Childbearing Postponement, its Option Value, and the Biological Clock
Online Appendix

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B Solving the Post-Birth Program

We note the variable part of the value function as $V(a_\tau) = W_2(a_\tau) - \omega$. The corresponding Bellman equation is

$$
\begin{aligned}
V(a_t) &= \max_{c_t} \left\{ u(c_t)dt + e^{-\rho\, dt} E\left[V(a_{t+dt})\right] \right\} \\
&\Leftrightarrow\ V(a_t) = \max_{c_t} \left\{ u(c_t)dt + (1-\rho\, dt)\left(V(a_t) + E\left[dV(a_t)\right]\right) \right\} \\
&\Leftrightarrow\ \rho V(a_t)dt = \max_{c_t} \left\{ u(c_t)dt + (1-\rho\, dt)E\left[dV(a_t)\right] \right\}
\end{aligned}
$$

using $e^{-\rho\, dt} \approx (1-\rho\, dt)$ for small dt. To solve for the value function, we make an educated guess, $V(a_t) = D_2 \frac{a_t^{1-\varepsilon}}{1-\varepsilon}$ where D_2 is a constant to be determined. According to Itô's lemma:

$$
\begin{aligned}
E\left[dV(a_t)\right] &= \frac{\partial V(a_t)}{\partial a_t} E[da_t] + \frac{1}{2}\frac{\partial^2 V(a_t)}{\partial a_t^2} E[(da_t)^2] \\
&= D_2 a_t^{1-\varepsilon} \left(r_2 - (1+\beta)\frac{c_t}{a_t} - \frac{\varepsilon}{2}\sigma_2^2 \right) dt
\end{aligned}
$$

as $E[da_t] = (r_2 a_t - (1+\beta)c_t)dt$ and $E[(da_t)^2] = \sigma_2^2 a_t^2 dt$ (as $(dt)^2 \approx 0$, $(dt)^{3/2} \approx 0$ and $E[(dz)^2] = dt$)

The Bellman equation then becomes:

$$
\rho V(a_t) = \max_{c_t} \left\{ \frac{c_t^{1-\varepsilon}}{1-\varepsilon} + D_2 a_t^{1-\varepsilon} \left(r_2 - (1+\beta)\frac{c_t}{a_t} - \frac{\varepsilon}{2}\sigma_2^2 \right) \right\}
$$

The first-order condition with respect to consumption is: $c_t = ((1+\beta)D_2)^{-1/\varepsilon} a_t$. In order to determine constant D_2, the Bellman equation can be rewritten:

$$\begin{aligned}
\rho D_2 \frac{a_t^{1-\varepsilon}}{1-\varepsilon} &= \frac{\left(((1+\beta)D_2)^{-1/\varepsilon} a_t\right)^{1-\varepsilon}}{1-\varepsilon} + D_2 a_t^{1-\varepsilon}\left(r_2 - (1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} D_2^{-1/\varepsilon} - \frac{\varepsilon}{2}\sigma_2^2\right) \\
&\Leftrightarrow \frac{\rho}{1-\varepsilon} = \frac{(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} D_2^{-1/\varepsilon}}{1-\varepsilon} + \left(r_2 - (1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} D_2^{-1/\varepsilon} - \frac{\varepsilon}{2}\sigma_2^2\right) \\
&\Leftrightarrow \frac{\rho}{1-\varepsilon} = (1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} \frac{\varepsilon}{1-\varepsilon} D_2^{-1/\varepsilon} + \left(r_2 - \frac{\varepsilon}{2}\sigma_2^2\right) \\
&\Leftrightarrow D_2 = \left[\frac{(1+\beta)^{\frac{1-\varepsilon}{\varepsilon}}}{\varepsilon}\left(\rho + (\varepsilon - 1)\left(r_2 - \frac{\varepsilon}{2}\sigma_2^2\right)\right)\right]^{-\varepsilon} = q^{-\varepsilon},
\end{aligned}$$

where q is defined in Equation (7). Therefore

$$W_2(a_\tau) = q^{-\varepsilon} \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + \omega,$$

and

$$c_t = ((1+\beta)D_2)^{-1/\varepsilon} a_t = (1+\beta)^{-1/\varepsilon} q a_t.$$

C Solving the Full Program

Standard integration by parts yields:

$$\int_0^\tau \mu_t \dot{a}_t dt = \mu_\tau a_\tau - \mu_0 a_0 - \int_0^\tau \dot{\mu}_t a_t \, dt,$$

which allows to rewrite $W(a_0)$ as:

$$W(a_0) = \int_0^\tau \left(H(c_t, a_t, \mu_t) + \dot{\mu}_t a_t\right) dt + \varphi(\tau, a_\tau) - \mu_\tau a_\tau + \mu_0 a_0.$$

The first-order variation of $W(a_0)$ with respect to the state and control variable's path for a given a_0 but for τ and a_τ free yields:

$$\begin{aligned}
dW(a_0) &= \int_0^\tau \left(\frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} da_t + \frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} dc_t + \dot{\mu}_t \, da_t\right) dt \\
&\quad + \left[H(c_\tau, a_\tau, \mu_\tau) + \dot{\mu}_\tau a_\tau\right] d\tau + \frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} da_\tau + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} d\tau \\
&\quad - \dot{\mu}_\tau a_\tau d\tau - \mu_\tau da_\tau.
\end{aligned}$$

