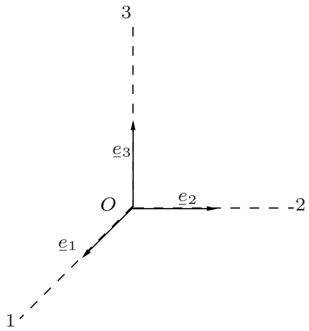
MECA 1901 - Mécanique des milieux continus

Formulaire

Septembre 2008

1 Calcul tensoriel

Repère cartésien orthonormé

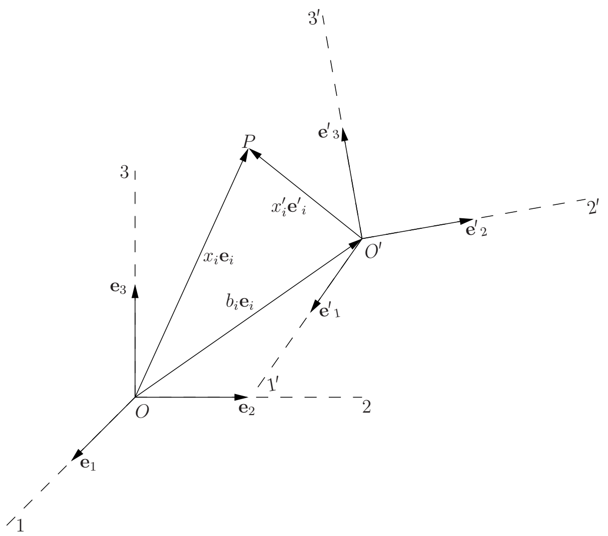


Base orthonormée : $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$

Orientation directe : $\underset{\sim}{e}_i \times \underset{\sim}{e}_j = \epsilon_{ijk} \underset{\sim}{e}_k$

"Vecteur" position : $\underset{\sim}{x} = x_i \underset{\sim}{e}_i$

Changement de repère cartésien orthonormé



Matrice de changement de base :

$$a_{ij} = \cos(\underset{\sim}{e}'_i, \underset{\sim}{e}_j) = \underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j$$

Propriété d'orthogonalité :

$$a_{ik}\, a_{jk} = \delta_{ij} \quad \text{et} \quad a_{ki}\, a_{kj} = \delta_{ij}$$

$$\begin{aligned} \underset{\sim}{e}'_i &= a_{ij}\, \underset{\sim}{e}_j \\ \underset{\sim}{e}_i &= a_{ji}\, \underset{\sim}{e}'_j \end{aligned}$$

$$\begin{aligned} x'_i &= a_{ij}\,(x_j - b_j) \\ x_i &= a_{ji}\, x'_j + b_i \end{aligned}$$

Invariance tensorielle

– scalaire (tenseur d'ordre 0)

$$\rho' = \rho$$

– vecteur (tenseur d'ordre 1)

$$\underline{v}' = v'_i\, \underline{e}'_i = v_i\, \underline{e}_i = \underline{v}$$

$$\begin{aligned} v'_i &= a_{ij} v_j \\ v_i &= a_{ji} v'_j \end{aligned}$$

– tenseur d'ordre 2

$$\underline{\underline{\sigma}}' = \sigma'_{ij}\, \underline{e}'_i\, \underline{e}'_j = \sigma_{ij}\, \underline{e}_i\, \underline{e}_j = \underline{\underline{\sigma}}$$

$$\begin{aligned} \sigma'_{ij} &= a_{im} a_{jn} \sigma_{mn} \\ \sigma_{ij} &= a_{mi} a_{nj} \sigma'_{mn} \end{aligned}$$

– tenseur d'ordre quelconque

$$\begin{aligned} T'_{ijk} &= a_{il}\, a_{jm}\, a_{kn}\, T_{lmn} && \text{(3 indices)} \\ T_{ijk} &= a_{li}\, a_{mj}\, a_{nk}\, T'_{lmn} && \text{(réciproque)} \end{aligned}$$

