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Technical Appendix to

THE CHILD IS FATHER OF THE MAN: IMPLICATIONS FOR THE DEMOGRAPHIC TRANSITION

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Appendix A. Proofs of Propositions

Proof of Proposition 1. After substituting the integral in (2) into (1), the objective becomes

$$\left(\beta \ln \hat{n} + \delta \ln \hat{A}\right) + \underbrace{\mu(\theta + T)^{\alpha}(A - T - \phi \hat{n}) - \hat{n}\left(\frac{\kappa}{2}\frac{\hat{A}^2}{A}\right)}_{C}$$

which is maximised under the restrictions $T \geq 0$ and $C \geq \bar{C}$.

First order conditions to this problem are (omitting the Kuhn–Tucker conditions):

$$(1+\eta)\hat{A}^2 = \frac{\delta}{\kappa \hat{n}} A \tag{A.1}$$

$$(1+\eta)\alpha\mu(\theta + T)^{\alpha-1}(A - T - \phi\hat{n}) = (1+\eta)\mu(\theta+T)^{\alpha} - \lambda \tag{A.2}$$

$$\frac{1}{\hat{n}}\left(\beta - \frac{\delta}{2}\right) = (1+\eta)\mu(\theta+T)^{\alpha}\phi \tag{A.3}$$

where λ and η are the Kuhn–Tucker multipliers associated with the constraints $T \geq 0$ and $C \geq \bar{C}$ respectively. The interior solution (4)–(6) is (A.1)–(A.3), under $\eta = \lambda = 0$. The corner solution (7)–(9) results from the same system under $\eta = T = 0$ and finally, the corner solution (10)–(12) results from the first order conditions under $T = 0$ and $C = \bar{C}$. Under Assumption 2, $\eta = 0$ and $C = \bar{C}$ are not optimal.

Interior Regime. The solution to the first order conditions (4)–(6) exists and is unique if and only if the loci in (5) and (6) cut once and only once for positive n and T and $C \geq 0$ at the solution. The locus in (5) is a straight line with negative slope and cuts the $\hat{n}$ axes at $(A - \theta/\alpha)/\phi \equiv n_0$, see Figure A1. The locus in (6) has a negative slope, is convex, and is such that $\hat{n}$ goes to zero when T goes to infinity and cuts the $\hat{n}$ axes at $(\beta - \delta/2)/\mu\phi\theta^{\alpha} \equiv n_1$. Comparing these two points and imposing $n_0 \geq n_1$ leads to the condition $A \geq \bar{A}$, where

$$\bar{A} = \frac{\beta - \frac{\delta}{2}}{\mu\theta^{\alpha}} + \frac{\theta}{\alpha}.$$

