Strategic Fertility, Education Choices, and Conflicts in Deeply Divided Societies
Online Appendix

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A The Indonesian Administrative System

Indonesia's administrative system is composed of four main levels: 34 provinces, about 500 regencies or cities, over 6,000 districts and 75,000 villages. It has been a rather centralized system since the Dutch era and after independence until the early 2000, with the central power in Jakarta appointing governors at the province level, based on their loyalty rather than their knowledge of the local context. The village administration level and its elected village chief (*lurah* or *kepala desa*) in contrast has probably the most direct influence on a citizen's daily life. However, residential segregation is very strong in Indonesia. As noted by Bazzi, Koehler-Derrick, and Marx (2020), the ethnic fractionalization index (the probability that any two residents belong to different ethnicities) is of around 0.81 nationally, while it drops to a median of 0.04 at the village level. For this reason, we believe that the village is not the most relevant level at which to measure religious divisions, as it is not where the conflict about resource appropriation by different groups happens.

Regencies (*kabupaten*) and cities (*kota*), led by regents (*bupati*) and mayors (*walikota*) respectively, have become chief administrative units since the early 2000, responsible for providing most government services, such as provision of public schools and public health facilities. Indeed, following the implementation of drastic regional autonomy measures in 2001, regencies have received the largest part of the decentralized competencies while the regents and the representative council members have become elected officials for 5-year terms. This makes it the ideal level of disaggregation to examine conflict over resource appropriation.

The decentralization movement has been accompanied by a massive wave of creation of new regencies, stemming from older regencies splitting up, known as the *pemekaran* or blossoming, taking the total number of regencies from about 300 to just over 500 in the interval of a few years. We acknowledge that this movement may have been influenced by existing power struggles across religious groups. It may also have inflected the dynamics of those struggles in return by creating more homogenous regencies. We however rely on regency boundaries from before the decentralization happened and consider that the division movement is too recent to have substantially influenced cultural norms of different groups.

B Additional descriptive statistics

Table 1 shows summary statistics on the size of religious groups at the regency level and decomposes its variance into a between observations and a within observation (across time) standard deviation. The overall shares are somewhat different from those at the population level because they are taken unweighted at the regency level.[1] The distribution of religion shares has not drastically changed over time. Most of the variation occurs across observations, while that within variation is limited to between 2 and 6 percentage points. In contrast, Figure 1 shows strong trends in educational attainments, with a doubling of the number of years of schooling over the period, and in child mortality, which has been divided by four.[2]

Religion	**Mean size**	**Std. Dev.**	**Min**	**Max**	**Between obs. Std. Dev.**	**Within obs. Std. Dev.**
Buddhist	0.01	0.032	0.00	0.45	0.024	0.021
Hindu	0.03	0.157	0.00	1.00	0.156	0.018
Muslim	0.82	0.293	0.00	1.00	0.291	0.058
Catholic	0.05	0.150	0.00	1.00	0.157	0.026
Protestant	0.08	0.180	0.00	0.99	0.175	0.042
Confucian	0.01	0.054	0.00	0.94	0.037	0.039
# of obs.	1336					
# of regencies	269					

Table 1: Summary statistics on size of religious group

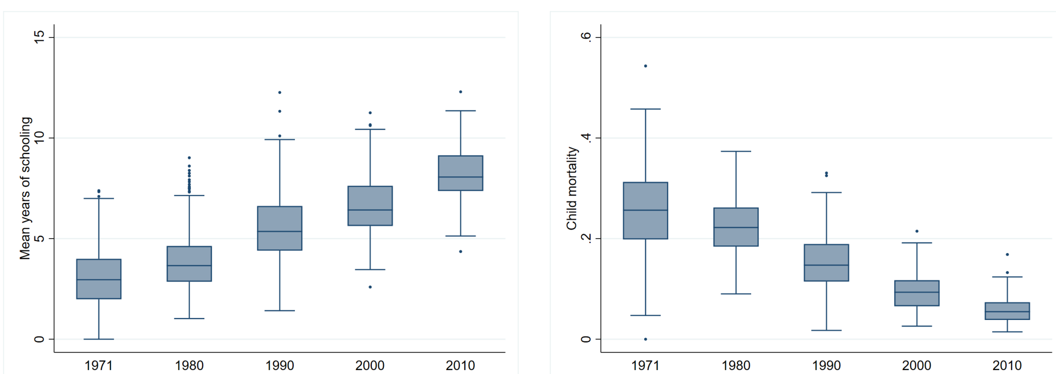


Figure 1: Distribution of regency-level controls

[1]Regencies with large shares of Protestants tend to be smaller in size, and conversely for Muslims, which explains the discrepancies between regency-level and population means.

[2]The child mortality rate is computed following Baudin, de la Croix, and Gobbi (2018) as one minus the share of surviving children among those ever born.

C Proofs of Propositions

C.1 Proof of Proposition 1

The first order conditions associated to the maximization program of any individual of group a are

$$\beta\tau N^{-\alpha}\left((1-\alpha)H^{-\alpha}(e^a)^\rho + \Pi^a\frac{\alpha H^{1-\alpha}}{xn^a}\right) - \gamma\, e^a - \lambda n^a = 0.$$

and

$$\beta\tau n^a N^{-\alpha}\rho(1-\alpha)H^{-\alpha}(e^a)^{\rho-1} - \gamma\, n^a = 0.$$

Suppose $x^a = x$. Using the second equation, we obtain

$$(e^a)^{\rho-1} = \frac{\gamma}{\beta\tau\rho(1-\alpha)}N^\alpha H^\alpha \quad \forall x \in [0,1].$$

For type b we obtain

$$(e^b)^{\rho-1} = \frac{\gamma}{\beta\tau\rho(1-\alpha)}N^\alpha H^\alpha \quad \forall x \in [0,1].$$

so that we deduce that if a Nash equilibrium exists, then, at this equilibrium we have $e^a(x) = e^b(x)$ $\forall x \in [0,1]$.

Using that equality in the first equation, we obtain

$$\beta\tau N^{-\alpha}(e^a)^{\rho(1-\alpha)}\left(n^a x + n^b(1-x)\right)^{-\alpha} - \gamma\, e^a = \lambda n^a.$$

For type b we obtain this equation becomes

$$\beta\tau N^{-\alpha}(e^b)^{\rho(1-\alpha)}\left(n^a x + n^b(1-x)\right)^{-\alpha} - \gamma\, e^b = \lambda n^b.$$

Since $e^a(x) = e^b(x)$ $\forall x \in [0,1]$, then we can deduce that $n^a(x) = n^b(x)$ $\forall x \in [0,1]$. Using that in the FOC, we easily deduce that $\forall i \in \{a,b\}$, n^i and e^i are constant with respect to x. Using these equations in the FOC, we easily deduce that a Nash equilibrium exists and is unique.