Opérations vectorielles et tensorielles

Opérations sur les vecteurs de base $\underline{e}_i$ et les dyades de base $\underline{e}_i\, \underline{e}_j$ d'une base orthonormée d'orientation directe :

$$\begin{aligned} \underline{e}_i \cdot \underline{e}_j &= \delta_{ij} \\ \underline{e}_i \times \underline{e}_j &= \epsilon_{ijk}\, \underline{e}_k \\ \underline{e}_i\, \underline{e}_j : \underline{e}_k\, \underline{e}_l &= (\underline{e}_j \cdot \underline{e}_k)(\underline{e}_i \cdot \underline{e}_l) = \delta_{jk}\, \delta_{il} \\ \underline{e}_i\, \underline{e}_j \cdot \underline{e}_k &= \underline{e}_i\, (\underline{e}_j \cdot \underline{e}_k) = \delta_{jk}\, \underline{e}_i \\ \underline{e}_i \cdot \underline{e}_j\, \underline{e}_k &= (\underline{e}_i \cdot \underline{e}_j)\, \underline{e}_k = \delta_{ij}\, \underline{e}_k \\ \underline{e}_i\, \underline{e}_j \cdot \underline{e}_k\, \underline{e}_l &= \underline{e}_i\, (\underline{e}_j \cdot \underline{e}_k)\, \underline{e}_l = \delta_{jk}\, \underline{e}_i\, \underline{e}_l \\ \underline{e}_i\, \underline{e}_j \times \underline{e}_k &= \underline{e}_i\, (\underline{e}_j \times \underline{e}_k) = \epsilon_{jkl}\, \underline{e}_i\, \underline{e}_l \\ \underline{e}_i \times \underline{e}_j\, \underline{e}_k &= (\underline{e}_i \times \underline{e}_j)\, \underline{e}_k = \epsilon_{ijl}\, \underline{e}_l\, \underline{e}_k \end{aligned}$$

Expression utile pour $\epsilon_{ijm}\, \epsilon_{klm}$:

$$\epsilon_{ijm}\, \epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

Déterminant d'un tenseur d'ordre 2 :

$$\det(\underline{\underline{T}}) = \epsilon_{ijk}\, T_{i1}\, T_{j2}\, T_{k3} = \epsilon_{ijk}\, T_{1i}\, T_{2j}\, T_{3k} \quad \text{ou encore} \quad \det(\underline{\underline{T}}) = \frac{1}{6}\, \epsilon_{ijk}\, \epsilon_{lmn}\, T_{il}\, T_{jm}\, T_{kn}$$

Opérations sur les vecteurs :

– expression d'un vecteur en termes de ses composantes :

$$\underline{u} = u_i\, \underline{e}_i$$

– addition de deux vecteurs :

$$\underline{u} + \underline{v} = (u_i + v_i)\, \underline{e}_i$$

– produit scalaire de deux vecteurs :

$$\underline{u} \cdot \underline{v} = u_i\, v_i$$

– produit vectoriel de deux vecteurs :

$$\underline{u} \times \underline{v} = \epsilon_{ijk}\, u_j\, v_k\, \underline{e}_i$$

– produit tensoriel ou dyadique de deux vecteurs :

$$\underline{u}\, \underline{v} = u_i\, v_j\, \underline{e}_i\, \underline{e}_j$$

– produit mixte de trois vecteurs :

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = (\underline{u} \times \underline{v}) \cdot \underline{w} = \epsilon_{ijk}\, u_i\, v_j\, w_k$$

Opérations sur les tenseurs :

– expression d'un tenseur en termes de composantes :

$$\begin{aligned} \underline{\underline{T}} &= T_{ij}\, \underline{e}_i\, \underline{e}_j \\ \underline{\underline{T}}^T &= T_{ji}\, \underline{e}_i\, \underline{e}_j \end{aligned}$$

– addition de deux tenseurs :

$$\underline{\underline{S}} + \underline{\underline{T}} = (S_{ij} + T_{ij})\, \underline{e}_i\, \underline{e}_j$$