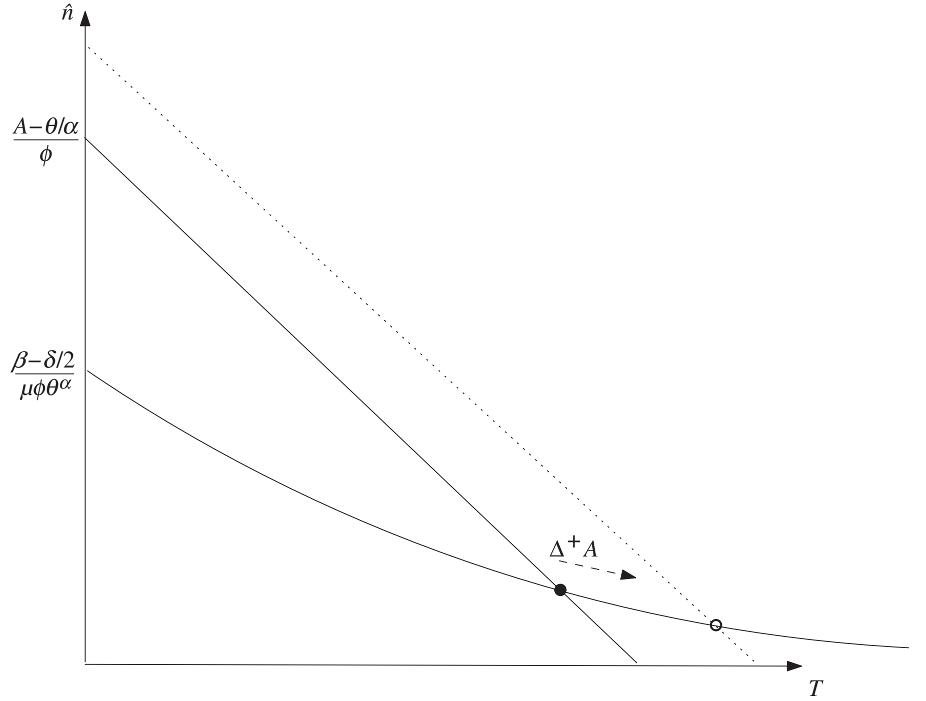

Fig. A1. *The Interior Solution*

Substituting (4) and (5) in the definition of C gives

$$C = \frac{\mu}{\alpha}(\theta + T)^{1+\alpha} - \frac{\delta}{2},$$

which is positive under Assumption 2 for all $T \geq 0$.

Corner regime $\underline{A} \leq A < \bar{A}$. If $A < \bar{A}$, the straight line is above the convex curve at $T = 0$ (see Figure A1). A sufficient condition for these two curves not to intersect in the positive plane is that the straight line is steeper than the convex curve at zero. This is guaranteed by Assumption 2. In that case, there is no interior solution as negative values for T are not feasible. Consequently, the solution must be corner with $T = 0$. From (7)–(9), at this corner solution

$$C = \mu\theta^{\alpha}A - \beta,$$

which is positive for $A \geq \underline{A}$, with

$$\underline{A} \equiv \frac{\beta + \bar{C}}{\mu\theta^{\alpha}}.$$

From Assumption 2, $\underline{A} < \bar{A}$. It is easy to see that the solution is unique.

Corner regime $\bar{C}/\mu\theta^{\alpha} < A < \underline{A}$. Finally, when $\bar{C}/\mu\theta^{\alpha} < A < \underline{A}$, the optimal solution is (10)–(12), with both inequality constraints being binding. Uniqueness is trivial.

Proof of Corollary 1. For the interior solution, we apply the implicit function theorem to (4)–(6), which leads to

$$f'_A = \mathrm{d}\hat{A}/\mathrm{d}A = \sqrt{\frac{A\delta}{\hat{n}\kappa}}\frac{\{T(\alpha+1) + \theta + \alpha[(A - \hat{n}\phi)\alpha + \theta]\}}{2A[(\alpha+1)(T+\theta) - \hat{n}\alpha^2\phi]}.$$

The numerator is positive. Under Assumption 2 the denominator is also positive. The results for f_n' and f_T' can be proved using the same arguments. For the corner solutions the result is straightforward.

Proof of Proposition 2. Let us denote the function $f_A(\cdot)$ by $f_{A1}(\cdot)$ when $A \geq \bar{A}$, $f_{A2}(\cdot)$ when $\underline{A} \leq A < \bar{A}$, and $f_{A3}(\cdot)$ when $\bar{C}/\mu\theta^\alpha < A < \underline{A}$.

The dynamics of life expectancy following $A_{i+1} = f_A(A_i)$ are monotonic because f_A is continuous and non-decreasing.

Let us first prove the *existence* of a stationary solution by considering the values of the function $f_A(\cdot)$ near 0 and as A goes to infinity.