C.2 Proof of Lemma 1

Existence of the function B.

First, because the function V_t is continuous and defined on a compact set, by Weirstrass's theorem we deduce that there exists a function $\mathbf{B}_{x_t} : [0,\bar{n}] \times [0,\bar{e}] \to [0,\bar{n}] \times [0,\bar{e}]$ and a function $\mathbf{B}_{1-x_t} : [0,\bar{n}] \times [0,\bar{e}] \to [0,\bar{n}] \times [0,\bar{e}]$ given by

$$\mathbf{B}_{x_t}(n_t^b, e_t^b) = \arg\max_{n_t^a, e_t^a} V_t(n_t^a, n_t^b, e_t^{ia}, e_t^b, x_t),$$

$$\mathbf{B}_{1-x_t}(n_t^a, e_t^a) = \arg\max_{n_t^b, e_t^b} V_t(n_t^b, n_t^a, e_t^b, e_t^a, 1-x_t).$$

Hence there exists a function $\mathbf{B} : [0,\bar{n}]^2 \times [0,\bar{e}]^2 \to [0,\bar{n}]^2 \times [0,\bar{e}]^2$ given by $\mathbf{B}(\mathbf{s}^a,\mathbf{s}^b) = \{\mathbf{B}_{x_t}(\mathbf{s}^b), \mathbf{B}_{1-x_t}(\mathbf{s}^a)\}$.

Convexity of the set $\{(\mathbf{s}^a,\mathbf{s}^b),\mathbf{B}(\mathbf{s}^a,\mathbf{s}^b)\}$.

Let us skip time indexation.

Define the following functions:

$f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(e) = e^\rho$, $\rho \in [0,1]$

$g : \mathbb{R}^+ \to \mathbb{R}^+$ given by $g(e) = e^{\rho\mu}$, $\rho \in [0,1]$, $\mu \in [1,+\infty)$.

$F : \mathbb{R}^{+2} \to \mathbb{R}^+$ given by

$$F(X,Y) = X(1-\alpha)(x^iX+B)^{-\alpha} + \frac{Y}{x^iY+B'}\alpha(x^iX+B)^{1-\alpha}$$

with $B = f(e^{-i})n^{-i}$, $B' = g(e^{-i})n^{-i}$.

$\tilde{F} : \mathbb{R}^{+2} \to \mathbb{R}^+$ given by $\tilde{F}(n^i,e^i) = F(f(e^i)n^i, g(e^i)n^i)$.

$\hat{F} : \mathbb{R}^{+4} \to \mathbb{R}^+$ given by $\hat{F}(n^i,e^i,\lambda,\gamma) = \tilde{F}(n^i,e^i) - \frac{\lambda}{2}(n^i)^2 - \gamma n^i e^i$.

Let us express the first and second derivatives of F with respect to X and Y.

$$\begin{aligned}
F_X &\equiv \frac{\partial F}{\partial X} = (1-\alpha)(x^iX+B)^{-\alpha}\left(1 - \alpha\frac{x^iX}{x^iX+B} + \alpha\frac{x^iY}{x^iY+B'}\right),\\
F_Y &\equiv \frac{\partial F}{\partial Y} = \alpha(x^iX+B)^{1-\alpha}\frac{B'}{(x^iY+B')^2},\\
F_{XX} &\equiv \frac{\partial^2 F}{\partial X^2} = x^i\alpha(1-\alpha)(x^iX+B)^{-\alpha-1}\left(-2 + (1+\alpha)\frac{x^iX}{x^iX+B} - \alpha\frac{x^iY}{x^iY+B'}\right),\\
F_{YY} &\equiv \frac{\partial^2 F}{\partial Y^2} = -2\alpha(x^iX+B)^{1-\alpha}\frac{x^iB'}{(x^iY+B')^3},\\
F_{XY} &\equiv \frac{\partial^2 F}{\partial X \partial Y} = x^i(1-\alpha)\alpha(x^iX+B)^{-\alpha}\frac{B'}{(x^iY+B')^2}.
\end{aligned}$$

We have $F_Y > 0$, $F_{YY} < 0$, $F_{XY} > 0$. Using $x^iX/(x^iX+B) < 1$ and $x^iY/(x^iY+B') < 1$, we easily find $F_X > 0$ $F_{XX} < 0$.

We have $\hat{F}(n^i,e^i,\lambda,\gamma) = V(n^i,n^{-i},e^i,e^{-i},x^i)$. Then, V is quasi-concave in (n^i,e^i), if and only if $\hat{F}$ is quasi-concave in (n^i,e^i). To determine whether $\hat{F}$ is quasi-concave, let us express the bordered Hessian matrix of $\hat{F}$ which we denote by $H(n^i,e^i)$.

$$H(n^i,e^i) = \begin{pmatrix} 0 & \hat{F}_n & \hat{F}_e \\ \hat{F}_n & \hat{F}_{nn} & \hat{F}_{ne} \\ \hat{F}_e & \hat{F}_{ne} & \hat{F}_{ee} \end{pmatrix}$$

where

$$\begin{aligned}
\hat{F}_n &= \tilde{F}_n - \lambda n^i - \gamma e^i, \\
\hat{F}_e &= \tilde{F}_e - \gamma n^i, \\
\hat{F}_{nn} &= \tilde{F}_{nn} - \lambda, \\
\hat{F}_{ne} &= \tilde{F}_{ne} - \gamma, \\
\hat{F}_{ee} &= \tilde{F}_{ee},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{F}_n &= F_X f(e^i) + F_Y g(e^i), \\
\tilde{F}_e &= F_X f^{'}(e^i) n^i + F_Y g^{'}(e^i) n^i, \\
\tilde{F}_{nn} &= F_{XX}(f(e^i))^2 + F_{YY}(g(e^i))^2 + 2F_{XY} f(e^i) g(e^i), \\
\tilde{F}_{ee} &= F_X f^{''}(e^i) n^i + F_Y g^{''}(e^i) n^i + F_{XX}(f^{'}(e^i) n^i)^2 + F_{YY}(g^{'}(e^i) n^i)^2 + 2F_{XY} f^{'}(e^i) g^{'}(e^i)(n^i)^2, \\
\tilde{F}_{en} &= F_X f^{'}(e^i) + F_Y g^{'}(e^i) + F_{XX} f^{'}(e^i) f(e^i) n^i + F_{YY} g^{'}(e^i) g(e^i) n^i + \\
&\qquad F_{XY} f^{'}(e^i) g(e^i) n^i + F_{XY} g^{'}(e^i) f(e^i) n^i.
\end{aligned}$$