Rearranging terms leads to:

$$
\begin{aligned}
dW(a_0) \;=\; & \int_0^{\tau} \left[\left(\frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} + \dot{\mu}_t \right) da_t + \frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} dc_t \right] dt \\
& + \left[H(c_\tau, a_\tau, \mu_\tau) + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} \right] d\tau \\
& + \left[\frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau \right] da_\tau .
\end{aligned}
$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function, that is $dW(a_0) \leq 0$ for any $da_t, t \in (0, \tau)$, for any $dc_t, t \in (0, \tau)$, and for any $d\tau$ and da_τ, which gives the following necessary conditions for an interior maximizer:

$$
\begin{aligned}
\frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} &= 0 \\
\frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} + \dot{\mu}_t &= 0
\end{aligned}
$$

$$
\begin{aligned}
H(c_\tau, a_\tau, \mu_\tau) + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} &= 0 \\
\frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau &= 0
\end{aligned}
$$

The first two conditions are standard Pontryagin conditions. The last two conditions may be interpreted as optimality conditions with respect to the switching time τ and the free state value a_τ. The third one equalizes the marginal benefit of waiting to the marginal cost of waiting. The last one is a continuity condition: it implies that the shadow price of the state variable at the time of the switch, μ_τ, is equal to the expected marginal value of the state variable in τ (derived from the programs after the switch).

The two standard Pontryagin conditions imply:

$$
\begin{aligned}
/c_t \quad &: \quad u'(c_t)e^{-\rho t} = \mu_t \\
/a_t \quad &: \quad \dot{\mu}_t / \mu_t = -r_1 \Rightarrow \mu_t = \mu_0 e^{-r_1 t} \\
&\Rightarrow \quad \dot{c}_t / c_t = (r_1 - \rho)/\varepsilon \Rightarrow c_t = c_0 e^{\frac{r_1 - \rho}{\varepsilon} t} \text{ and } c_t = \left(\mu_t e^{\rho t} \right)^{-1/\varepsilon}
\end{aligned}
$$

Therefore, the dynamics of assets can be rewritten as:

$$
\dot{a}_t = r_1 \; a_t - c_0 e^{\frac{r_1 - \rho}{\varepsilon} t}
$$

Using a variable change $x_t = a_t e^{-r_1 t}$ we solve for $x_t = (c_0/p) e^{\left(\frac{r_1 - \rho}{\varepsilon} - r_1 \right) t} + \overline{x}$, where p is defined in Equation (6) and $\overline{x}$ is a constant to be determined. Therefore:

$$
a_t = \frac{c_0}{p} \; e^{\left(\frac{r_1 - \rho}{\varepsilon} \right) t} + \overline{x} e^{r_1 t}
$$

Without the procreation option, the transversality condition would imply $\overline{x} = 0$ and c_0 would be determined by a_0.

Moreover at $t = 0$:

$$
\begin{aligned}
a_0 &= \tfrac{c_0}{p} + \overline{x} \\
&\Leftrightarrow \qquad \overline{x} = a_0 - \frac{c_0}{p} \\
&\Rightarrow \qquad a_t = \frac{c_0}{p} e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} + \left[a_0 - \frac{c_0}{p}\right] e^{r_1 t}
\end{aligned}
$$

$$
a_t = a_0 e^{r_1 t} + \frac{c_0}{p}\left[e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} - e^{r_1 t}\right] \tag{1}
$$

The fourth condition allows to find a_τ as a function of τ:

$$
\frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau = 0
$$

$$
\begin{aligned}
&\Leftrightarrow \quad \mu_\tau = e^{-\rho\tau}\left[\pi(\tau) q^{-\varepsilon} + (1-\pi(\tau)) p^{-\varepsilon}\right] a_\tau^{-\varepsilon} \\
&\Leftrightarrow \quad \mu_\tau = e^{-\rho\tau}(s(\tau) a_\tau)^{-\varepsilon} \\
&\Rightarrow \quad \mu_0 = \mu_\tau e^{r_1\tau} = e^{(r_1-\rho)\tau}(s(\tau) a_\tau)^{-\varepsilon}
\end{aligned}
$$

where we have introduced the following notation:

$$
s(t) = \left(\pi(t) q^{-\varepsilon} + (1-\pi(t)) p^{-\varepsilon}\right)^{-1/\varepsilon}
$$

which is valid in particular for $t = \tau$. This allows to identify c_0 as a function of τ and a_τ:

$$
c_0 = (\mu_0)^{-1/\varepsilon} = \left(e^{(r_1-\rho)\tau}(s(\tau) a_\tau)^{-\varepsilon}\right)^{-1/\varepsilon} = \left(e^{(r_1-\rho)\tau}\right)^{-1/\varepsilon} s(\tau) a_\tau \tag{2}
$$

Using (1) and (2), it is then possible to obtain an expression for a_τ as a function of τ and a_0:

$$
a_\tau = \frac{e^{r_1\tau}}{1 + s(\tau)\left[e^{p\tau} - 1\right]/p} a_0 \tag{3}
$$

Note that without the procreation option, this expression would become:

$$
a_\tau = e^{\frac{r_1-\rho}{\varepsilon}\tau} a_0
$$

Therefore, it is possible to express $X(\tau)$ the effect of the procreation option as follows:

$$
a_\tau = e^{\frac{r_1-\rho}{\varepsilon}\tau} X(\tau) a_0
$$

with

$$
X(\tau) = \frac{e^{p\tau}}{1 + s(\tau)\left[e^{p\tau} - 1\right]/p}
$$

$X(\tau) \geq 1$ if $\varepsilon \geq 1$.
It is now possible to express the dynamics of a_t:

$$a_t = a_0 \left[e^{r_1 t} - \frac{e^{p\tau} \left[e^{r_1 t} - e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} \right]}{p/s(\tau) + [e^{p\tau} - 1]} \right] \tag{4}$$

