Fast oriented bounding box optimization on the rotation group $SO(3, \mathbb{R})$

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Coming up next...

The problem

Given a set of n points $\mathcal{X}$ in 3D, find the minimum-volume arbitrarily oriented bounding box enclosing $\mathcal{X}$.

Collision detection, intersection tests, object representation, data approximation... (BV trees...)

In 2D: the rotating calipers method

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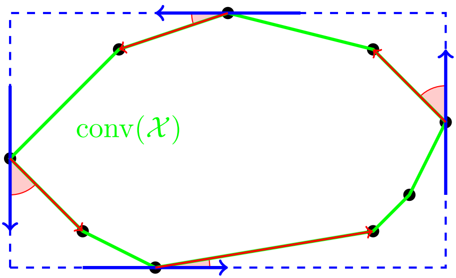

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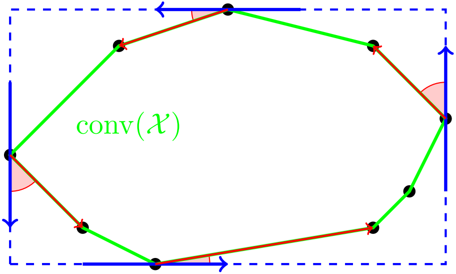

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Loop on all edges:
$\mathcal{O}(n) \rightarrow$ easy and efficient

In 3D: generalization of the rotating calipers?

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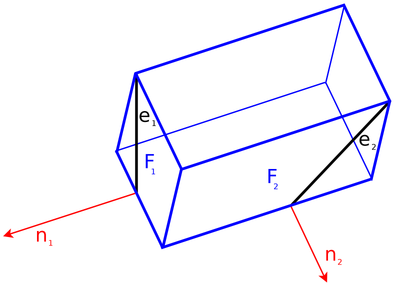

In 3D: generalization of the rotating calipers?

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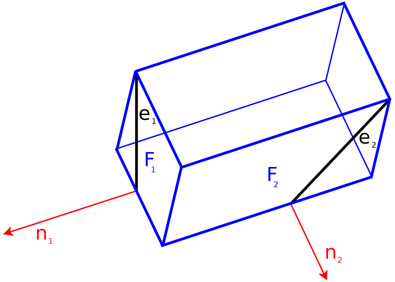
Problem:

Loop on all pairs of edges and rotate the box while keeping edges flush → $\mathcal{O}(n^3)$ time complexity...



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- Guaranteed quality approximation methods:
  same problem...

What do we want?

Goal:

- ◇ Very good accuracy: find an optimal OBB in (nearly?) all cases
- ◇ If a suboptimal solution is returned, it should be close to the best one
- ◇ Computational cost has to be low

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Our approach: iterative algorithm based on optimization methods

Coming up next...

A first, direct, formulation

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Smooth but constrained optimization problem

Unconstrained formulation

$$\min_{R\in SO(3,\mathbb{R})} \underbrace{\begin{pmatrix} \displaystyle\min_{\Delta\in\mathbb{R}^3,\Xi\in\mathbb{R}^3} & \Delta_\xi\Delta_\eta\Delta_\zeta \\ \text{s.t.} & -\frac{1}{2}\Delta \leq R\mathbf{X}_i - \Xi \leq \frac{1}{2}\Delta \ \forall i = 1,\ldots,N \end{pmatrix}}_{f(R)}$$

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Unconstrained but non-differentiable optimization problem

Solving this problem requires...

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- ... a derivative-free method

$f(R)$ is not differentiable everywhere...

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- ... a global search technique

$f(R)$ has many local minima...

Solving this problem requires...

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- ... a global search technique
- ... a fast convergence rate

That was the point!

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- ◇ GA alone would be very slow to converge (GA more suitable for discrete search spaces)
- ◇ NM alone would be stuck in local minima (even with restarts)

Coming up next...

Global exploration: genetic algorithms

Stochastic population-based evolutionary method

(original variant proposed by Holland in the 1970s)

- ◇ Population-based: keep a large set of candidates at each iteration
- ◇ Evolutionary: generate new candidates by combining current ones depending on their performance

The general framework

Start with a set of candidates (*population*)
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- Selection: *parents* are selected depending on their fitness
- Crossover: selected parents produce *offsprings*
- Mutation: offsprings can be subject to *mutations*
  (random modification, gradient step, SA step, ...)