The function $\hat{F}$ is quasi-concave if the determinant of H is positive. The determinant of H is given by

$$\begin{aligned}
Det(H) =& -\left(\hat{F}_n\right)^2 \hat{F}_{ee} + 2\hat{F}_n \hat{F}_e \hat{F}_{en} - \left(\hat{F}_e\right)^2 \hat{F}_{nn} \\
=& -\left(\tilde{F}_n - \lambda n^i - \gamma e^i\right)^2 \tilde{F}_{ee} + 2\left(\tilde{F}_n - \lambda n^i - \gamma e^i\right)\left(\tilde{F}_e - \gamma n^i\right)\left(\tilde{F}_{en} - \gamma\right) \\
& - \left(\tilde{F}_e - \gamma n^i\right)^2 \left(\tilde{F}_{nn} - \gamma\right)
\end{aligned}$$

Define $D : \mathbb{R}^{+2} \to \mathbb{R}^+$ given by $D(\lambda, \gamma) = Det(H)$.

We will first interest in $D(0,0)$. Note that we have $\tilde{F}_n > 0$ and $\tilde{F}_e > 0$ so that sufficient conditions to have $D(0,0) > 0$ are

$$\tilde{F}_{nn} < 0, \text{ and } \tilde{F}_{ee} < 0, \text{ and } \tilde{F}_{en} > 0.$$

Now, we will show there exist $\hat{\mu} \in (1, +\infty]$, $\bar{\mu} \in (1, +\infty]$, such that $\forall \mu \leq \min\{\hat{\mu}, \bar{\mu}\}$ and $\alpha < 1/2$, then those conditions are satisfied.

1. We start with $\tilde{F}_{nn}$ for which we give two expressions.

$$\begin{aligned}
\tilde{F}_{nn} =& \frac{f(e^i)}{n^i} x^i \alpha(1-\alpha)(x^i X + B)^{-\alpha} \left(2\frac{B^{'}Y}{(x^i Y + B^{'})^2}\right. \\
& \left. + \frac{x^i X}{x^i X + B}\left(-2 + (1+\alpha)\frac{x^i X}{x^i X + B} - \alpha \frac{x^i Y}{x^i Y + B^{'}}\right)\right) + F_{YY}(g(e^i))^2,
\end{aligned}$$

or

$$\begin{aligned}\tilde{F}_{nn} =& 2g(e^i)x^i\alpha\frac{(x^iX+B)^{-\alpha}B'}{(x^iY+B')^2}\left((1-\alpha)-\frac{(x^iX+B)/x^iX}{(x^iY+B')/x^iY}\right).\\ &+F_{XX}(f(e^i))^2,\end{aligned}$$

where $X \equiv f(e^i)n^i$ and $Y \equiv g(e^i)n^i$.

Since $F_{XX} < 0$ and $F_{YY} < 0$. A sufficient condition to have $\tilde{F}_{nn} < 0$ is

$$2\frac{x^iB'Y}{(x^iY+B')^2}+\frac{x^iX}{x^iX+B}\left(-2+(1+\alpha)\frac{x^iX}{x^iX+B}-\alpha\frac{x^iY}{x^iY+B'}\right)<0,$$

or

$$(1-\alpha)-\frac{(x^iX+B)/x^iX}{(x^iY+B')/x^iY}<0.$$

Let us introduce new notations. We set $z \equiv (n^ix^i)/(n^{-i}x^{-i})$ and $c \equiv (e^i/e^{-i})^\rho$. The above conditions rewrites as

$$2\frac{1}{(1+c^{-\mu}z^{-1})(1+c^\mu z)}+\frac{1}{(1+c^{-1}z^{-1})}\left(-2+(1+\alpha)\frac{1}{(1+c^{-1}z^{-1})}-\alpha\frac{1}{(1+c^{-\mu}z^{-1})}\right)<0,$$

or

$$(1-\alpha)-\frac{(1+c^{-1}z^{-1})}{(1+c^{-\mu}z^{-1})}<0.$$

We will consider several cases : (i) $c > 1$, (ii)-a $c < 1$ and $cz < 1$, (ii)-b $c < 1$ and $cz > 1$.

(i) The second inequality holds whenever $(1+c^{-1}z^{-1})/(1+c^{-\mu}z^{-1}) > 1$ which is equivalent to $c > 1$. Hence when $c < 1$ we have $\tilde{F}_{nn} < 0$.

Consider the case $c < 1$ and look at the first inequality.

Define the functions $Z : [1,+\infty) \to \mathbb{R}^+$, $L : \mathbb{R}^+ \to \mathbb{R}^+$ and $G : \mathbb{R}^+ \to \mathbb{R}^+$, $\tilde{G} : [1,+\infty) \to \mathbb{R}^+$ respectively given by

$$\begin{aligned}Z(\mu) &= c^\mu z\\ L(\tilde{z}) &= \frac{1}{(1+\tilde{z}^{-1})}\left(-2+(1+\alpha)\frac{1}{(1+\tilde{z}^{-1})}\right),\end{aligned}$$

$$\begin{aligned}G(Z) &= 2\frac{1}{(1+Z)(1+Z^{-1})}+L(\tilde{z}),\\ \tilde{G}(\mu) &= G(Z(\mu)).\end{aligned}$$

(ii)- a A sufficient condition for the first inequality to hold is $\tilde{G}(\mu) < 0$. Suppose that $c < 1$. We show that provided that $cz < 1$, $\forall \mu \in [1,+\infty)$, $\tilde{G}(\mu) < 0$.