– produit scalaire ou deux points de deux tenseurs :

$$\underline{\underline{S}} : \underline{\underline{T}} = S_{ij}\, T_{ji}$$

– produit tensoriel de deux tenseurs :

$$\underline{\underline{S}} \cdot \underline{\underline{T}} = S_{ij}\, T_{jk}\, \underline{e}_i\, \underline{e}_k$$

– produit vectoriel d'un tenseur et d'un vecteur :

$$\underline{\underline{T}} \cdot \underline{u} = T_{ij}\, u_j\, \underline{e}_i$$

Parties symétrique et antisymétrique d'un tenseur

Tout tenseur $\underline{\underline{T}}$ peut se décomposer en la somme d'un tenseur symétrique $\underline{\underline{S}}$ et d'un tenseur antisymétrique $\underline{\underline{A}}$:

$$\underline{\underline{T}} = \underline{\underline{S}} + \underline{\underline{A}}$$

avec

$$\begin{aligned} \underline{\underline{S}} &= \frac{1}{2}\left(\underline{\underline{T}} + \underline{\underline{T}}^T\right) \\ \underline{\underline{A}} &= \frac{1}{2}\left(\underline{\underline{T}} - \underline{\underline{T}}^T\right) \end{aligned}$$

Propriété : A tout tenseur antisymétrique $\underline{\underline{A}}$ peut être associé un vecteur $\underline{a}$, et inversément, tel que :

$$a_i = \frac{1}{2}\, \epsilon_{ijk}\, A_{kj} \quad \text{et} \quad A_{ij} = -\, \epsilon_{ijk}\, a_k$$

Parties sphérique et déviatoire d'un tenseur

Tout tenseur $\underline{\underline{T}}$ peut se décomposer de la manière suivante :

$$\underline{\underline{T}} = T^s \underline{\underline{\delta}} + \underline{\underline{T}}^d$$

où T^s est la partie sphérique de $\underline{\underline{T}}$:

$$T^s = \frac{1}{3} \mathrm{tr}\, \underline{\underline{T}}$$

et $\underline{\underline{T}}^d$ sa partie déviatoire :

$$\underline{\underline{T}}^d = \underline{\underline{T}} - T^s \, \underline{\underline{\delta}}$$

Remarque :

$$\mathrm{tr}\, \underline{\underline{T}}^d = 0$$

Le tenseur $\underline{\underline{\delta}} = \delta_{ij} \, e_i \, e_j$ est le tenseur unité

Invariants scalaires

Invariant d'un vecteur :

$$\| \underline{v} \|^2 = v_i \, v_i$$

Invariants d'un tenseur :

$$\begin{aligned} I_T &= tr\ \underline{\underline{T}} = T_{ii} \\ II_T &= tr\ \underline{\underline{T}}^2 = T_{ij} T_{ji} \\ III_T &= tr\ \underline{\underline{T}}^3 = T_{ij} T_{jk} T_{ki} \end{aligned}$$

A partir de ces invariants, on peut définir d'autres invariants, par exemple :

$$\begin{aligned} I_1^T &= I_T \\ I_2^T &= \frac{1}{2} \left(I_T^2 - II_T \right) \\ I_3^T &= \frac{1}{6} \left(I_T^3 - 3 I_T \cdot II_T + 2 III_T \right) \; = \; \det\ \underline{\underline{T}} \end{aligned}$$

Opérations différentielles en coordonnées cartésiennes orthonormées

– opérateur nabla :

$$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}$$

– opérateur laplacien :

$$\Delta = \underline{\nabla}^2 = \underline{\nabla} \cdot \underline{\nabla} = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i^2}$$

– gradient d'un champ scalaire :

$$\underline{\nabla} f = \underline{e}_i \, \frac{\partial f}{\partial x_i}$$

– laplacien d'un champ scalaire :

$$\underline{\nabla} \cdot \underline{\nabla} f = \frac{\partial^2 f}{\partial x_i^2}$$