In addition, using $c_\tau = c_0 e^{\frac{r_1-\rho}{\varepsilon}\tau}$, Equations (2) and (4), it is possible to express consumption as a function of τ only:

$$c_\tau = \frac{e^{p\tau} s(\tau)}{1 + s(\tau) \left[e^{p\tau} - 1 \right] / p} a_0 e^{\frac{r_1-\rho}{\varepsilon}\tau}$$

Note that absent the procreation option, this expression would become:

$$c_\tau = p e^{\frac{r_1-\rho}{\varepsilon}\tau} a_0$$

Finally, the third condition gives a second relation between a_τ and τ:

$$\begin{aligned}
& H(c_\tau, a_\tau, \mu_\tau) + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} = 0 \\
\Leftrightarrow\ & \frac{c_\tau^{1-\varepsilon}}{1-\varepsilon} e^{-\rho\tau} + \mu_\tau \left(r_1 a_\tau - c_\tau \right) - \rho \varphi(\tau, a_\tau) \\
& \qquad + e^{-\rho\tau} \left[\pi'(\tau) \left(q^{-\varepsilon} \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + \omega \right) - \pi'(\tau) \left(p^{-\varepsilon} \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} \right) \right] = 0 \\
\Leftrightarrow\ & \frac{c_\tau^{1-\varepsilon}}{1-\varepsilon} e^{-\rho\tau} + \mu_\tau \left(r_1 a_\tau - c_\tau \right) - \rho \varphi(\tau, a_\tau) \\
& \qquad + e^{-\rho\tau} \left[\pi'(\tau) \left(q^{-\varepsilon} \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + \omega \right) - \pi'(\tau) \left(p^{-\varepsilon} \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} \right) \right] = 0 \\
\Leftrightarrow\ & \frac{c_\tau^{1-\varepsilon}}{1-\varepsilon} e^{-\rho\tau} + \mu_\tau \left(r_1 a_\tau - c_\tau \right) - \rho \varphi(\tau, a_\tau) \\
& \qquad + e^{-\rho\tau} \pi'(\tau) \left[q^{-\varepsilon} \ - p^{-\varepsilon} \right] \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + e^{-\rho\tau} \pi'(\tau) \omega = 0.
\end{aligned}$$

To obtain a function of τ only, recall that

$$\begin{aligned}
s(\tau)^{-\varepsilon} &= \pi(\tau) q^{-\varepsilon} + (1 - \pi(\tau)) p^{-\varepsilon} \\
&= \pi(\tau)(q^{-\varepsilon} - p^{-\varepsilon}) + p^{-\varepsilon} \\
&\Rightarrow \frac{s(\tau)^{-\varepsilon} - p^{-\varepsilon}}{\pi(\tau)} = q^{-\varepsilon} - p^{-\varepsilon}
\end{aligned}$$

and

$$c_\tau^{-\varepsilon} e^{-\rho\tau} = \mu_\tau$$

then

$$\begin{aligned}
& \frac{c_\tau^{1-\varepsilon}}{1-\varepsilon} e^{-\rho\tau} + r_1 e^{-\rho\tau} s(\tau)^{-\varepsilon} a_\tau^{1-\varepsilon} - c_\tau^{1-\varepsilon} e^{-\rho\tau} - \rho \varphi(\tau, a_\tau) \\
& \qquad + e^{-\rho\tau} \frac{\pi'(\tau)}{\pi(\tau)} \left(s(\tau)^{-\varepsilon} - p^{-\varepsilon} \right) \frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + e^{-\rho\tau} \pi'(\tau) \omega = 0
\end{aligned}$$

$$\begin{aligned}
\text{where } \varphi(\tau,a_\tau) &= e^{-\rho\tau}\left[\pi(\tau)\left(q^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}+\omega\right)+(1-\pi(\tau))p^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}\right] \\
&= e^{-\rho\tau}\left[\pi(t)q^{-\varepsilon}+(1-\pi(\tau))p^{-\varepsilon}\right]\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}+e^{-\rho\tau}\pi(\tau)\omega \\
&= e^{-\rho\tau}s(\tau)^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}+e^{-\rho\tau}\pi(\tau)\omega
\end{aligned}$$

therefore

$$\frac{\varepsilon c_\tau^{1-\varepsilon}}{1-\varepsilon}+r_1 s(\tau)^{-1/\varepsilon}a_\tau^{1-\varepsilon}-\rho\left[s(\tau)^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}+\pi(\tau)\omega\right]+\frac{\pi'(\tau)}{\pi(\tau)}\left(s(\tau)^{-\varepsilon}-p^{-\varepsilon}\right)\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}+\pi'(\tau)\omega=0$$

$$\Leftrightarrow \frac{\varepsilon c_\tau^{1-\varepsilon}}{1-\varepsilon}+r_1 s(\tau)^{-\varepsilon}a_\tau^{1-\varepsilon}-\rho\left[s(\tau)^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon}\right]+\frac{\pi'(\tau)}{\pi(\tau)}\left(s(\tau)^{-\varepsilon}-p^{-\varepsilon}\right)\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} = \left(\rho\pi(\tau)-\pi'(\tau)\right)\omega$$

Recall

$$c_\tau = c_0 e^{\frac{r_1-\rho}{\varepsilon}\tau} = s(\tau)a_\tau$$

Therefore:

$$a_\tau^{1-\varepsilon}\left(\frac{\varepsilon\left(s(\tau)\right)^{1-\varepsilon}}{1-\varepsilon}+r_1 s(\tau)^{-\varepsilon}-\frac{\rho s(\tau)^{-\varepsilon}}{1-\varepsilon}+\frac{\pi'(\tau)}{\pi(\tau)}\frac{s(\tau)^{-\varepsilon}-p^{-\varepsilon}}{1-\varepsilon}\right) = \left(\rho\pi(\tau)-\pi'(\tau)\right)\omega$$

and using the expression for a_τ we obtain an implicit expression for τ, as a function of a_0.