To do so first we can compute $G(1)$. We find

$$\begin{aligned}\tilde{G}(1) &= \frac{1}{(1+c^{-1}z^{-1})}\left(\frac{2}{(1+cz)} - 2 + \frac{1}{(1+c^{-1}z^{-1})}\right) \\ &= -\frac{1}{(1+c^{-1}z^{-1})(1+c^{-1}z^{-1})(1+cz)} < 0.\end{aligned}$$

Second, let us compute the derivative of $\tilde{G}$ with respect to μ. We obtain

$$\tilde{G}^{'}(\mu) = G^{'}(Z(\mu))Z^{'}(\mu).$$

We easily show that $G^{'}(Z(\mu)) > 0$ if and only if $Z(\mu) < 1$ and $Z^{'}(\mu) > 0$ if and only if $c > 1$. By assumption $c < 1$ so that $Z^{'}(\mu) < 0$. Suppose, in addition that $cz < 1$. It implies $c^{\mu}z < 1$ equivalent to $G^{'}(Z(\mu)) > 0$ so that we deduce $\tilde{G}^{'}(\mu) < 0$ which, with the continuity of G, allows us to deduce that $\tilde{G}(\mu) < 0$ $\forall \mu \in [1, +\infty)$.

(ii)- b Finally consider the case $c < 1$ and $cz > 1$. Note that $G(Z) < 1/2 + L(\tilde{z})$ $\forall Z \in \mathbb{R}^{+}$. A sufficient condition to have $G(Z) < 0$ (which in turn implies that the first inequality holds) is $L(\tilde{z}) < -1/2$. The function L is convex, decreasing for all $\tilde{z} \in [0, 1/\alpha]$ and increasing for all $\tilde{z} > 1/\alpha$. We have $L(1) = -1/4(3+\alpha) < -1/2$ and $\lim_{\tilde{z}\to\infty} L = -(1-\alpha)$ which is lower than one half whenever $\alpha \leq 1/2$. Hence we deduce that $L(\tilde{z}) < -1/2$ which implies that the second inequality holds.

Finally we can deduce that $\tilde{F}_{nn} < 0$.

2. Consider the case $\tilde{F}_{en}$.

$$\begin{aligned}\tilde{F}_{en} =& f^{'}(e^{i})(1-\alpha)(x^{i}X+B)^{-\alpha}\left[1 - \alpha\frac{x^{i}X}{x^{i}X+B} + \alpha\frac{x^{i}Y}{x^{i}Y+B^{'}}\right. . \\ &\left. +\alpha\frac{x^{i}X}{x^{i}X+B}\left(-2 + (1+\alpha)\frac{x^{i}X}{x^{i}X+B} - \alpha\frac{x^{i}Y}{x^{i}Y+B^{'}}\right) + \alpha\frac{x^{i}YB^{'}}{(x^{i}Y+B^{'})^{2}}\right] \\ &+ \alpha g^{'}(e^{i})\frac{(x^{i}X+B)^{1-\alpha}B^{'}}{(x^{i}Y+B^{'})^{2}}\left(1 - 2\frac{x^{i}Y}{x^{i}Y+B^{'}} + (1-\alpha)\frac{x^{i}X}{x^{i}X+B}\right).\end{aligned}$$

Use the notations $z \equiv (n^{i}x^{i})/(n^{-i}x^{-i})$ and $c \equiv (e^{i}/e^{-i})^{\rho}$ and define the function $M : [1, +\infty) \to \mathbb{R}$ given by

$$\begin{aligned}M(\mu) =& \rho(1-\alpha)\frac{1}{1+c^{-1}z^{-1}}\left[1 - \alpha\frac{1}{1+c^{-1}z^{-1}} + \alpha\frac{1}{1+c^{-\mu}z^{-1}}\right. . \\ &\left. +\alpha\frac{1}{1+c^{-1}z^{-1}}\left(-2 + (1+\alpha)\frac{1}{1+c^{-1}z^{-1}} - \alpha\frac{1}{1+c^{-\mu}z^{-1}}\right) + \alpha\frac{1}{1+c^{-\mu}z^{-1}}\frac{1}{1+c^{\mu}z}\right] \\ &+ \alpha\rho\mu\frac{1}{1+c^{-\mu}z^{-1}}\frac{1}{1+c^{\mu}z}\left(1 - 2\frac{1}{1+c^{-\mu}z^{-1}} + (1-\alpha)\frac{1}{1+c^{-1}z^{-1}}\right).\end{aligned}$$

The inequality $\tilde{F}_{en} > 0$ is equivalent to $M(\mu) > 0$.

Let us compute $M(1)$. We obtain

$$M(1) = \rho(1-\alpha)\frac{1}{1+c^{-1}z^{-1}}\left[1-2\alpha\frac{1}{1+c^{-1}z^{-1}}+\alpha\left(\frac{1}{1+c^{-1}z^{-1}}\right)^2\right]$$
$$+\alpha\rho\frac{1}{1+c^{-1}z^{-1}}\frac{1}{1+cz}\left(2-\alpha-(1+\alpha)\frac{1}{1+c^{-1}z^{-1}}\right).$$

Since $\frac{1}{1+c^{-1}z^{-1}} < 1$ we deduce that

$$M(1) > \rho(1-\alpha)\frac{1}{1+c^{-1}z^{-1}}\left[1-2\alpha+\alpha\left(\frac{1}{1+c^{-1}z^{-1}}\right)^2\right]$$
$$+\alpha\rho\frac{1}{1+c^{-1}z^{-1}}\frac{1}{1+cz}\left(1-2\alpha\right).$$

Suppose that $\alpha \leq 1/2$. Then we the RHS is positive so that $M(1) > 0$. Given that the function M is continuous on $[1,+\infty)$, we deduce that there exists $\hat{\mu} \in]1,+\infty]$ such that $\forall \mu \leq \hat{\mu}$ $M(\mu) \geq 0$ which is equivalent to $\tilde{F}_{en} \geq 0$.

3. Consider the case $\tilde{F}_{ee}$.

$$\tilde{F}_{ee} = F_X f^{''}(e^i)n^i + F_{XX}(f^{'}(e^i)n^i)^2 +$$
$$\rho\mu\alpha\frac{B^{'}Y}{(xY+B^{'})^2}\left(2\rho(1-\alpha)\frac{xX}{(xX+B)}-2\rho\mu\frac{xY}{(xY+B^{'})}+\rho\mu-1\right)$$

The terms of the first line are all negative. Hence, $\tilde{F}_{ee}$ is negative if the term of the second line is negative. Again, we set $z \equiv (n^i x^i)/(n^{-i}x^{-i})$ and $c \equiv (e^i/e^{-i})^\rho$ and we define the function $N : [1,+\infty) \to \mathbb{R}$ given by

$$N(\mu) = \rho\mu\alpha\frac{1}{x^i}\frac{1}{(1+c^{-\mu}z^{-1})(1+c^{\mu}z)}\left(2\rho(1-\alpha)\frac{1}{(1+c^{-1}z^{-1})}-2\rho\mu\frac{1}{(1+c^{-\mu}z^{-1})}+\rho\mu-1\right)$$

We have $N(1) < 0$. Given that the function N is continuous on $[1,+\infty)$, we deduce that there exists $\bar{\mu} \in]1,+\infty]$ such that $\forall \mu \leq \bar{\mu}$ $N(\mu) < 0$ which implies $\tilde{F}_{ee} < 0$.