The Nelder-Mead simplex algorithm

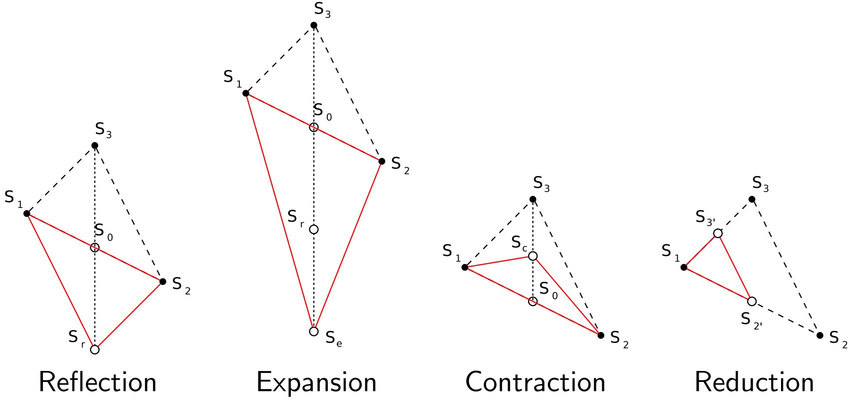
Derivative-free simplicial optimization method

(original algorithm proposed by Nelder & Mead in 1965)

Simplex (in $\mathbb{R}^n$) $=$ set of $n+1$ affinely independent points

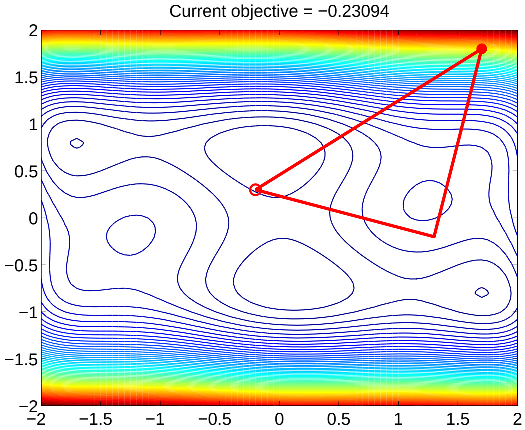
- $n = 2$: triangle
- $n = 3$: tetrahedron
- ...

Ideas of the algorithm (details omitted)



Four main ways to move/transform the simplex depending on the performance of its vertices: affine combinations

Nelder-Mead and the six-hump camel back...



The initial simplex

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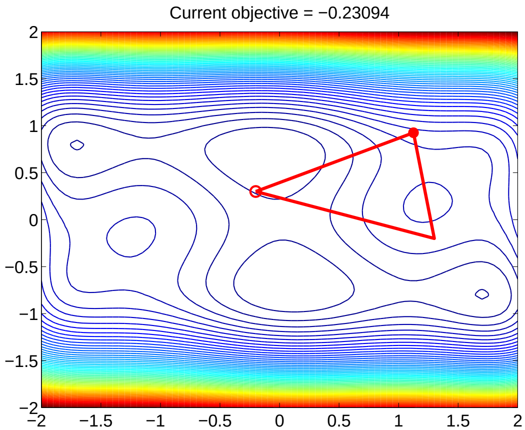

Iteration 1: contraction

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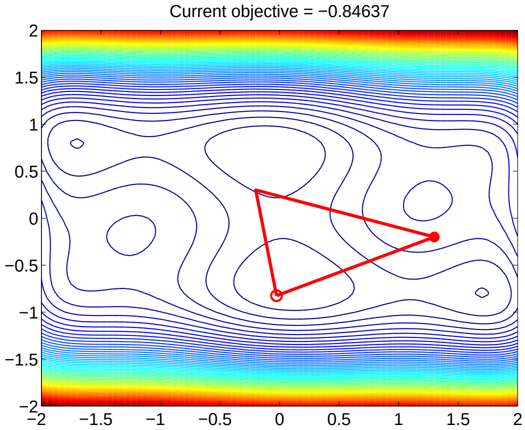
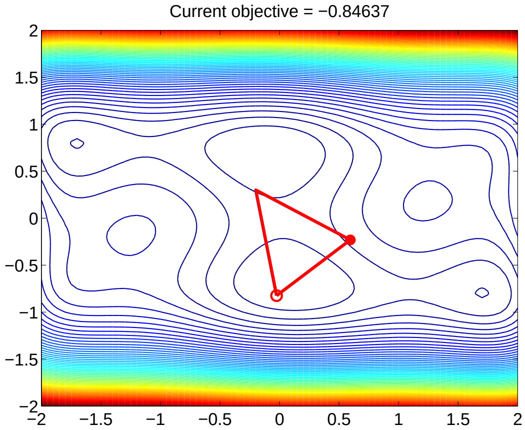


