text{in } L^1(\Omega) \text{ as } i \to \infty.$$ (22)

Finally, for any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ in $\Omega$, by (16) and (21) we have

$$\int T_k(u)\Delta \varphi - aT_k(u)\varphi \leq \int T_k(u)\Delta \varphi - a_iT_k(u)\varphi \leq \int (\Delta u - a_iu)^+\varphi \to 0 \quad \text{as } i \to \infty,$$

so that (15) holds.
Proof of Theorem 1 completed

It follows from Lemma 1 in Section 2 that, under the hypotheses of the theorem, there exists \( \tilde{u} : \Omega \to \mathbb{R} \) quasicontinuous such that \( u = \tilde{u} \) a.e. in \( \Omega \). Let us assume that \( \tilde{u} = 0 \) on a set of positive capacity \( E \subset \Omega \). We shall prove that \( u = 0 \) a.e. in \( \Omega \).

We split the proof into two steps:

Step 1. Under the hypotheses of the theorem, if we assume in addition that \( u \in L^\infty (\Omega) \), then \( u = 0 \) a.e. in \( \Omega \).

**Proof.** Since \( u \in L^\infty (\Omega) \), we have \( au \in L^1 (\Omega) \). It follows from (1) and Remark 3 that

\[
-\Delta u + au \geq 0 \quad \text{in} \mathcal{D}'(\Omega).
\]

Recall that for \( \varepsilon, \lambda > 0 \) we have defined

\[
\Omega_\lambda := \{x \in \Omega : \text{dist} (x, \partial \Omega) > \lambda\}
\]

and

\[
u_\varepsilon(x) := \rho_\varepsilon * u(x) = \int_\Omega \rho_\varepsilon(x-y)u(y) \, dy \quad \forall x \in \Omega,
\]

where \( \rho \in C_0^\infty(B_1) \), \( \rho \geq 0 \) in \( B_1 \), is a radial mollifier.

Using the above notation, for \( 0 < \varepsilon < \lambda \), we have in \( \Omega_\lambda \) that

\[
\Delta u_\varepsilon \leq (au)_\varepsilon = au_\varepsilon + [(au)_\varepsilon - au_\varepsilon] \\
\leq au_\varepsilon + [(au)_\varepsilon - au_\varepsilon]^+ \\
=: au_\varepsilon + f_\varepsilon.
\]

Since \( (au)_\varepsilon \to au \) in \( L^1 (\Omega) \), \( u_\varepsilon \to u \) a.e. in \( \Omega \) and \( u \) is bounded,

\[
f_\varepsilon \to 0 \quad \text{in} \ L^1 (\Omega).
\]

Let \( \delta > 0 \) be a fixed number. Multiplying (22) by \( \frac{1}{u_\varepsilon + \delta} \), we get

\[
\frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \leq a + \frac{f_\varepsilon}{\delta} \quad \text{in} \ \Omega_\lambda \quad \forall \varepsilon \in (0, \lambda).
\]

We also remark that

\[
\frac{\nabla u_\varepsilon}{(u_\varepsilon + \delta)^2} = -\nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \quad \text{in} \ \Omega.
\]

Let \( \varphi \in C_0^\infty (\Omega) \) and \( 0 < \varepsilon < \text{dist}(\text{supp} \varphi, \partial \Omega) \). We now use (25), integration by parts, estimate
(24) and Cauchy-Schwarz, to get
\[ \int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 = -\int \nabla u_\varepsilon \cdot \nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \varphi^2 \]
\[ = \int \frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \varphi^2 + \int \frac{2 \varphi \nabla \varphi \cdot \nabla u_\varepsilon}{u_\varepsilon + \delta} \]
\[ \leq \int \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + \frac{1}{2} \int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 + 2 \int |\nabla \varphi|^2. \]

Therefore,
\[ \frac{1}{2} \int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 \leq \int \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + 2 \int |\nabla \varphi|^2. \]