Last step.

It is easy to check that D is continuous in $(0,0)$ that is, if $(\lambda^n,\gamma^n) \to (0,0)$ then $D(\lambda^n,\gamma^n) \to D(0,0)$. Using that fact and $D(0,0) > 0$ we can deduce that there exist two thresholds $\bar{\lambda}$, $\bar{\gamma}$ such that when $\lambda < \bar{\lambda}$ and $\gamma < \bar{\gamma}$ then $D(\lambda,\gamma) > 0$. It is equivalent to say that for $\lambda < \bar{\lambda}$ and $\gamma < \bar{\gamma}$, $\forall x^i \in [0,1]$, the function V_t is quasi-concave in $\mathbf{s}^i$.

C.3 Proof of item (ii) of Proposition 2

Define $\mathbf{f_s^a}$ and $\mathbf{f_s^b}$ the functions going from $[0,1]$ into $[0,\bar{n}]\times[0,\bar{e}]$ which for each value of x gives the vector of strategy of one group at the Nash equilibrium. That is ,

$$\mathbf{f_s^a}(x) \equiv (n^a(x_t), e^a(x_t)),$$
$$\mathbf{f_s^b}(x) \equiv (n^b(x_t), e^b(x_t)).$$

By definition,

$$\mathbf{f_s^a}(x) = \mathbf{B}_x(\mathbf{f_s^b}(x)),$$
$$\mathbf{f_s^b}(x) = \mathbf{B}_{1-x}(\mathbf{f_s^a}(x)).$$

Therefore, the functions $\mathbf{f_s^a}$ and $\mathbf{f_s^b}$ are implicitly given by

$$\mathbf{f_s^a}(x) - \mathbf{B}_x(\mathbf{B}_{1-x}(\mathbf{f_s^a}(x))) = 0,$$
$$\mathbf{f_s^b}(x) - \mathbf{B}_{1-x}(\mathbf{B}_x(\mathbf{f_s^b}(x))) = 0.$$

Hence, we have

$$\mathbf{f_s^a}(1-x) - \mathbf{B}_{1-x}(\mathbf{B}_x(\mathbf{f_s^a}(1-x))) = 0,$$

so that we deduce $\mathbf{f_s^a}(1-x) = \mathbf{f_s^b}(x)$.

C.4 Proof of Proposition 3

Suppose that $\mu = 1$. The first order conditions can be rewritten as

$$\frac{x_t(e_t^a)^\rho}{n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho}\Pi_{t+1}^b N_t^{-\alpha}\frac{(n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{(1-\alpha)}}{x_t}$$
$$+\Pi_{t+1}^a N_t^{-\alpha}(1-\alpha)\frac{(n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{-\alpha}}{x_t}(e_t^a)^\rho x_t = \gamma e_t^a + \lambda n_t^a$$

$$\frac{\rho\,(e_t^a)^{\rho-1}\,n_t^a x_t}{n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho}\Pi_{t+1}^b N_t^{-\alpha}\frac{(n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{(1-\alpha)}}{x_t}$$
$$+\rho\Pi_{t+1}^a(1-\alpha)N_t^{1-\alpha}\frac{(n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{-\alpha}}{x_t}n_t^a e_t^{a\rho-1}x_t = \gamma n_t^a$$

Equalizing, the second FOC for group 1 and 2, we obtain

$$\left(\frac{e_t^a}{e_t^b}\right)^{-(1-\rho)} = \frac{1-\alpha\Pi_{t+1}^b}{1-\alpha\Pi_{t+1}^a}$$

Also using the two FOC together (for group 1) we deduce that

$$\frac{x_t(e_t^a)^\rho}{n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho}\Pi_{t+1}^b N_t^{-\alpha}\frac{n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{(1-\alpha)}}{x_t}$$
$$+\Pi_{t+1}^a N_t^{-\alpha}(1-\alpha)\frac{(n_t^a x_t(e_t^a)^\rho + n_t^b(1-x_t)(e_t^b)^\rho)^{-\alpha}}{x_t}(e_t^a)^\rho x_t = \frac{\lambda}{1-p}n_t^a$$

Equalizing, this equation for for group 1 and 2, we obtain

$$\left(\frac{e_t^a}{e_t^b}\right)^\rho = \frac{1-\alpha\Pi_{t+1}^b}{1-\alpha\Pi_{t+1}^a}\frac{n_t^a}{n_t^b}.$$

We deduce

$$\frac{e_t^a}{e_t^b} = \frac{n_t^a}{n_t^b}.$$

Hence, $\frac{n_t^{1*}}{n_t^{2*}}$ is implicitely given by

$$\left(\frac{n_t^{1*}}{n_t^{2*}}\right)^{-(1-\rho)} - \frac{1-\alpha\Pi_{t+1}^{2*}}{1-\alpha\Pi_{t+1}^{1*}} \equiv k(\frac{n_t^{1*}}{n_t^{2*}}) = 0,$$

where

$$\Pi_{t+1}^{1*} = \frac{1}{1+\frac{n_t^a}{n_t^b}^{-1-\rho}(1-x_t)},$$
$$\Pi_{t+1}^{2*} = \frac{1}{1+\frac{n_t^a}{n_t^b}^{1+\rho}x_t}$$

One has

$$\frac{d\frac{n_t^{1*}}{n_t^{2*}}}{dx_t} = -\frac{\frac{\partial k}{\partial x_t}}{\frac{\partial k}{\partial\frac{n_t^{1*}}{n_t^{2*}}}}.$$

One easily finds that

$$\frac{\partial k}{\partial x_t} < 0,$$
$$\frac{\partial k}{\partial\frac{n_t^{1*}}{n_t^{2*}}} < 0,$$

so that we deduce $\frac{d\frac{n_t^{1*}}{n_t^{2*}}}{dx_t} = \frac{d\frac{e_t^{1*}}{e_t^{2*}}}{dx_t} < 0$.

Finally, using Proposition 2, we deduce that $de_t^a/dx_t < 0$, $de_t^b/dx_t > 0$, $dn_t^a/dx_<0$, $dn_t^b/dx_t > 0$.