$$\begin{aligned}
&e^{(1-\varepsilon)r_1\tau}\left[1+\frac{s(\tau)}{p}\left[e^{p\tau}-1\right]\right]^{\varepsilon-1} \\
&\qquad\times\frac{\varepsilon}{\varepsilon-1}s(\tau)^{-\varepsilon}\left(p-s(\tau)-\frac{1}{\epsilon}\frac{\pi'(\tau)}{\pi(\tau)}\left(1-\frac{p^{-\varepsilon}}{s(\tau)^{-\varepsilon}}\right)\right)=\pi(\tau)\left(\rho-\frac{\pi'(\tau)}{\pi(\tau)}\right)\omega a_0^{\varepsilon-1}
\end{aligned}$$

The value function of the full program is:

$$\begin{aligned}
W(a_0) &= \int_0^\tau \frac{\left(s(\tau)X(\tau)a_0 e^{\frac{r_1-\rho}{\varepsilon}t}\right)^{1-\varepsilon}}{1-\varepsilon}e^{-\rho t}dt+\varphi(\tau,a_\tau) \\
&= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon}\int_0^\tau e^{-pt}dt+\varphi(\tau,a_\tau) \\
&= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon}\left[\frac{-1}{p}e^{-pt}\right]_0^\tau+\varphi(\tau,a_\tau) \\
&= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon}\left[\frac{-1}{p}e^{-p\tau}+\frac{1}{p}\right]+\varphi(\tau,a_\tau)
\end{aligned}$$

D Proofs

D.1 Proof of Proposition 2

For $r_2 \geq r_1$, $\beta \geq 0$ and $\sigma = 0$, it is possible to show that $\frac{\partial W(a_0)}{\partial \tau} < 0$. It is then optimal to attempt to get pregnant as soon as possible. This result is also valid in the more restricted case where $r_2 \approx r_1$ thus proving the first part of the proposition.

To prove the second part of the proposition, we rewrite Equation (23) as (see Appendix B):

$$e^{(1-\varepsilon)r_1\tau}\left[1+\frac{s(\tau)}{p}\left[e^{p\tau}-1\right]\right]^{\varepsilon-1} \times \frac{\varepsilon}{\varepsilon-1}s(\tau)^{-\varepsilon}\left(p-s(\tau)-\frac{1}{\epsilon}\frac{\pi'(\tau)}{\pi(\tau)}\left(1-\frac{p^{-\varepsilon}}{s(\tau)^{-\varepsilon}}\right)\right)=\pi(\tau)\left(\rho-\frac{\pi'(\tau)}{\pi(\tau)}\right)\omega a_0^{\varepsilon-1} \quad (5)$$

In the neighborhood of $\tau = 0$, since $\pi'(\tau) = 0$, it can be simplified to:

$$e^{(1-\varepsilon)r_1\tau}\left[1+\frac{s(\tau)}{p}\left[e^{p\tau}-1\right]\right]^{\varepsilon-1}\frac{\varepsilon}{\varepsilon-1}s(\tau)^{-\varepsilon}\left(p-s(\tau)\right)=\pi(\tau)\rho\omega a_0^{\varepsilon-1}$$

The right-hand side of this equation is always positive and finite. For $r_2 = r_1$ and $\sigma = 0$, the left-hand side (LHS) of the equation is equal to zero. In addition, $\partial LHS(\tau)/\partial\sigma > 0$ and $\lim_{\sigma\to\sigma_{\max}} LHS \to \infty$ with $\sigma_{\max} = \left[2(r_2 - \rho/(1-\varepsilon)/\varepsilon\right]^{1/2}$. Therefore, there exists $\underline{\sigma} > 0$ such that $\sigma > \underline{\sigma} \Leftrightarrow \tau^* > 0$. This proves the second part of the proposition.

To prove the third part of the proposition, we compare the functions $W_1(a_\tau)$ and $W_2(a_\tau)$. We have $\lim_{\sigma\to\sigma_{\max}} W_2(a_\tau) \to -\infty$ for a finite a_τ, while $W_1(a_\tau)$ remains finite as it is not affected by σ. Therefore, $\lim_{\sigma\to\sigma_{\max}} \tau^* \to +\infty$. In addition, $\frac{\partial W_2(a_\tau)}{\partial\sigma} < 0$ and we know from the first part of the proposition that for $r_2 = r_1$ and $\sigma = 0$, it is optimal to get pregnant as soon as possible. Therefore, there exists a value $\bar{\sigma}$ of σ that is sufficiently large to have $\tau^* \geq T$ if $\sigma > \bar{\sigma}$.

D.2 Proof of Proposition 3

For $\tau \geq T$ and $\tau \leq 0$, we have $\pi'(\tau) = 0$. Under the assumption $\pi'(\tau) = 0$ and τ finite, the derivatives of the LHS and of the RHS of Equation (5) with respect to τ and ω can be written:

$$\frac{\partial LHS}{\partial\tau} = e^{(1-\varepsilon)r_1\tau}\varepsilon s(\tau)^{-\epsilon}(p-s(\tau))\left(-r_1\left[1+\frac{s(\tau)}{p}\left(e^{p\tau}-1\right)\right]+s(\tau)e^{p\tau}\right)$$

$$\frac{\partial LHS}{\partial\omega} = 0, \frac{\partial RHS}{\partial\tau} = 0, \frac{\partial RHS}{\partial\omega} = \pi(\tau)\rho a_0^{\varepsilon-1} > 0$$

For $\rho < r_1$ and $\pi'(\tau) = 0$, the LHS is then a decreasing function for all τ which ensures $\bar{\omega} < \tilde{\omega}$. $\rho < r_1$ and τ in the neighborhood of 0 and T (which ensures $\pi'(\tau) = 0$) are sufficient, together with continuity, to yield the existence and uniqueness of $\bar{\omega}$ and $\tilde{\omega}$.