- Near 0, we have that $\lim_{A\to 0} f_A(A) = \lim_{A\to 0} f_{A3}(A) = 0$ and $\lim_{A\to 0} f_A'(A) = \lim_{A\to 0} f_{A3}'(A) = +\infty$. Hence, we have $f_A(A) > A$ for small A.
- For large A we will show that

$$\lim_{A\to\infty} \frac{f_A(A)}{A} = \lim_{A\to\infty} \frac{f_{A1}(A)}{A} = 0.$$

From (5) and (6)

$$\hat{n} = \text{cste}(A - \phi\hat{n} + \theta)^{-\alpha}.$$

Substituting in (4) and dividing by A^2 gives

$$\left(\frac{\hat{A}}{A}\right)^2 = \text{cste}\frac{(A - \phi\hat{n} + \theta)^\alpha}{A}.$$

As $\lim_{A\to\infty} f_n(A) = 0$, we have that $\lim_{A\to\infty} f_{A1}(A)/A = 0$. Hence, we have $f_A(A) < A$ for large A.

By continuity of $f_A(A)$ there is at least one value of A such that $f_A(A) = A$.

Let us now prove the *uniqueness* of the stationary solution. For $0 < A < \underline{A}$, the function $f_{A3}(\cdot)$ is increasing and concave, with $f_{A3}(0) = 0$ and $f_{A3}'(0) = \infty$. Function $f_{A2}(\cdot)$ is increasing and concave, with $f_{A2}(0) = 0$ and $f_{A2}'(0) = \infty$. As the two functions have the same slope where there is a regime shift, $f_{A3}'(\underline{A}) = f_{A2}'(\underline{A})$, the function f_A is concave on the interval $[0, \bar{A})$ implying that if it crosses the diagonal on the interval $[0, \bar{A})$, it crosses it only once.

Uniqueness would be guaranteed if the following property is satisfied: the existence of a stationary solution on $[0, \bar{A})$ excludes the existence of another one on $[\bar{A}, +\infty)$. Let us demonstrate this property by a reductio ad absurdum. Suppose we have a stationary solution on $[0, \bar{A}]$. Then, we have necessarily $f_A(\bar{A}) < A$ (f_A is below the 45 degrees line). Thus, if there exists other stationary solutions larger than $\bar{A}$, the slope of f_A should at least be larger than or equal to one at one of these solutions.

From the implicit function theorem applied to (4)–(6),

$$\frac{d\hat{A}}{\hat{A}} \Big/ \frac{dA}{A} = \frac{1}{2}\frac{(1+\alpha)(A - \phi\hat{n})}{A - (1+\alpha)\hat{n}}.$$

At a fixed point of f_{A1}, as Corollary 1 shows that $f_A'(A) > 0$ in this interval, the denominator must be strictly positive. It is then easy to see that $f_A'(A) < 1$ iff $A > [(1+\alpha)/(1-\alpha)]\phi\hat{n}$. As from Corollary 1, $f_n'(\cdot) < 0$ in this interval, $f_A'(A) < 1$ for all $A \geq \bar{A}$ iff $\bar{A} > [(1+\alpha)/(1-\alpha)]\phi\hat{n}$, which holds under Assumption 2. Hence, $f_{A1}'(A) < 1$ for any fixed point of $f_{A1}(\cdot)$ in $A \geq \bar{A}$, which excludes the existence of additional solutions if there is already one on the interval $[0, \bar{A}]$.

If there is no solution on $[0, \bar{A}]$, there could be only one on $[\bar{A}, +\infty)$ as, in such a case, $f_{A3}(\bar{A}) > \bar{A}$ and f_{A3} is concave. f_A is above the diagonal before the unique steady state equilibrium and below it afterwards, which ensures *global stability*.

Proof of Remark 1. A steady state for A in the interior regime exists if there is a solution to the system (4)–(6), evaluated at the steady state. Eliminating A and n from Equation (5) using Equations (4) and (6), we find that the steady state T should satisfy:

$$T(1+\alpha)+\theta=\alpha\left[\frac{2\delta\phi\mu(T+\theta)^{\alpha}}{\kappa(2\beta-\delta)}-\frac{(2\beta-\delta)(T+\theta)^{-\alpha}}{2\mu}\right].$$

The left-hand side is a linear increasing function of T. The right-hand side is a concave function of T, with a slope going to zero as T goes to infinity. A sufficient condition for existence and uniqueness of a stationary solution is that the right-hand side is larger than the left-hand side at $T=0$. This leads to Condition (13).