C.5 Proof of Proposition 4

Let us drop time indexation and define

$$H \equiv \left(n^a(e^a)^\rho x + n^b(e^b)^\rho(1-x)\right).$$

Remind that $\Pi^a = \Pi^a(N^a, N^b, h^a, h^b)$ with

$$\Pi^a(N^a, N^b, h^a, h^b) = \begin{cases} \dfrac{(h^a)^\mu n^a x}{(h^a)^\mu n^a x + (h^b)^\mu n^b(1-x)}, & \text{if } h^i \neq 0 \text{ and } n^i \neq 0 \quad \forall i \in \{a,b\}, \\ \dfrac{n^a x}{n^a x + n^b(1-x)}, & \text{if } h^i = 0 \text{ and } n^i \neq 0 \quad \forall i \in \{a,b\}, \\ \dfrac{(h^a)^\mu}{(h^a)^\mu + (h^b)^\mu}, & \text{if } h^i \neq 0 \text{ and } n_t^i = 0 \quad \forall i \in \{a,b\}, \\ \frac{1}{2}, & \text{if } h^i = 0 \text{ and } n_t^i = 0 \quad \forall i \in \{a,b\}, \end{cases}$$

Each household in group a solves the following program

$$\max_{n^a, e^a} \beta\tau n^a N^{-\alpha}\left((1-\alpha)H^{-\alpha}(e^a)^\rho + \Pi^a \frac{\alpha H^{1-\alpha}}{x n^a}\right) - \gamma\, n^a e^a - \frac{\lambda}{2}\left(n^a\right)^2.$$

The first order conditions associated to this program are

$$\begin{aligned} &\beta\tau N^{-\alpha}\left((1-\alpha)H^{-\alpha}(e^a)^\rho + \Pi^a \frac{\alpha H^{1-\alpha}}{x n^a}\right) \\ &-\beta\tau n^a N^{-\alpha}\alpha(1-\alpha)H^{-\alpha-1}(e^a)^{2\rho}x + \beta\tau n^a N^{-\alpha}\alpha(1-\alpha)\Pi^a \frac{H^{-\alpha}}{x n^a}(e^a)^\rho x \\ &-\beta\tau n^a N^{-\alpha}\alpha H^{1-\alpha}\frac{(h^a)^{2\mu} x}{\left((h^a)^\mu n^a x + (h^b)^\mu n^b(1-x)\right)^2} - \gamma\, e^a - \lambda n^a = 0. \end{aligned}$$

and

$$\begin{aligned} &\beta\tau n^a N^{-\alpha}\rho(1-\alpha)H^{-\alpha}(e^a)^{\rho-1} \\ &-\beta\tau n^a N^{-\alpha}\alpha(1-\alpha)H^{-\alpha-1}(e^a)^\rho x n^a \rho(e^a)^{\rho-1} + \beta\tau n^a N^{-\alpha}\alpha(1-\alpha)\Pi^a \frac{H^{-\alpha}}{x n^a} n^a x\rho(e^a)^{\rho-1} \\ &+\beta\tau n^a N^{-\alpha}\alpha H^{1-\alpha}\frac{\rho\mu(e^a)^{\rho\mu-1}\left(h^b\right)^\mu n^b(1-x)}{\left((h^a)^\mu n^a x + (h^b)^\mu n^b(1-x)\right)^2} - \gamma\, n^a = 0. \end{aligned}$$

Set $x = 1$ and denote (n^{a1}, e^{a1}) the vector which solves the above system of equation. We find

$$\beta\tau N^{-\alpha}\left(\left(n^{a1}\right)^{-\alpha}\left(e^{a1}\right)^{(1-\alpha)\rho}\right)(1-\alpha) - \gamma\, e^{a1} - \lambda n^{a1} = 0,$$

and

$$\rho\beta\tau N^{-\alpha}\left(\left(n^{a1}\right)^{-\alpha+1}\left(e^{a1}\right)^{(1-\alpha)\rho-1}\right)(1-\alpha) - \gamma\, n^{a1} = 0.$$

We obtain

$$(e^{a1})^{1-\rho(1-\alpha)} = \frac{\rho\beta\tau N^{-\alpha}(1-\alpha)}{\gamma}\left(n^{a1}\right)^{-\alpha},$$

$$(n^{a1})^{1+\alpha} = \frac{\beta\tau N^{-\alpha}}{\lambda}(e^{a1})^{\rho(1-\alpha)}(1-\alpha)(1-\rho).$$

Set $x = 0$ and denote (n^{a0}, e^{a0}) the vector which solves the above system of equation. In that case, we obtain

$$n^{a0} = \frac{\beta\tau N^{-\alpha}}{\lambda}(e^{a0})^{\rho}(n^{a1})^{-\alpha}(e^{a1})^{-\alpha\rho}\left((1-\alpha)(1-\rho) - \alpha\frac{(e^{a0})^{\rho(\mu-1)}}{(e^{a1})^{\rho(\mu-1)}}(\rho\mu - 1)\right)$$

and

$$(e^{a0})^{1-\rho} = \frac{\rho\beta\tau N^{-\alpha}}{\gamma}(n^{a1})^{-\alpha}(e^{a1})^{-\alpha\rho}\left((1-\alpha) + \alpha\mu\frac{(e^{a0})^{\rho(\mu-1)}}{(e^{a1})^{\rho(\mu-1)}}\right).$$

Set $x = 1/2$ and denote $(n^{a1/2}, e^{a1/2})$ the solution to the above system of equation. We find

$$(e^{a1/2})^{1-\rho(1-\alpha)} = \frac{\rho\beta\tau N^{-\alpha}}{\gamma}\left(n^{a1/2}\right)^{-\alpha}\left(1-\alpha+\frac{\alpha\mu}{2}\right),$$

$$(n^{a1/2})^{1+\alpha} = \frac{\beta\tau N^{-\alpha}}{\lambda}(e^{a1/2})^{\rho(1-\alpha)}\left((1-\frac{\alpha}{2}) - \rho(1-\alpha) - \frac{\rho\alpha\mu}{2}\right)$$

Now we will show that provided that $\mu \in (\mu^*, \tilde{\mu})$, we have

$$e^{a0} > e^{a1/2} > e^{a1} \quad \text{and} \quad n^{a1/2} > n^{a1} > n^{a0}.$$

Step 1. We start by comparing choices at $x = 1/2$ and $x = 1$.