E Kernel Density Estimations of Income Growth Distribution - Alternative Samples

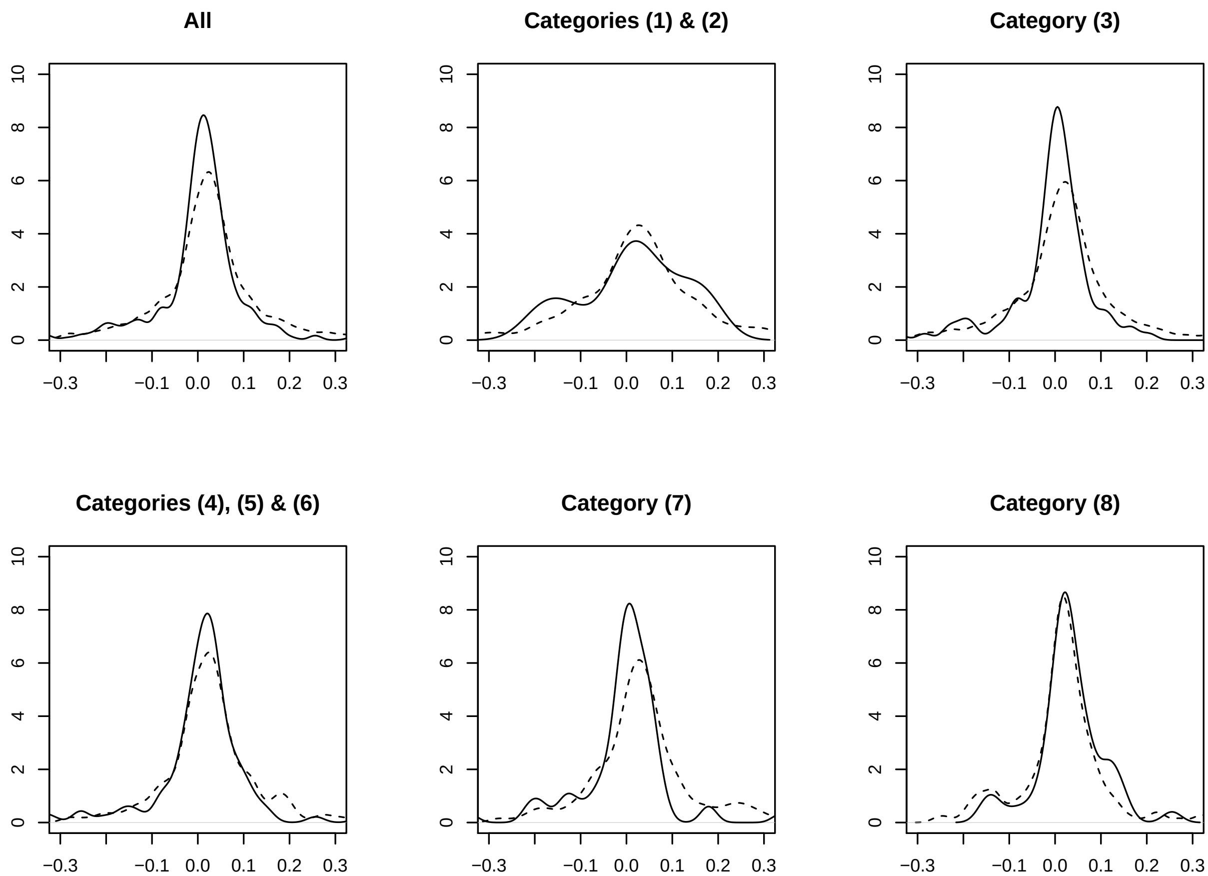

Figure 1: Kernel density estimations of individual income growth distribution by education category. Married women. Childless women (solid) and mothers (dashed)