– gradient d'un champ vectoriel :

$$\begin{aligned} \underline{\nabla}\, \underline{u} &= \frac{\partial u_j}{\partial x_i} \, \underline{e}_i \, \underline{e}_j \\ \underline{\nabla} \underline{u}^T &= \frac{\partial u_i}{\partial x_j} \, \underline{e}_i \, \underline{e}_j \end{aligned}$$

– divergence d'un champ vectoriel :

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial u_i}{\partial x_i}$$

– rotationnel d'un champ vectoriel :

$$\underline{\nabla} \times \underline{u} = \epsilon_{ijk} \, \frac{\partial u_k}{\partial x_j} \, \underline{e}_i$$

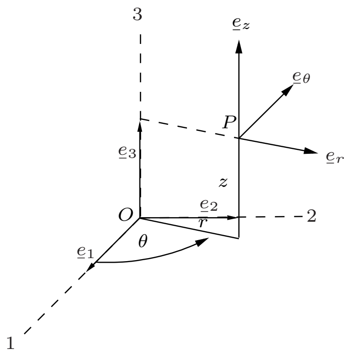
– laplacien d'un champ vectoriel :

$$\underline{\nabla} \cdot \underline{\nabla} \underline{u} = \frac{\partial^2 u_j}{\partial x_i^2} \, \underline{e}_j$$

– divergence d'un champ tensoriel :

$$\underline{\nabla} \cdot \underline{\underline{T}} = \frac{\partial T_{ij}}{\partial x_i} \, \underline{e}_j$$

Système de coordonnées-composantes cylindriques



$$r = \sqrt{{x_1}^2 + {x_2}^2} \qquad x_1 = r\ \cos\theta$$

$$\theta = \arctan\left(\frac{x_2}{x_1}\right) \qquad x_2 = r\ \sin\theta$$

$$z = x_3 \qquad x_3 = z$$

"Vecteur" position :

$$\underline{x} = r\ \cos\theta\ \underline{e}_1 + r\ \sin\theta\ \underline{e}_2 + z\ \underline{e}_3 = r\ \underline{e}_r + z\ \underline{e}_z$$

Vecteurs de base :

$$\begin{aligned}
\underline{e}_r &= \frac{\partial \underline{x}}{\partial r} / \left\| \frac{\partial \underline{x}}{\partial r} \right\| = \frac{\partial \underline{x}}{\partial r} = \cos\theta\ \underline{e}_1 + \sin\theta\ \underline{e}_2 \\
\underline{e}_\theta &= \frac{\partial \underline{x}}{\partial \theta} / \left\| \frac{\partial \underline{x}}{\partial \theta} \right\| = \frac{1}{r}\frac{\partial \underline{x}}{\partial \theta} = -\sin\theta\ \underline{e}_1 + \cos\theta\ \underline{e}_2 \\
\underline{e}_z &= \frac{\partial \underline{x}}{\partial z} / \left\| \frac{\partial \underline{x}}{\partial z} \right\| = \frac{\partial \underline{x}}{\partial z} = \underline{e}_3
\end{aligned}$$

Dérivées des vecteurs de base :

$$\begin{array}{ccc}
\dfrac{\partial \underline{e}_r}{\partial r} = \underline{0} & \dfrac{\partial \underline{e}_\theta}{\partial r} = \underline{0} & \dfrac{\partial \underline{e}_z}{\partial r} = \underline{0} \\
\dfrac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta & \dfrac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r & \dfrac{\partial \underline{e}_z}{\partial \theta} = \underline{0} \\
\dfrac{\partial \underline{e}_r}{\partial z} = \underline{0} & \dfrac{\partial \underline{e}_\theta}{\partial z} = \underline{0} & \dfrac{\partial \underline{e}_z}{\partial z} = \underline{0}
\end{array}$$

Matrice de changement de base (cartésien → cylindrique)

$$[a_{ij}(r,\theta,z)] = {}_{\text{cyl}}\mathcal{A}_{\text{car}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Opérations différentielles en coordonnées cylindriques :

