On the growth rate of matrices with row uncertainties

Chia-Tche Chang, Raphaël M. Jungers and Vincent D. Blondel
Department of Mathematical Engineering
Université catholique de Louvain
Avenue Georges Lemaître, 4, B-1348 Louvain-la-Neuve, Belgium
{chia-tche.chang,raphael.jungers,vincent.blondel}@uclouvain.be

Abstract

Let A and B be two $n \times n$ matrices with rows denoted by a_i and b_i. We consider the discrete linear system characterized by the equation :

$$x(t+1) = M_t x(t), \quad M_t \in \Sigma(A,B),$$

where the set $\Sigma(A,B)$ consists of all 2^n matrices such that the ith row is either a_i or b_i.

Here, we are interested in the growth rate and the boundedness of such systems, i.e. the question whether the matrix products $M_t M_{t-1} \ldots M_0$ remain bounded for all products. In particular, we are interested in the case where B is the identity matrix. Indeed, this particular case can be interpreted as an asynchronous linear iteration with zero-delay, where only a subset $\sigma \subset \{1,\ldots,n\}$ of the variables are updated with the matrix A at each time step:

$$\begin{aligned} x_k(t+1) &= \sum_j a_{kj} x_j(t) \quad && \text{if } k \in \sigma, \\ &= x_k(t) && \text{if } k \notin \sigma. \end{aligned}$$

Such iterations arise in several contexts, for example in parallel and distributed computation [1], where a processor i needs the value $x_j(t)$ to be transferred from processor j in order to compute $x_i(t+1)$. Update is thus impossible if the value is not yet available.

Another example is consensus problems where groups of agents update their opinions asynchronously.

The maximal growth rate of such a discrete linear system can be measured by a quantity called joint spectral radius (JSR). The JSR of a set Σ of matrices is defined by the following expression:

$$\rho(\Sigma) = \lim_{k\to\infty} \max\{\|A_{i_1} \ldots A_{i_k}\|^{1/k} \mid A_i \in \Sigma\},$$

independently of the matrix norm used. For bounded sets Σ, the JSR is also equal to the so-called generalized spectral radius $\bar{\rho}$, defined by the following equation:

$$\bar{\rho}(\Sigma) = \limsup_{k\to\infty} \max\{\rho(A_{i_1} \ldots A_{i_k})^{1/k} \mid A_i \in \Sigma\}.$$

For the linear iterations we consider, all trajectories converge thus to the origin if and only if the JSR of the corresponding set of matrices is strictly less than 1.

The computation of the JSR is notoriously difficult in the general case: the problem of checking whether $\rho \leq 1$ has been proved undecidable [2] and the decidability of the question $\rho < 1$ is currently unknown (see [3] for a survey). However, in our case, i.e. matrices with row uncertainties, if A is nonnegative, then the JSR $\rho(\Sigma(A,I))$ is easy to compute [4] and is in fact even equal to $\max\{\rho(A), 1\}$.

If we allow negative entries in the matrix A, it can be seen that this does not hold anymore, and we show with a simple example that the JSR can be strictly greater than the spectral radii of all matrices in $\Sigma(A,I)$.

A related open question is thus whether it is possible to compute the joint spectral radius and to decide boundedness of $\Sigma(A,I)$ in polynomial time. In this talk, we discuss properties and complexity issues involving such sets of matrices.

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