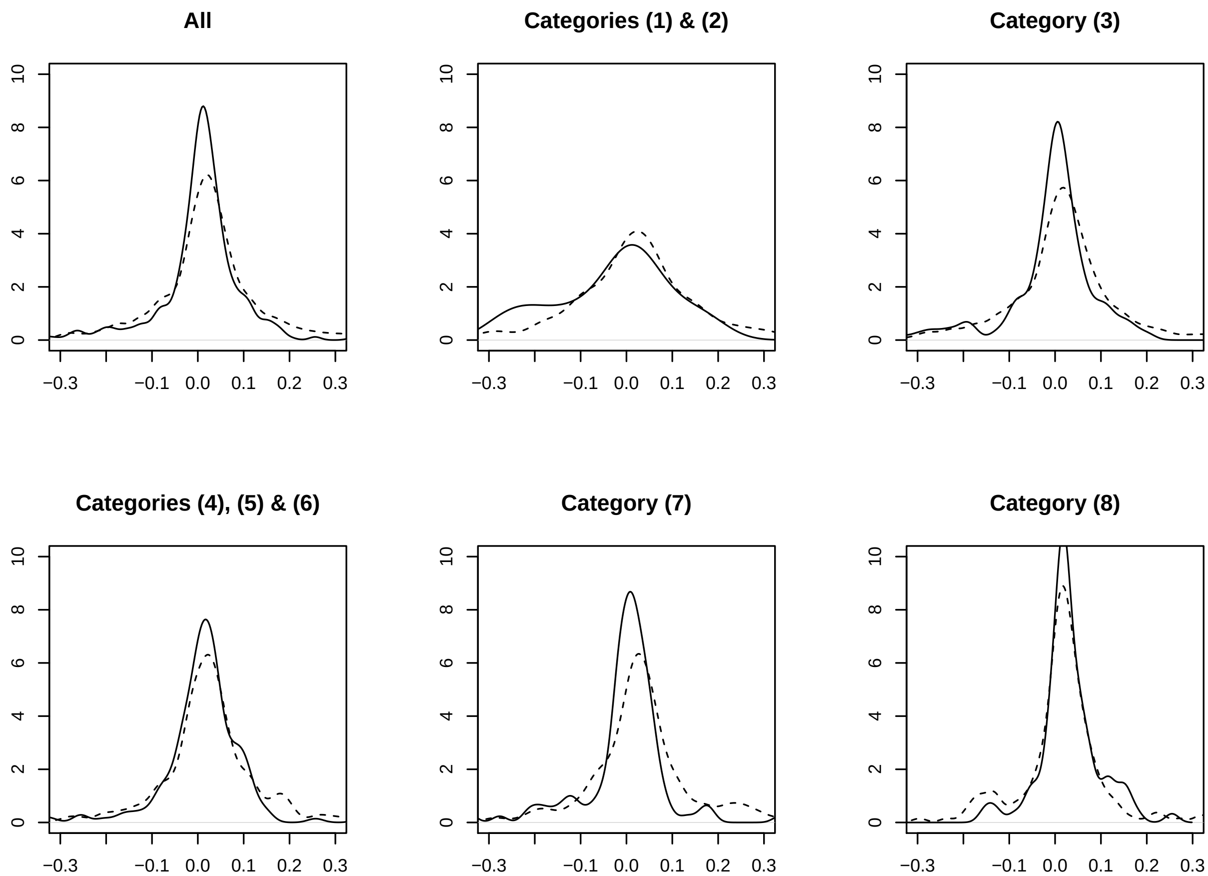

Figure 2: Kernel density estimations of individual income growth distribution by education category. All women except teenage (< 16) mothers. Childless women (solid) and mothers (dashed)

F Solving the Full Program for mothers of two children

The full maximization program can be written:

$$W(a_0) = \max_{\{c_t,\tau,\theta,a_t\}} \int_0^\tau u(c_t)e^{-\rho t}dt + \varphi(\tau, a_\tau)$$

$$\begin{aligned}
\text{where } \varphi(\tau, a_\tau) &= e^{-\rho\tau}\left[\pi(\tau)W_2(a_\tau,\theta) + (1-\pi(\tau))W_1(a_\tau)\right] \\
\text{with } W_1(a_\tau) &= p^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} \\
W_2(a_\tau,\theta) &= q^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + \omega + e^{-\rho\theta}\pi(\theta)\mathrm{E}_0\left[W_3(a_\theta) - q^{-\varepsilon}\frac{a_\theta^{1-\varepsilon}}{1-\varepsilon} - \omega\right], \\
\text{and } \; W_3(a_\theta) &= \arg\max_{c_t,a_t} \mathbb{E}\left[\int_\theta^\infty u(c_t)\, e^{-\rho(t-\theta)}dt + (1+\delta)\omega\right]
\end{aligned}$$

$$\text{subject to } da_t = \begin{cases} (r_1\ a_t - c_t)dt & \text{if } t \leq \gamma \\ (r_2\ a_t - (1+\beta)c_t)dt + \sigma\ \ a_t\ dz_t & \text{if } \gamma < t \leq \varphi \\ (r_2\ a_t - (1+2\beta)c_t)dt + \sigma\ \ a_t\ dz_t & \text{otherwise} \end{cases}$$

where the age at second birth is denoted φ. It is given by:

$$\varphi = \begin{cases} \theta & \text{with proba.}\pi(\theta) \\ +\infty & \text{with proba.}1-\pi(\theta) \end{cases}$$

and subject to

$$\theta \geq \tau + \zeta.$$

In the text (see section 5), we have first solved for $W_3(a_\theta)$, and then computed the expectation in $W_2(a_\tau,\theta)$ to derive the optimal age at second birth as a function of the optimal age at first birth. To derive the optimal age at first birth, the programme can then be rewritten:

$$W(a_0) = \max_{\{c_t,\tau,a_t\}} \int_0^\tau u(c_t)e^{-\rho t}dt + \varphi(\tau, a_\tau)$$

$$\begin{aligned}
\text{where } \varphi(\tau, a_\tau) &= e^{-\rho\tau}\left[\pi(\tau)W_2(a_\tau,\theta^*(\tau)) + (1-\pi(\tau))W_1(a_\tau)\right] \\
\text{with } W_2(a_\tau,\theta^*(\tau)) &= q^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} + \omega \\
&\quad + e^{-\rho\theta^*}\pi(\theta^*(\tau))\left(\frac{v^{-\varepsilon} - q^{-\varepsilon}}{1-\varepsilon}a_\tau^{1-\varepsilon}e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta^*(\tau)-\tau)} + \delta\omega\right) \\
\text{and } \; W_1(a_\tau) &= p^{-\varepsilon}\frac{a_\tau^{1-\varepsilon}}{1-\varepsilon} \\
\text{subject to } &: \; \dot{a}_t = r_1\ a_t - c_t \;\text{ and }\; a_0 \text{ given}
\end{aligned}$$

where $\theta^*(\tau)$ is the optimal θ as a function of τ.