– opérateur nabla :

$$\underline{\nabla} = \underline{e}_r\ \frac{\partial}{\partial r} + \underline{e}_\theta\ \frac{1}{r}\ \frac{\partial}{\partial \theta} + \underline{e}_z\ \frac{\partial}{\partial z}$$

– gradient d'un champ scalaire :

$$\underline{\nabla} f = \underline{e}_r\ \frac{\partial f}{\partial r} + \underline{e}_\theta\ \frac{1}{r}\ \frac{\partial f}{\partial \theta} + \underline{e}_z\ \frac{\partial f}{\partial z}$$

– laplacien d'un champ scalaire :

$$\Delta f = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

– gradient d'un champ vectoriel :

$$\begin{array}{lll}
(\underset{\sim}{\nabla}\underset{\sim}{u})_{rr} = \dfrac{\partial u_r}{\partial r} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{r\theta} = \dfrac{\partial u_\theta}{\partial r} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{rz} = \dfrac{\partial u_z}{\partial r} \\
(\underset{\sim}{\nabla}\underset{\sim}{u})_{\theta r} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{\theta z} = \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} \\
(\underset{\sim}{\nabla}\underset{\sim}{u})_{zr} = \dfrac{\partial u_r}{\partial z} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{z\theta} = \dfrac{\partial u_\theta}{\partial z} & (\underset{\sim}{\nabla}\underset{\sim}{u})_{zz} = \dfrac{\partial u_z}{\partial z}
\end{array}$$

– divergence d'un champ vectoriel :

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

– rotationnel d'un champ vectoriel :

$$\begin{aligned}
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_z &= \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}
\end{aligned}$$

– laplacien d'un champ vectoriel :

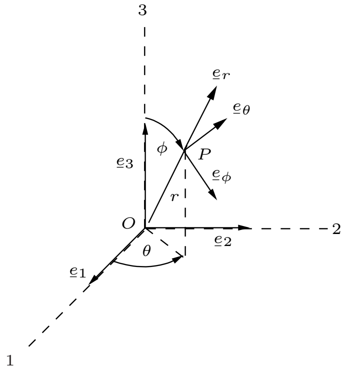
$$\begin{aligned}
(\Delta \underset{\sim}{v})_r &= \Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \\
(\Delta \underset{\sim}{v})_\theta &= \Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \\
(\Delta \underset{\sim}{v})_z &= \Delta v_z
\end{aligned}$$

$$\text{avec} \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

– divergence d'un champ tensoriel :

$$\begin{aligned}
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_r &= \frac{\partial}{\partial r} T_{rr} + \frac{1}{r}\frac{\partial}{\partial \theta} T_{\theta r} + \frac{\partial}{\partial z} T_{zr} + \frac{T_{rr} - T_{\theta\theta}}{r} \\
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_\theta &= \frac{\partial}{\partial r} T_{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta} T_{\theta\theta} + \frac{\partial}{\partial z} T_{z\theta} + \frac{T_{\theta r} + T_{r\theta}}{r} \\
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_z &= \frac{\partial}{\partial r} T_{rz} + \frac{1}{r}\frac{\partial}{\partial \theta} T_{\theta z} + \frac{\partial}{\partial z} T_{zz} + \frac{T_{rz}}{r}
\end{aligned}$$

Système de coordonnées-composantes sphériques



$$r = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2} \qquad x_1 = r\ \sin\phi\ \cos\theta$$

$$\phi = \arctan\left(\frac{\sqrt{{x_1}^2 + {x_2}^2}}{x_3}\right) \qquad x_2 = r\ \sin\phi\ \sin\theta$$

$$\theta = \arctan\left(\frac{x_2}{x_1}\right) \qquad x_3 = r\ \cos\phi$$

"Vecteur" position :

$$\underline{x} = r\ \sin\phi\ \cos\theta\ \underline{e}_1 + r\ \sin\phi\ \sin\theta\ \underline{e}_2 + r\ \cos\phi\ \underline{e}_3 = r\ \underline{e}_r$$