Iteration 2: reflection

Nelder-Mead and the six-hump camel back...



Iteration 3: contraction

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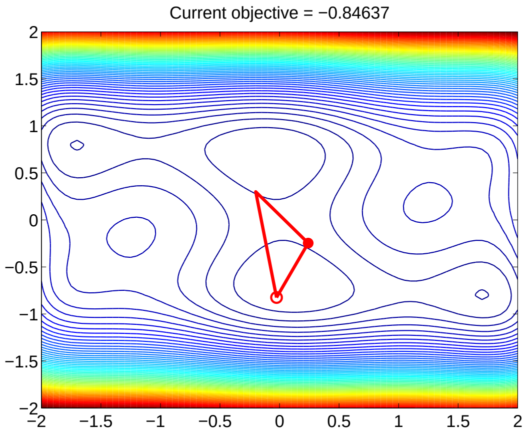

Iteration 4: contraction

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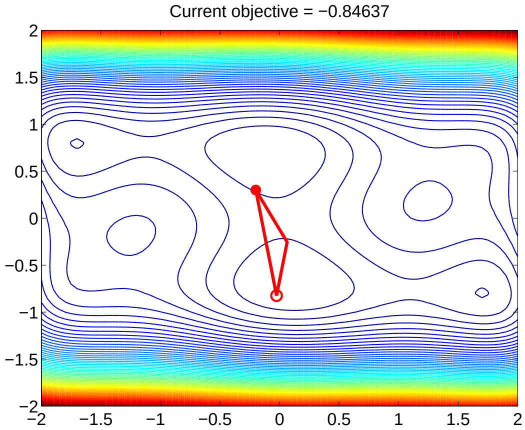
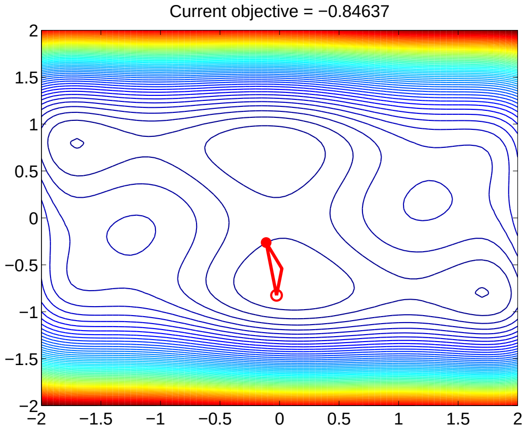


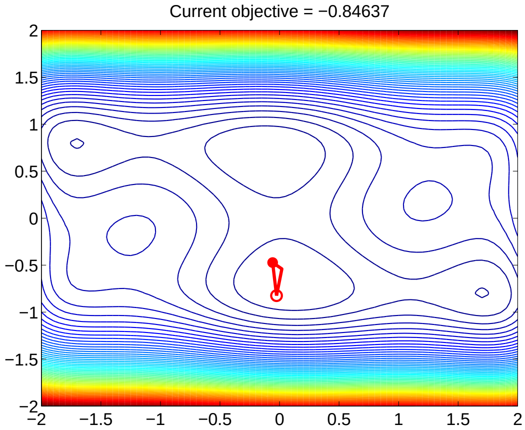
Iteration 5: contraction

Nelder-Mead and the six-hump camel back...