Since
\[ \nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) = \frac{\nabla u_\varepsilon}{u_\varepsilon + \delta}, \]
the estimate above may be rewritten as
\[ \frac{1}{2} \int \left| \nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + 2 \int |\nabla \varphi|^2. \] (26)

We now let \( \varepsilon \downarrow 0 \) in (26). It follows from (23) that (see also Lemma 1)
\[ \log \left( \frac{u}{\delta} + 1 \right) \in H^{1}_{loc}(\Omega) \forall \delta > 0 \]
and
\[ \frac{1}{2} \int \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int \left( a \varphi^2 + 2 |\nabla \varphi|^2 \right) \forall \varphi \in C^\infty_0(\Omega). \] (27)

Let \( E \subset \Omega \) be a set of positive capacity such that \( \tilde{u} = 0 \) on \( E \). Without any loss of generality, we may assume that \( E \subset \Omega_\lambda \) for some \( \lambda > 0 \) sufficiently small.

Assume \( \omega \subset \subset \Omega \) is an open connected set containing \( E \). Let \( \varphi_0 \in C^\infty_0(\Omega) \) be a fixed test function such that \( \varphi \equiv 1 \) on \( \omega \).

By (27), we have
\[ \int_\omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \leq 2 \int \left( a \varphi_0^2 + 2 |\nabla \varphi_0|^2 \right). \] (28)

Since the quasicontinuous representative \( \tilde{\log} \left( \frac{u}{\delta} + 1 \right) = \log \left( \frac{\tilde{u}}{\delta} + 1 \right) \) of \( \log \left( \frac{u}{\delta} + 1 \right) \) equals 0 on \( E \subset \Omega \) with \( \text{cap } E > 0 \), it follows from a variant of Poincaré’s inequality (easily proved by contradiction) that there exists \( C > 0 \) (depending only on \( E \) and \( \Omega \)) such that
\[ \int_\omega \log^2 \left( \frac{u}{\delta} + 1 \right) \leq C \int_\omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \forall \delta > 0. \] (29)

(28) and (29) yield
\[ \int_\omega \log^2 \left( \frac{u}{\delta} + 1 \right) \leq 2C \int \left( a \varphi_0^2 + 2 |\nabla \varphi_0|^2 \right) \forall \delta > 0. \] (30)
In particular, the integral in the left-hand side remains bounded as $\delta \downarrow 0$.

On the other hand,

$$\log^2 \left( \frac{u}{\delta} + 1 \right) \to +\infty \quad \text{a.e. in } \omega \setminus \{u = 0\} \text{ as } \delta \downarrow 0.$$ \hspace{1cm} (31)

By (30) and (31), we conclude that $u = 0$ a.e. in $\omega$. Since $\omega$ is an arbitrary connected neighborhood of $E$ in $\Omega_\lambda$ for all $\lambda > 0$ small, we conclude that $u = 0$ a.e. in $\Omega$.

**Step 2.** Proof of Theorem 1 completed.

From Lemma 3, we know that $\Delta T_1(u)$ is a Radon measure and

$$-\Delta T_1(u) + aT_1(u) \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

In addition, $\widetilde{T}_1(u) = T_1(\tilde{u}) = 0$ on $E \subset \Omega$ with $\operatorname{cap} E > 0$.

By Step 1, we have $T_1(u) = 0$ a.e. in $\Omega$, and so $u = 0$ a.e. in $\Omega$.

**Acknowledgments:** The authors thank A. Ancona for interesting discussions. The first author (H.B.) is partially supported by a European grant ERB FMRX CT98 0201. He is also a member of the Institut Universitaire de France. The second author (A.C.P.) is supported by CAPES, Brazil, under the grant BEX1187/99-6.

**Bibliography**


(1) Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie, B.C. 187
4 Pl. Jussieu
75252 Paris Cedex 05
France

(2) Rutgers University
Dept. of Math., Hill Center, Busch Campus
110 Frelinghuysen Rd, Piscataway, NJ 08854
USA