Vecteurs de base :

$$\underline{e}_r = \frac{\partial \underline{x}}{\partial r} / \left\| \frac{\partial \underline{x}}{\partial r} \right\| = \frac{\partial \underline{x}}{\partial r} = \sin\phi\ \cos\theta\ \underline{e}_1 + \sin\phi\ \sin\theta\ \underline{e}_2 + \cos\phi\ \underline{e}_3$$

$$\underline{e}_\phi = \frac{\partial \underline{x}}{\partial \phi} / \left\| \frac{\partial \underline{x}}{\partial \phi} \right\| = \frac{1}{r}\frac{\partial \underline{x}}{\partial \phi} = \cos\phi\ \cos\theta\ \underline{e}_1 + \cos\phi\ \sin\theta\ \underline{e}_2 - \sin\phi\ \underline{e}_3$$

$$\underline{e}_\theta = \frac{\partial \underline{x}}{\partial \theta} / \left\| \frac{\partial \underline{x}}{\partial \theta} \right\| = \frac{1}{r\sin\phi}\frac{\partial \underline{x}}{\partial \theta} = -\sin\theta\ \underline{e}_1 + \cos\theta\ \underline{e}_2$$

Dérivées des vecteurs de base :

$$\frac{\partial \underline{e}_r}{\partial r} = \underline{0} \qquad \frac{\partial \underline{e}_\phi}{\partial r} = \underline{0} \qquad \frac{\partial \underline{e}_\theta}{\partial r} = \underline{0}$$

$$\frac{\partial \underline{e}_r}{\partial \phi} = \underline{e}_\phi \qquad \frac{\partial \underline{e}_\phi}{\partial \phi} = -\underline{e}_r \qquad \frac{\partial \underline{e}_\theta}{\partial \phi} = \underline{0}$$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta\ \sin\phi \qquad \frac{\partial \underline{e}_\phi}{\partial \theta} = \underline{e}_\theta\ \cos\phi \qquad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r\ \sin\phi - \underline{e}_\phi\ \cos\phi$$

Matrice de changement de base (cartésien → sphérique)

$$[a_{ij}(r,\phi,\theta)] = {}_{\text{sph}}\mathcal{A}_{\text{car}} = \begin{bmatrix} \sin\phi\ \cos\theta & \sin\phi\ \sin\theta & \cos\phi \\ \cos\phi\ \cos\theta & \cos\phi\ \sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

Opérations différentielles en coordonnées sphériques

– opérateur nabla :

$$\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \underline{e}_\theta \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}$$

– gradient d'un champ scalaire :

$$\underline{\nabla} f = \underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\phi \frac{1}{r}\frac{\partial f}{\partial \phi} + \underline{e}_\theta \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}$$

– laplacien d'un champ scalaire :

$$\Delta f \ = \ \underline{\nabla}\cdot\underline{\nabla} f \ = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2\ \tan\phi}\frac{\partial f}{\partial \phi} + \frac{1}{r^2\ \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

– gradient d'un champ vectoriel :

$$(\underset{\sim}{\nabla} \underset{\sim}{u})_{rr} = \frac{\partial u_r}{\partial r} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{r\phi} = \frac{\partial u_\phi}{\partial r} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{r\theta} = \frac{\partial u_\theta}{\partial r}$$

$$(\underset{\sim}{\nabla} \underset{\sim}{u})_{\phi r} = \frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{\phi\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \phi}$$

$$(\underset{\sim}{\nabla} \underset{\sim}{u})_{\theta r} = \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{\theta\phi} = \frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{r\tan\phi} \qquad (\underset{\sim}{\nabla} \underset{\sim}{u})_{\theta\theta} = \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{u_\phi}{r\tan\phi}$$

– divergence d'un champ vectoriel :

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \frac{2}{r}\, u_r + \frac{\partial u_r}{\partial r} + \frac{1}{r\,\tan\phi}\, u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r\,\sin\phi}\frac{\partial u_\theta}{\partial \theta}$$