Standard integration by parts yields:

$$\int_0^\tau \mu_t \dot{a}_t dt = \mu_\tau a_\tau - \mu_0 a_0 - \int_0^\tau \dot{\mu}_t a_t \, dt,$$

which allows to rewrite $W(a_0)$ as:

$$W(a_0) = \int_0^\tau (H(c_t, a_t, \mu_t) + \dot{\mu}_t a_t) \, dt + \varphi(\tau, a_\tau) - \mu_\tau a_\tau + \mu_0 a_0.$$

The first-order variation of $W(a_0)$ with respect to the state and control variable's path for a given a_0 but for τ and a_τ free yields:

$$\begin{aligned} dW(a_0) &= \int_0^\tau \left(\frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} da_t + \frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} dc_t + \dot{\mu}_t \, da_t \right) dt \\ &\quad + [H(c_\tau, a_\tau, \mu_\tau) + \dot{\mu}_\tau a_\tau] \, d\tau + \frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} da_\tau + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} d\tau \\ &\quad - \dot{\mu}_\tau a_\tau d\tau - \mu_\tau da_\tau. \end{aligned}$$

Rearranging terms leads to:

$$\begin{aligned} dW(a_0) &= \int_0^\tau \left[\left(\frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} + \dot{\mu}_t \right) da_t + \frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} dc_t \right] dt \\ &\quad + \left[H(c_\tau, a_\tau, \mu_\tau) + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} \right] d\tau \\ &\quad + \left[\frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau \right] da_\tau. \end{aligned}$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function, that is $dW(a_0) \leq 0$ for any $da_t, t \in (0, \tau)$, for any $dc_t, t \in (0, \tau)$, and for any $d\tau$ and da_τ, which gives the following necessary conditions for an interior maximizer:

$$\begin{aligned} \frac{\partial H(c_t, a_t, \mu_t)}{\partial c_t} &= 0 \\ \frac{\partial H(c_t, a_t, \mu_t)}{\partial a_t} + \dot{\mu}_t &= 0 \end{aligned}$$

$$\begin{aligned} H(c_\tau, a_\tau, \mu_\tau) + \frac{\partial \varphi(\tau, a_\tau)}{\partial \tau} &= 0 \\ \frac{\partial \varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau &= 0 \end{aligned}$$

The first two conditions are standard Pontryagin conditions. The last two conditions may be interpreted as optimality conditions with respect to the switching time τ and the free state value a_τ. The third one equalizes the marginal benefit of waiting to the marginal cost of waiting. The last one is a continuity condition: it implies that the shadow price of the state variable at the time of the switch, μ_τ, is equal to the expected marginal value of the state variable in τ (derived from the programs after the switch).

The two standard Pontryagin conditions imply:

$$\begin{aligned} /c_t &\quad : \quad u'(c_t)e^{-\rho t} = \mu_t \\ /a_t &\quad : \quad \dot{\mu}_t/\mu_t = -r_1 \Rightarrow \mu_t = \mu_0 e^{-r_1 t} \\ &\Rightarrow \quad \dot{c}_t/c_t = (r_1-\rho)/\varepsilon \Rightarrow c_t = c_0 e^{\frac{r_1-\rho}{\varepsilon}t} \text{ and } c_t = \left(\mu_t e^{\rho t}\right)^{-1/\varepsilon} \end{aligned}$$

Therefore, the dynamics of assets can be rewritten as:

$$\dot{a}_t = r_1\, a_t - c_0 e^{\frac{r_1-\rho}{\varepsilon}t}$$

Using a variable change $x_t = a_t e^{-r_1 t}$ we solve for $x_t = (c_0/p)e^{\left(\frac{r_1-\rho}{\varepsilon}-r_1\right)t} + \overline{x}$, where p is defined in Equation (6) and $\overline{x}$ is a constant to be determined. Therefore:

$$a_t = \frac{c_0}{p}\, e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} + \overline{x}e^{r_1 t}$$

Without the procreation option, the transversality condition would imply $\overline{x} = 0$ and c_0 would be determined by a_0.

Moreover at $t = 0$:

$$\begin{aligned} a_0 &= \tfrac{c_0}{p} + \overline{x} \\ &\Leftrightarrow \qquad \overline{x} = a_0 - \frac{c_0}{p} \\ &\Rightarrow \qquad a_t = \frac{c_0}{p}e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} + \left[a_0 - \frac{c_0}{p}\right]e^{r_1\, t} \\ &a_t = a_0 e^{r_1 t} + \frac{c_0}{p}\left[e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t} - e^{r_1 t}\right] \qquad (6) \end{aligned}$$

The fourth condition allows to find a_τ as a function of τ:

$$\frac{\partial\varphi(\tau, a_\tau)}{\partial a_\tau} - \mu_\tau = 0$$

$$\begin{aligned} &\Leftrightarrow \quad \mu_\tau = e^{-\rho\tau}\left[\pi(\tau)\left(q^{-\varepsilon} + f(\tau)\right) + (1-\pi(\tau))p^{-\varepsilon}\right]a_\tau^{-\varepsilon} \\ &\Leftrightarrow \quad \mu_\tau = e^{-\rho\tau}(s(\tau)a_\tau)^{-\varepsilon} \\ &\Rightarrow \quad \mu_0 = \mu_\tau e^{r_1\tau} = e^{(r_1-\rho)\tau}(s(\tau)a_\tau)^{-\varepsilon} \end{aligned}$$

with $f(\tau) = e^{-\rho T}\pi(T)(v^{-\epsilon} - q^{-\epsilon})e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(T-\tau)}$ and where we have introduced the following notation:

$$s(t) = \left(\pi(t)\left(q^{-\varepsilon} + f(\tau, T)\right) + (1-\pi(t))p^{-\varepsilon}\right)^{-1/\varepsilon}$$

which is valid in particular for $t = \tau$. This allows to identify c_0 as a function of τ and a_τ:

$$c_0 = (\mu_0)^{-1/\varepsilon} = \left(e^{(r_1-\rho)\tau}(s(\tau)a_\tau)^{-\varepsilon}\right)^{-1/\varepsilon} = \left(e^{(r_1-\rho)\tau}\right)^{-1/\varepsilon} s(\tau)a_\tau \tag{7}$$