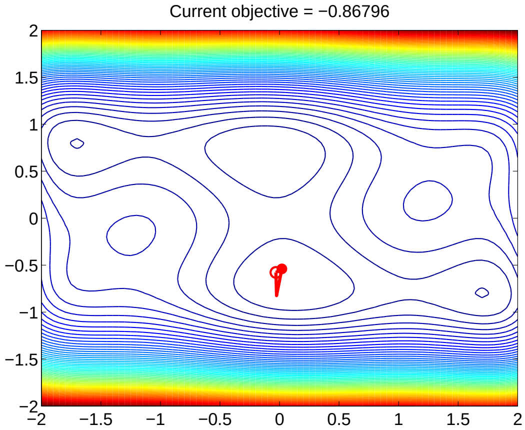
Iteration 6: reduction

Nelder-Mead and the six-hump camel back...



Iteration 7: contraction

Nelder-Mead and the six-hump camel back...



Some more iterations...

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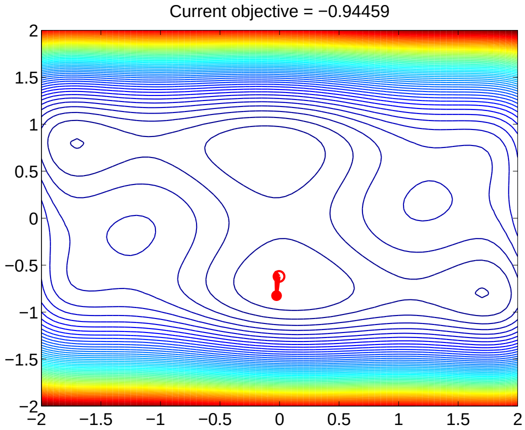

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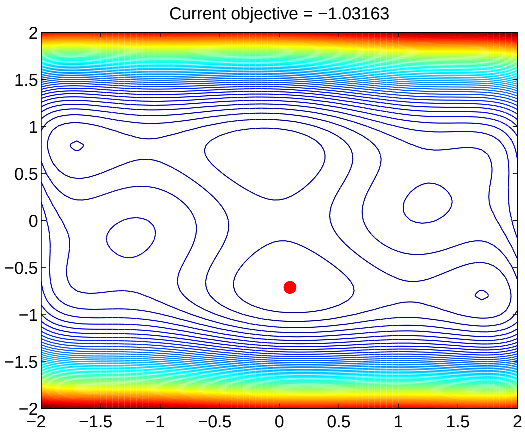

The final result

Mixing NM and GA together (simplified version)

Population of M simplices (simplex = set of 4 rotation matrices)
Fitness function $f(R)$ (volume of corresponding OBB)

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$$A_1 B_1 C_1 D_1 \otimes A_2 B_2 C_2 D_2 \to A_{i_1} B_{i_2} C_{i_3} D_{i_4}, \quad i_k \in \{1, 2\}$$

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- Mutation: Apply K Nelder-Mead iterations on each offspring

HYBBRID

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$$=$$

HYbrid Bounding Box Rotation IDentification algorithm

Coming up next...

Behavior of HYBBRID

All algorithms tested on a benchmark set of ~ 300 objects (Gamma db)
Implementations done in MATLAB (built-in functions are used)

Behavior of HYBBRID: yes, it works!

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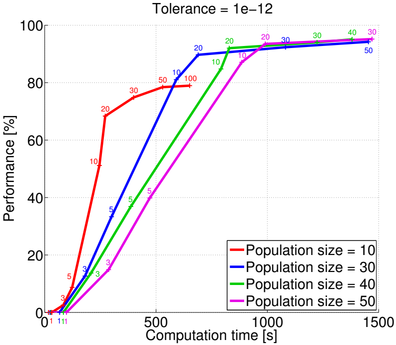
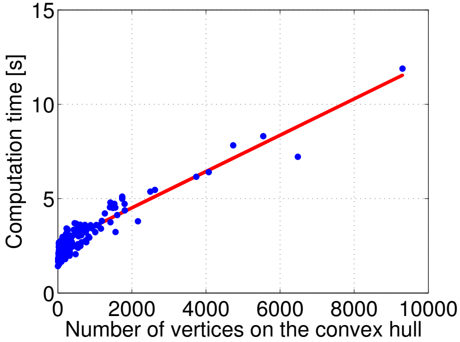


