A comparison of approximation algorithms for the joint spectral radius

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Abstract

Let $\Sigma$ be a set of $n \times n$ matrices and let us consider the following linear iteration:

$$x(t+1) = A_t x(t), \quad A_t \in \Sigma \text{ for all } t.$$  

These kinds of switched linear systems arise in many applications such as asynchronous systems, hybrid systems, switching control, ...

The stability under arbitrary switchings of this system depends on a quantity called the joint spectral radius (JSR), which represents the maximal growth rate of such a discrete linear system. More precisely, the JSR of a set $\Sigma$ of matrices is defined by the following expression:

$$\rho(\Sigma) = \lim_{k \to \infty} \max \left\{ \| A_{t_1} \ldots A_{t_k} \|^{1/k} \mid A_t \in \Sigma \right\},$$

independently of the matrix norm used. For bounded sets $\Sigma$, the JSR is also equal to the so-called generalized spectral radius $\rho$ defined by the following equation:

$$\rho(\Sigma) = \limsup_{k \to \infty} \max \left\{ \rho(A_{t_1} \ldots A_{t_k})^{1/k} \mid A_t \in \Sigma \right\}.$$

For the linear iterations we consider, all trajectories converge to the origin if and only if the JSR of the corresponding set of matrices is strictly less than 1.

Another quantity of interest is the joint spectral subradius (JSS), also called lower spectral radius, which represents the minimal achievable growth rate:

$$\rho(\Sigma) = \lim_{k \to \infty} \min \left\{ \| A_{t_1} \ldots A_{t_k} \|^{1/k} \mid A_t \in \Sigma \right\}.$$  

The JSS is related to the mortality problem, where we ask if the zero matrix can be expressed as a finite product of matrices in $\Sigma$.

The joint spectral quantities are notoriously difficult to compute. Indeed, it is NP-Hard to approximate the JSR: moreover, the problem of checking whether $\rho \leq 1$ is undecidable, and the decidability of the question $\rho < 1$ is currently unknown. Furthermore, the problem of approximating the JSS is undecidable in the general case (see [1] for a survey on the joint spectral quantities). Despite these negative theoretical results, several algorithms have been designed in order to approximate the JSR. Indeed, in the case of the JSR, the following easy property holds:

$$\rho(\Sigma) = \inf \max \{ \| A \| \mid A \in \Sigma \}.$$  

Thus, one way to compute the JSR is to try to find a norm such that $\rho(\Sigma) = \max \{ \| A \| \mid A \in \Sigma \}$, i.e., an extremal norm. Several methods have been designed using this fact (see for example [2], [3], [4]). In this paper, we will study and compare several techniques for the computation of the joint spectral quantities, such as semidefinite programming that computes the infimum of a class of norms, or geometric algorithms that tries to approximate the unit ball of an extremal norm using polytopes.

Acknowledgements

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with the authors. Chia-Tche Chang is a Research Fellow of the Fonds National de la Recherche Scientifique (F.R.S.-FNRS).

References