– rotationnel d'un champ vectoriel :

$$\begin{aligned}
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_r &= \frac{1}{r\,\tan\phi}\, u_\theta + \frac{1}{r}\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r\,\sin\phi}\frac{\partial u_\phi}{\partial \theta} \\
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_\phi &= \frac{1}{r\,\sin\phi}\frac{\partial u_r}{\partial \theta} - \frac{1}{r}\, u_\theta - \frac{\partial u_\theta}{\partial r} \\
(\underset{\sim}{\nabla} \times \underset{\sim}{u})_\theta &= \frac{1}{r}\, u_\phi + \frac{\partial u_\phi}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \phi}
\end{aligned}$$

– laplacien d'un champ vectoriel :

$$\begin{aligned}
(\Delta \underset{\sim}{v})_r &= \Delta v_r - 2\frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{2v_\phi}{r^2\tan\phi} - \frac{2}{r^2\sin\phi}\frac{\partial v_\theta}{\partial \theta} \\
(\Delta \underset{\sim}{v})_\phi &= \Delta v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2\sin^2\phi} - \frac{2\cos\phi}{r^2\sin^2\phi}\frac{\partial v_\theta}{\partial \theta} \\
(\Delta \underset{\sim}{v})_\theta &= \Delta v_\theta + \frac{2}{r^2\sin\phi}\frac{\partial v_r}{\partial \theta} + \frac{2\cos\phi}{r^2\sin^2\phi}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\phi}
\end{aligned}$$

avec $$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2\tan\phi}\frac{\partial}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

– divergence d'un champ tensoriel :

$$\begin{aligned}
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_r &= \frac{\partial}{\partial r}\, T_{rr} + \frac{1}{r}\frac{\partial}{\partial \phi}\, T_{\phi r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\, T_{\theta r} + \frac{2\,T_{rr} - T_{\phi\phi} - T_{\theta\theta} + T_{\phi r}\cot\phi}{r} \\
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_\phi &= \frac{\partial}{\partial r}\, T_{r\phi} + \frac{1}{r}\frac{\partial}{\partial \phi}\, T_{\phi\phi} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\, T_{\theta\phi} + \frac{2\,T_{r\phi} + T_{\phi r} + (T_{\phi\phi} - T_{\theta\theta})\cot\phi}{r} \\
(\underset{\sim}{\nabla} \cdot \underset{\approx}{T})_\theta &= \frac{\partial}{\partial r}\, T_{r\theta} + \frac{1}{r}\frac{\partial}{\partial \phi}\, T_{\phi\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\, T_{\theta\theta} + \frac{2\,T_{r\theta} + T_{\theta r} + (T_{\phi\theta} + T_{\theta\phi})\cot\phi}{r}
\end{aligned}$$

Théorèmes sur les intégrales

– Théorème de la divergence (ou de Green) :

$$\int_{\partial V} \underline{v} \cdot \underline{n} \, d\sigma = \int_V \underline{\nabla} \cdot \underline{v} \, d\nu$$

– Théorème de Stokes :

$$\int_C \underline{u} \cdot \underline{t} \, dc = \int_S (\underline{\nabla} \times \underline{u}) \cdot \underline{n} \, d\sigma$$

– Théorème du transport (ou de Reynolds) :

$$\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \left(\frac{Df}{Dt} + f \underline{\nabla} \cdot v \right) dV$$

Variante :

$$\frac{d}{dt} \int_{V(t)} \rho g \, dV = \int_{V(t)} \rho \frac{Dg}{Dt} \, dV$$

2 Thermoélasticité infinitésimale isotrope

Petits déplacements

$$\underline{u} = \underline{x} - \underline{X} = O(\epsilon), \quad \epsilon \ll 1$$

Tenseur des déformations infinitésimales

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\Big(\underline{\nabla}\,\underline{u}^T + \underline{\nabla}\,\underline{u}\Big)$$