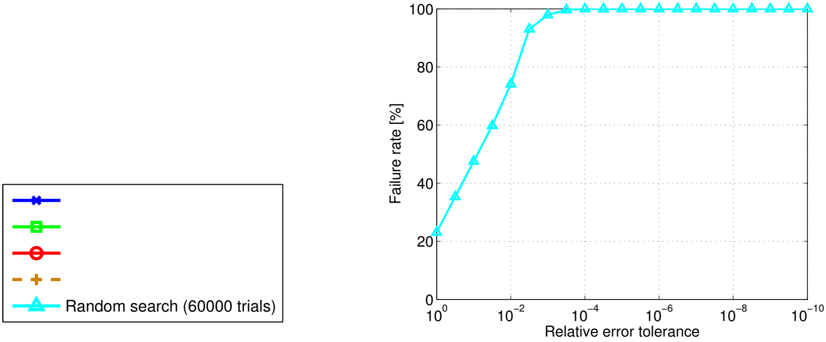
Error is less than 10^{-12} in $90\%+$ of the cases!

How does it scale?



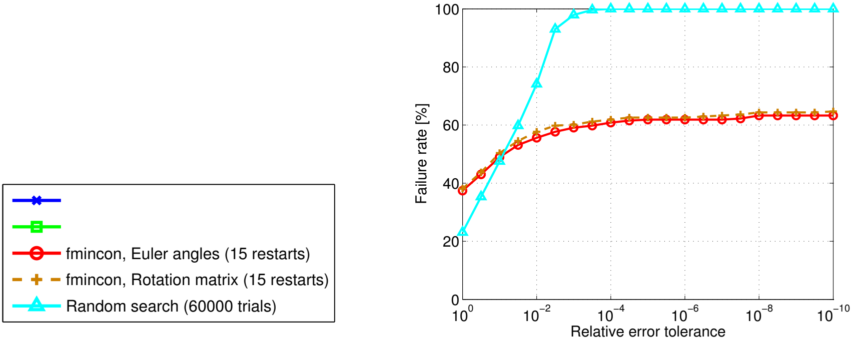
Experimental results show a roughly linear complexity!

Comparison to other iterative approaches



Trying random orientations does not work...

Comparison to other iterative approaches



Constrained smooth opti: success rate $\sim 40\%$ but mainly AABBs...

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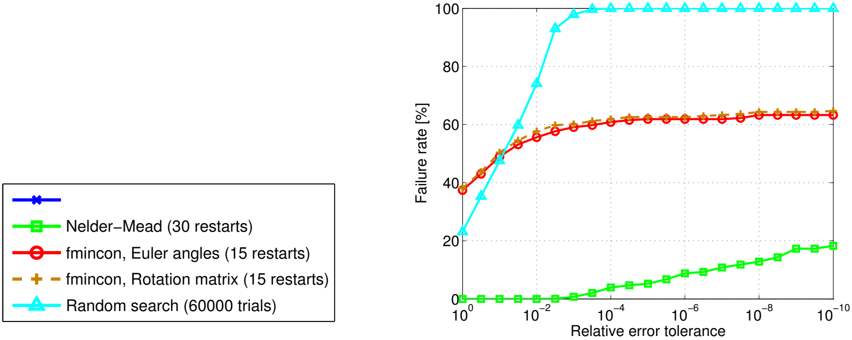
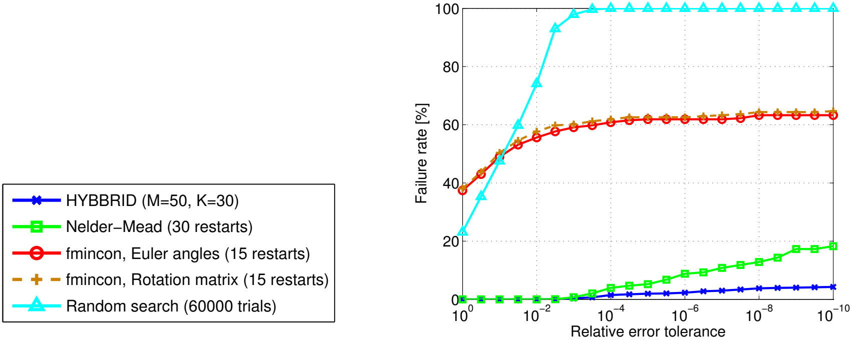


Unconstrained non-diff. opti, random initializations: much better results!