Using the FOC at $x = 1$ and $x = 1/2$, we can perform the following quantities

$$e^{a1} = (\beta\tau)^{\frac{1}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{\rho(1-\alpha)}{\gamma}\right)^{\frac{(1+\alpha)}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{(1-\alpha)(1-\rho)}{\lambda}\right)^{\frac{-\alpha}{(1+\alpha-\rho(1-\alpha))}},$$

$$n^{a1} = (\beta\tau)^{\frac{1}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{\rho(1-\alpha)}{\gamma}\right)^{\frac{\rho(1-\alpha)}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{(1-\alpha)(1-\rho)}{\lambda}\right)^{\frac{(1-\rho(1-\alpha))}{(1+\alpha-\rho(1-\alpha))}},$$

$$e^{a1/2} = (\beta\tau)^{\frac{1}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{\rho(1-\alpha+\alpha\mu/2)}{\gamma}\right)^{\frac{(1+\alpha)}{(1+\alpha-\rho(1-\alpha))}} \times\left(\frac{(1-\alpha/2-\rho(1-\alpha+\alpha\mu/2)}{\lambda}\right)^{\frac{-\alpha}{(1+\alpha-\rho(1-\alpha))}},$$

and

$$n^{a1/2} = (\beta\tau)^{\frac{1}{(1+\alpha-\rho(1-\alpha))}}\left(\frac{\rho(1-\alpha+\alpha\mu/2)}{\gamma}\right)^{\frac{\rho(1-\alpha)}{(1+\alpha-\rho(1-\alpha))}} \times\left(\frac{(1-\alpha/2-\rho(1-\alpha+\alpha\mu/2)}{\lambda}\right)^{\frac{(1-\rho(1-\alpha))}{(1+\alpha-\rho(1-\alpha))}}.$$

Note that at $\mu = 1$ we have $n^{a1/2} > n^{a1}$ and $e^{a1/2} > e^{a1}$. One easily shows that $\partial e^{a1/2}/\partial\mu > 0$ $\forall \mu \in \mathbb{R}^+$. Let us perform the derivative of $n^{a1/2}$ with respect to μ. We find

$$
\begin{aligned}
\frac{\partial n^{a1/2}}{\partial \mu} = \frac{\rho\alpha}{2\lambda(1+\alpha-\rho(1-\alpha))} \left(\frac{\rho(1-\alpha+\alpha\mu/2)}{\gamma}\right)^{\frac{\rho(1-\alpha)}{(1+\alpha-\rho(1-\alpha))}} \\
\times \left(\frac{(1-\alpha/2-\rho(1-\alpha+\alpha\mu/2)}{\lambda}\right)^{\frac{1-\rho(1-\alpha)}{(1+\alpha-\rho(1-\alpha))}-1} \\
\times \left[(1-\alpha)\left(\frac{(1-\alpha/2-\rho(1-\alpha)+\alpha\mu/2)}{(1-\alpha+\alpha\mu/2)}\right) - (1-\rho(1-\alpha))\right]
\end{aligned}
$$

which has the same sign as

$$
(1-\alpha)\left(\frac{(1-\alpha/2-\rho(1-\alpha+\alpha\mu/2)}{(1-\alpha+\alpha\mu/2)}\right) - (1-\rho(1-\alpha)).
$$

Note that

$$
\begin{aligned}
&(1-\alpha)\left(\frac{(1-\alpha/2-\rho(1-\alpha)+\alpha\mu/2)}{(1-\alpha+\alpha\mu/2)}\right) - (1-\rho(1-\alpha)) \\
&< 1-\alpha/2-\rho(1-\alpha)-\alpha\mu/2-1+\rho(1-\alpha) = -\alpha/2-\alpha\mu/2 < 0.
\end{aligned}
$$

Hence, $\partial n^{a1/2}/\partial\mu < 0$ $\forall \mu \in \mathbb{R}^+$.

When μ is such that $1-\alpha/2-\rho(1-\alpha)-\alpha\mu/2 = 0$, then $n^{a1/2} = 0 < n^{a1}$. We can deduce that there exists a unique $\tilde{\mu}$ implicitly given by

$$
\left(\frac{1-\alpha+\alpha\tilde{\mu}/2}{1-\alpha}\right)^{\rho(1-\alpha)} \left(\frac{(1-\alpha/2-\rho(1-\alpha+\alpha\tilde{\mu}/2)}{(1-\alpha)(1-\rho)}\right)^{(1-\rho(1-\alpha))} - 1 = 0,
$$

such that $\mu < \tilde{\mu} \Leftrightarrow n^{a1/2} > n^{a1}$.

Hence for any $\mu \in [1, \tilde{\mu})$ we have $n^{a1/2} > n^{a1}$ and $e^{a1/2} > e^{a1}$.

Step 2. Now we show that $n^{a1} > n^{a0}$.

First, we easily find that $e^{a0} > e^{a1}$. We have

$$
\begin{aligned}
&n^{a0} < n^{a1} \\
\Leftrightarrow &\frac{(e^{a0})^\rho}{(e^{a1})^\rho}\left((1-\alpha)(1-\rho) + \alpha\frac{(e^{a0})^{\rho(\mu-1)}}{(e^{a1})^{\rho(\mu-1)}}(1-\rho\mu)\right) - (1-\alpha)(1-\rho) < 0.
\end{aligned}
$$

Define the function $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ given by

$$
\Gamma(x) = x^\rho\left((1-\alpha)(1-\rho) - \alpha x^{\rho(\mu-1)}(\rho\mu-1)\right) - (1-\alpha)(1-\rho).
$$

One can compute

$$
\Gamma'(x) = \rho x^{\rho-1}\left((1-\alpha)(1-\rho) - \alpha x^{\rho(\mu-1)}(\mu\rho-1)\right) - \alpha x^\rho \rho(\mu-1)x^{\rho(\mu-1)-1}(\rho\mu-1).
$$

Suppose that $\rho\mu > 1$. Then, the function Γ reaches a maximum at $x_m = \left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right)^{\frac{1}{\rho(\mu-1)}}$. A sufficient condition for $\Gamma(e^{a0}/e^{a1}) < 0$ is $\Gamma(x_m) < 0$ which is equivalent to $\Lambda(\mu) < 0$ where $\Lambda : [1, +\infty) \to \mathbb{R}$ is given by

$$\Lambda(\mu) = (1-\alpha)(1-\rho)\left(\left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right)^{\frac{1}{(\mu-1)}}(1-\frac{1}{\mu}) - 1.\right)$$

First, suppose that $(1-\alpha)(1-\rho)/\left(\alpha(\rho\mu-1)\mu\right) < 1$ which is equivalent to $\mu > \mu_h$ where μ_h is such that

$$\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu_h-1)\mu_h} = 1.$$

In that case, one finds $\Gamma(x_m) < 0$. Second, suppose that $1/\rho < \mu < \mu_h$. Let us perform $\Lambda'(\mu)$. We find

$$\Lambda'(\mu) = (1-\alpha)(1-\rho)\left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right)^{\frac{1}{(\mu-1)}}\frac{1}{\mu} \\ \times\left(-\frac{1}{(\mu-1)}\ln\left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right) - \frac{2(\rho\mu-1)}{\mu}\right) < 0.$$

Since $\Lambda(1/\rho) = +\infty$, $\Lambda(\mu_h) < 0$ and Λ is continuous, we deduce that there exists a unique $\mu^* \in [1/\rho, \mu_h]$ such that $\mu \geq \mu^*$ is equivalent to $\Lambda(\mu) \leq 0$ which implies $n^a < n^b$.