Using (6) and (7), it is then possible to obtain an expression for a_τ as a function of τ and a_0:

$$a_\tau = \frac{e^{r_1\tau}}{1 + s(\tau)\left[e^{p\tau} - 1\right]/p} a_0 \tag{8}$$

Note that without the procreation option, this expression would become:

$$a_\tau = e^{\frac{r_1-\rho}{\varepsilon}\tau} a_0$$

Therefore, it is possible to express $X(\tau)$ the effect of the procreation option as follows:

$$a_\tau = e^{\frac{r_1-\rho}{\varepsilon}\tau} X(\tau) a_0$$

with

$$X(\tau) = \frac{e^{p\tau}}{1 + s(\tau)\left[e^{p\tau} - 1\right]/p}$$

$X(\tau) \geq 1$ if $\varepsilon \geq 1$.
It is now possible to express the dynamics of a_t:

$$a_t = a_0 \left[e^{r_1 t} - \frac{e^{p\tau}\left[e^{r_1 t} - e^{\left(\frac{r_1-\rho}{\varepsilon}\right)t}\right]}{p/s(\tau) + \left[e^{p\tau} - 1\right]} \right] \tag{9}$$

In addition, using $c_\tau = c_0 e^{\frac{r_1-\rho}{\varepsilon}\tau}$, Equations (7) and (9), it is possible to express consumption as a function of τ only:

$$c_\tau = \frac{e^{p\tau}s(\tau)}{1 + s(\tau)\left[e^{p\tau} - 1\right]/p} a_0 e^{\frac{r_1-\rho}{\varepsilon}\tau}$$

Note that absent the procreation option, this expression would become:

$$c_\tau = pe^{\frac{r_1-\rho}{\varepsilon}\tau} a_0$$

Finally, the third condition gives a second relation between a_τ and τ. Do note that $\partial\varphi(\tau, a_\tau)/\partial\tau$ needs to account for the fact that θ^*, the optimal age at second birth, is a function of τ.

The value function of the full program is:

$$\begin{aligned} W(a_0) &= \int_0^\tau \frac{\left(s(\tau)X(\tau)a_0 e^{\frac{r_1-\rho}{\varepsilon}t}\right)^{1-\varepsilon}}{1-\varepsilon} e^{-\rho t} dt + \varphi(\tau, a_\tau) \\ &= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon} \int_0^\tau e^{-pt} dt + \varphi(\tau, a_\tau) \\ &= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon} \left[\frac{-1}{p} e^{-pt}\right]_0^\tau + \varphi(\tau, a_\tau) \\ &= \frac{\left(s(\tau)X(\tau)a_0\right)^{1-\varepsilon}}{1-\varepsilon} \left[\frac{-1}{p} e^{-p\tau} + \frac{1}{p}\right] + \varphi(\tau, a_\tau) \end{aligned}$$

This is the new expression for $W(a_0)$ which depends on τ only.

G Proof of proposition 4

The FOC with respect to the age of the second child is:

$$0 = -\rho e^{-\rho\theta^*} \pi(\theta^*) \left(\frac{v^{-\varepsilon} - q^{-\varepsilon}}{1-\varepsilon} a_\tau^{1-\varepsilon} e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta^*-\tau)} + \delta\omega \right)$$

$$+ e^{-\rho\theta^*} \pi'(\theta^*) \left(\frac{v^{-\varepsilon} - q^{-\varepsilon}}{1-\varepsilon} a_\tau^{1-\varepsilon} e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta^*-\tau)} + \delta\omega \right)$$

$$+ e^{-\rho\theta^*} \pi(\theta^*) \left((1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2}) \frac{v^{-\varepsilon} - q^{-\varepsilon}}{1-\varepsilon} a_\tau^{1-\varepsilon} e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta^*-\tau)} \right)$$

that can be rewritten: $f(\theta^*) = g(\theta^*)$ with

$$f(\theta^*) = \rho - \frac{\pi'(\theta)}{\pi(\theta)},$$

and

$$g(\theta^*) = \frac{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2}) \frac{v^{-\varepsilon}-q^{-\varepsilon}}{1-\varepsilon} a_\tau^{1-\varepsilon} e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta-\tau)}}{\frac{v^{-\varepsilon}-q^{-\varepsilon}}{1-\varepsilon} a_\tau^{1-\varepsilon} e^{(1-\varepsilon)(r_2 - q(1+\beta)^{\frac{\varepsilon-1}{\varepsilon}} - \varepsilon\frac{\sigma^2}{2})(\theta-\tau)} + \delta\omega}$$

We have $f(\theta^*) > 0$ hence $f(\tau+\zeta) > 0$ and $\lim_{\theta^* \to T} f(\theta^*) = -\infty$. In addition, we assume that $(\pi'(\theta)/\pi(\theta))$ increases with θ which is consistent with the specification used for $\pi(t)$ in section 4; then, $f'(\theta^*) < 0$.

Moreover, $g'(\theta^*) < 0$ and $\lim_{\theta^* \to T} g(\theta^*) = l$ where l is finite. Therefore there exist at least one interior solution (between $\tau + \zeta$ and T) satisfying iff $f(\tau+\zeta) > g(\tau+\zeta)$. This provides a condition on $\delta\omega$ as stated in proposition (4).