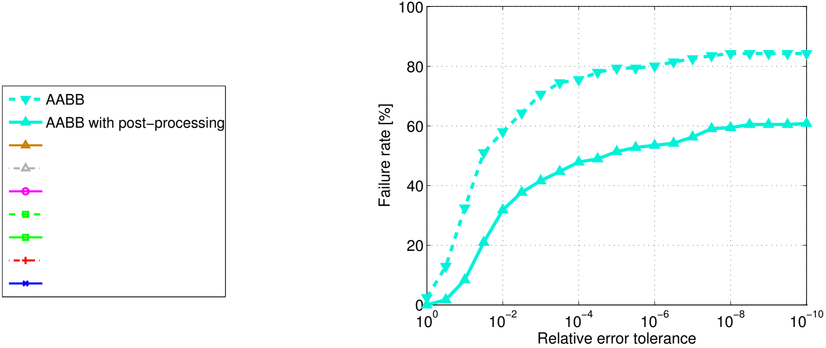
Comparison to other iterative approaches



HYBBRID: combining potential solutions does improve the success rate!

Comparison to other algorithms

First, let's ignore the computational cost and look at the failure rates...



A reference point: the simple AABB

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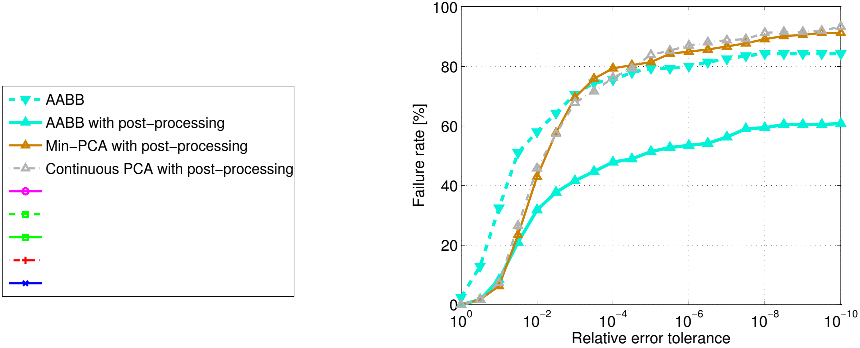

PCA-based methods: limited accuracy

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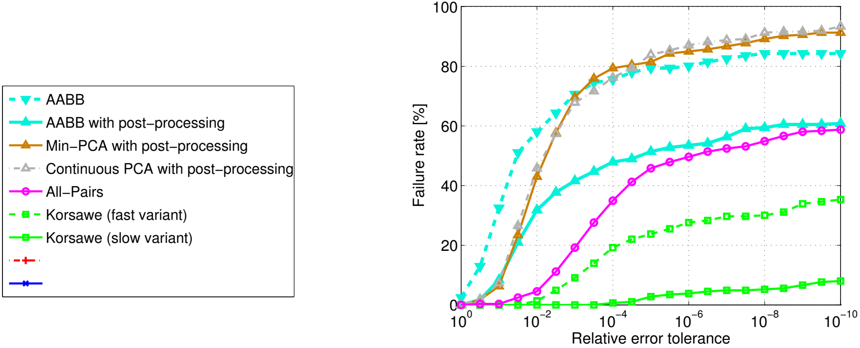

Brute-forcing on a set of orientations *may* be OK... if well chosen!

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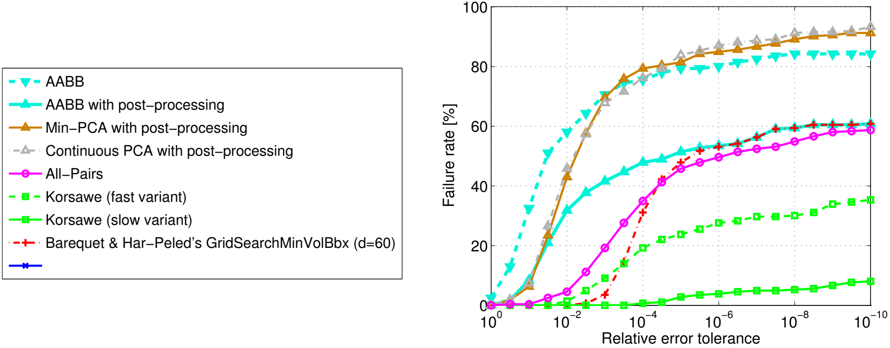