Energie libre

$$\rho_0\, F = \frac{\lambda(T)}{2}\,(\mathrm{tr}\,\underline{\underline{\epsilon}})^2 \;+ \mu(T)\,\underline{\underline{\epsilon}} : \underline{\underline{\epsilon}} + \rho_0\,\Phi(T) - \Big(3\lambda(T) + 2\mu(T)\Big)\,\alpha(T)\,\Big(T - T_0\Big)\,\mathrm{tr}\,\underline{\underline{\epsilon}}$$

Lien avec le second principe

$$\rho_0\Big(T\frac{\partial S}{\partial t} - \frac{\partial U}{\partial t}\Big) \geq -\underline{\underline{\sigma}} : \frac{\partial \underline{\underline{\epsilon}}}{\partial t} + \frac{1}{T}\,\underline{q}\cdot\underline{\nabla}T$$

Relations entre contraintes et déformations

$$\begin{aligned}
\underline{\underline{\sigma}} &= \lambda(T)\,(\mathrm{tr}\,\underline{\underline{\varepsilon}})\,\underline{\underline{\delta}} + 2\mu(T)\,\underline{\underline{\varepsilon}} - \alpha(T)\Big(T - T_0\Big)\,\Big(3\lambda(T) + 2\mu(T)\Big)\,\underline{\underline{\delta}} \\
\underline{\underline{\varepsilon}} &= \frac{1 + \nu(T)}{E(T)}\,\underline{\underline{\sigma}} - \frac{\nu(T)}{E(T)}\,(\mathrm{tr}\,\underline{\underline{\sigma}})\,\underline{\underline{\delta}} + \alpha(T)\Big(T - T_0\Big)\,\underline{\underline{\delta}}
\end{aligned}$$

Loi de Fourier

$$\underline{q} = -k(T)\,\underline{\nabla}T$$

Modules élastiques

$E(T)$	$=$	$\dfrac{\mu(T)(3\lambda(T) + 2\mu(T))}{\lambda(T) + \mu(T)}$	$E(T) > 0$	Module de Young
$\nu(T)$	$=$	$\dfrac{\lambda(T)}{2(\lambda(T) + \mu(T))}$	$-1 < \nu(T) < 1/2$	Coefficient de Poisson
$\lambda(T)$	$=$	$\dfrac{E(T)\nu(T)}{(1 + \nu(T))(1 - 2\nu(T))}$		1^{er} coefficient de Lamé
$\mu(T)$	$=$	$\dfrac{E(T)}{2(1 + \nu(T))}$	$\mu(T) > 0$	2^{nd} coefficient de Lamé
$\kappa(T)$	$=$	$\dfrac{E(T)}{3(1 - 2\nu(T))} = \lambda(T) + \dfrac{2}{3}\mu(T)$	$\kappa(T) > 0$	Module de compressibilité

3 Fluide visqueux newtonien

Tenseur des taux de déformation

$$\underline{\underline{d}} = \frac{1}{2}\left(\underline{\nabla}\,\underline{v}^T + \underline{\nabla}\,\underline{v}\right)$$

Equations de constitution

$$\begin{aligned} \underline{\underline{\sigma}} &= -p\,\underline{\underline{\delta}} + 2\eta(T)\,\underline{\underline{d}} \\ \underline{q} &= -k(T)\,\underline{\nabla}T \end{aligned}$$

Fonctions matérielles

$\eta(T)$		$\eta(T) > 0$	Viscosité dynamique
$\nu(T)$	$= \dfrac{\eta(T)}{\rho}$	$\nu(T) > 0$	Viscosité cinématique
$k(T)$		$k(T) > 0$	Conductivité thermique
$C_v(T)$	$= \dfrac{d\hat{U}}{dT} = T\,\dfrac{d\hat{S}}{dT}$	$C_v(T) > 0$	Chaleur spécifique à volume constant
$\dfrac{k(T)}{\rho\,C_v(T)}$		$\dfrac{k(T)}{\rho\,C_v(T)} > 0$	Diffusivité thermique