A genetic algorithm approach for the approximation of the joint spectral radius

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1 Introduction

The *joint spectral radius (jsr)* $\rho(\Sigma)$ of a set of matrices $\Sigma \subset \mathbb{R}^{n\times n}$ characterizes the maximal asymptotic growth rate of products of matrices in the set. It is defined by:

$$\rho(\Sigma) = \lim_{k\to\infty} \rho_k(\Sigma),\ \rho_k(\Sigma) = \max\left\{ \|M\|^{1/k} \,\middle|\, M \in \Sigma^k \right\}, \quad (1)$$

where Σ^k is the set of products of length k of matrices in Σ, and indepedently of the chosen submultiplicative matrix norm. When the set Σ is bounded, the jsr is also equal to the *generalized spectral radius* $\bar{\rho}(\Sigma)$:

$$\bar{\rho}(\Sigma) = \limsup_{k\to\infty} \bar{\rho}_k(\Sigma),\ \bar{\rho}_k(\Sigma) = \max\left\{ \rho(M)^{1/k} \,\middle|\, M \in \Sigma^k \right\}.$$

The jsr appears in many applications such as stability of switched systems, combinatorics, graph theory, ... For example, let Σ be a bounded set of $\mathbb{R}^{n\times n}$, then the discrete-time system $x(t+1) = A_t x(t)$ with $A_t \in \Sigma$ for all t is stable under arbitrary switchings if and only if $\rho(\Sigma) < 1$.

2 Computation of the joint spectral radius

The jsr is notoriously difficult to compute. Indeed, the problem of approximating the jsr is NP-Hard [1]. Several approximation methods have been designed, some of which even allowing arbitrarily accurate approximations but this is thus at the expense of a high computation time.

A first class of methods considers the definition of the jsr and uses the relation $\bar{\rho}_k(\Sigma) \le \bar{\rho}(\Sigma) \le \rho(\Sigma) \le \rho_k(\Sigma)$, which holds for all k. The idea is thus to generate sets of products in order to compute $\bar{\rho}_k(\Sigma)$ and $\rho_k(\Sigma)$ while discarding as many products as possible, using a branch-and-bound technique. Unfortunately, this sequence of bounds is generally slow to converge, in particular the upper bound.

Hence, a second approach is to carefully choose a norm giving a faster convergence rate. We are searching for an *extremal norm*, i.e., a norm such that the limit in (1) is reached for $k = 1$. Such a norm can be obtained either by an explicit iterative construction, i.e., by building a sequence of norms converging to an extremal norm, or by optimizing on the set of norms, e.g., using the relation $\rho(\Sigma) = \inf \max\{\|A\| \mid A \in \Sigma\}$, where the infimum is taken on the set of norms. However, the size of the optimization problems may grow too fast, and there may be numerical issues when directly dealing with norms. We have thus several algorithms that are guaranteed to converge to the jsr, but that are either too slow to reach a good accuracy, or subject to numerical problems.

3 Genetic algorithm approach

In our approach, we are willing to drop the guaranteed convergence to the exact value in order to obtain bounds of good quality with low computation cost. The algorithm we propose finds a lower bound on the jsr by considering a subset of all products of matrices of given lengths. There is no guarantee that the optimal product will be found as in the previously mentioned branch-and-bound methods but here, the emphasis is put on the low computation time. More precisely, the algorithm starts with a set of products of small length and iteratively generates new products by heuristically combining existing ones depending on their performance. The maximal allowed product length is slowly increased during the computation and each new product may thus provide a better bound. Experimental results tend to show that the bounds obtained by this method are tighter than those obtained by other algorithms and are often optimal on examples of small size. The required computation time is also much smaller, however, there is no a priori guarantee on the quality of the bounds returned by our method.

References

[1] J.N. Tsitsiklis and V.D. Blondel, "The Lyapunov exponent and joint spectral radius of pairs of matrices are hard — when not impossible — to compute and to approximate", *Math. of Control, Signals and Systems*, 10(1), 31–40, 1997.

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