Guaranteed approximation algorithms: limited by computational resources

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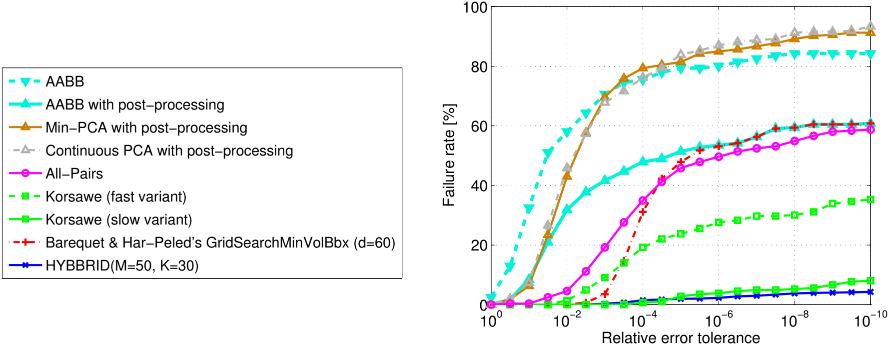
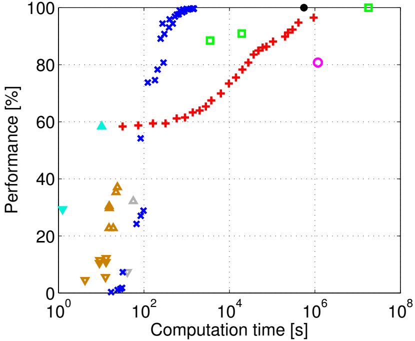
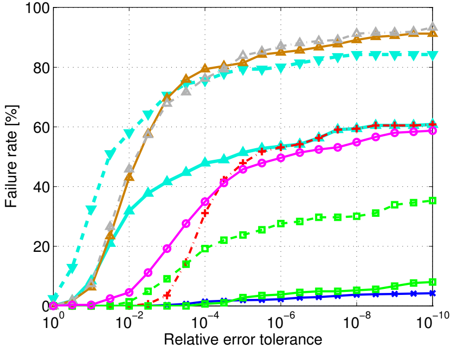


HYBBRID: more accurate than these other methods

Comparison to other algorithms

What are the computation times? (Tolerance: 10^{-3})

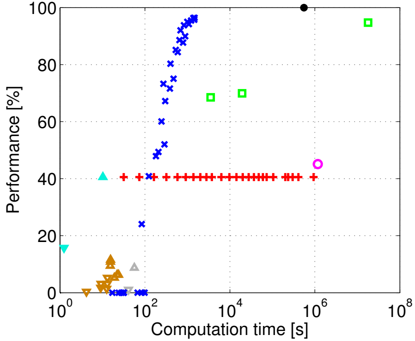


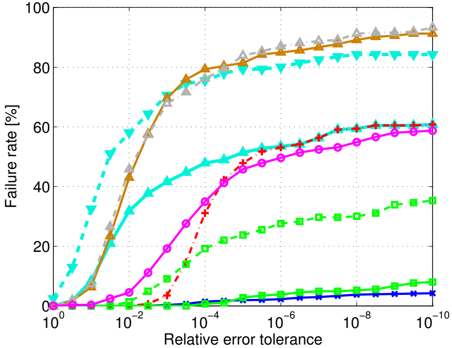


AABB – PCA – Continuous PCA – Brute-force on a set of orientations
O'Rourke's exact algorithm – Guaranteed approximation – HYBBRID

Comparison to other algorithms

What are the computation times? (Tolerance: 10^{-6})





AABB – PCA – Continuous PCA – Brute-force on a set of orientations
O'Rourke's exact algorithm – Guaranteed approximation – HYBBRID

Coming up next...

Conclusion

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  able to approximate optimal OBBs using optimization on $SO(3)$

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Thank you for your attention!