Step 3. Finally, we show that $e^{a0} > e^{a1/2}$.

$$\begin{aligned} & e^{a0} > e^{a1/2}, \\ \Leftrightarrow & (n^{a1})^{-\alpha}(e^{a1})^{-\alpha\rho}\left((1-\alpha) + \alpha\mu\frac{(e^{a0})^{\rho(\mu-1)}}{(e^{a1})^{\rho(\mu-1)}}\right) \\ & > \left(n^{a1/2}\right)^{-\alpha}(e^{a1/2})^{-\alpha\rho}\left(1-\alpha+\frac{\alpha\mu}{2}\right), \end{aligned}$$

In Step 1 we showed that $n^{a1/2} > n^{a1}$ and $e^{a1/2} > e^{a1}$ which implies $\left(n^{a1/2}\right)^{-\alpha}(e^{a1/2})^{-\alpha\rho} < (n^{a1})^{-\alpha}(e^{a1})^{-\alpha\rho}$. A sufficient condition for the above inequality to hold is

$$\alpha\mu\frac{(e^{a0})^{\rho(\mu-1)}}{(e^{a1})^{\rho(\mu-1)}} > \frac{\alpha\mu}{2},$$

which is true since we showed in Step 2. that $e^{a0}/e^{a1} > 1$. Hence we deduce that $e^{a0} > e^{a1/2}$.

C.6 Proof of Proposition 5

This proof is divided in two steps.

In step 1, we compare (i) $\Omega(1, d^{-i}, x^i)$ and $\Omega(0, d^{-i}, x^i)$, (ii) $\Omega(1, 1, 0)$ and $\Omega(1, 1, 1)$, (iii) $\Omega(1, 0, 1)$ and $\Omega(1, 1, 1)$, (iv) $\Omega(1, 1, 1)$ and $\Omega(1, 0, 0)$.

In step 2, we show that the Stackelberg-Nash equilibrium is unique and we determine the values of d_0 and d_1 at the equilibrium. We deduce the value of education and fertility at the equilibrium.

Step 1:

We start by defining the function $\Omega^i : \{0, 1\} \times \{0, 1\} \times [0, 1] \to \mathbb{R}^+$ which is given by $\forall i \in \{a, b\}$

$$\Omega(d^i, d^{-i}, x^i) = V(\hat{n}^i(d^i, d^{-i}), \hat{n}^{-i}(d^{-i}, d^i), \hat{e}^i(d^i, d^{-i}), \hat{e}^{-i}(d^{-i}, d^i), x^i) \quad \forall x^i \in [0, 1].$$

where the functions $\hat{n}^i$ and $\hat{e}^i$ are as in Definition 4.

(i) Comparison between $\Omega(1, d^{-i}, x^i)$ and $\Omega(0, d^{-i}, x^i)$. By definition, we have

$$\begin{aligned}\Omega(1, d^{-i}, x^i) &= \max_{(n^i, e^i) \in [0, \bar{n}] \times [0, \bar{e}]} V(n^i, \hat{n}^{-i}(d^{-i}, 1), e^i, \hat{e}^{-i}(d^{-i}, 1), x^i) \\ &= V(\hat{n}^i(1, d^{-i}), \hat{n}^{-i}(d^{-i}, 1), \hat{e}^i(1, d^{-i}), \hat{e}^{-i}(d^{-i}, 1), x^i),\end{aligned}$$

and

$$\begin{aligned}\Omega(0, d^{-i}, x^i) &= \max_{(n^{ij}, e^{ji}) \in [0, \bar{n}] \times [0, \bar{e}]} W(n^{ji}, \hat{n}^i(0, d^{-i}), \hat{n}^{-i}(d^{-i}, 0), e^{ji}, \hat{e}^i(0, d^{-i}), \hat{e}^{-i}(d^{-i}, 0), x^i) \\ &= W(\hat{n}^i(0, d^{-i}), \hat{n}^i(0, d^{-i}), \hat{n}^{-i}(d^{-i}, 0), \hat{e}^i(0, d^{-i}), \hat{e}^i(0, d^{-i}), \hat{e}^{-i}(d^{-i}, 0), x^i)\end{aligned}$$

Also, by definition, we have

$$\begin{aligned}&W(\hat{n}^i(0, d^{-i}), \hat{n}^i(0, d^{-i}), \hat{n}^{-i}(d^{-i}, 0), \hat{e}^i(0, d^{-i}), \hat{e}^i(0, d^{-i}), \hat{e}^{-i}(d^{-i}, 0), x^i) \\ &= V(\hat{n}^i(0, d^{-i}), \hat{n}^{-i}(d^{-i}, 0), \hat{e}^i(0, d^{-i}), \hat{e}^{-i}(d^{-i}, 0), x^i) \\ &< V(\hat{n}^i(1, d^{-i}), \hat{n}^{-i}(d^{-i}, 1), \hat{e}^i(1, d^{-i}), \hat{e}^{-i}(d^{-i}, 1), x^i)\end{aligned}$$

which is equivalent to

$$\Omega(1, d^{-i}, x^i) > \Omega(0, d^{-i}, x^i) \quad \forall d^{-i} \in \{0, 1\} \quad \forall x^i \in [0, 1].$$

(ii) Comparison between $\Omega(1, 1, 0)$ and $\Omega(1, 1, 1)$. Remind that we set $x^a = 0$.

Define the function $\Psi : [0, \bar{n}] \times [0, \bar{e}] \to \mathbb{R}$ given by

$$\Psi(n, e) = \beta\tau n \left((1 - \alpha) H^{-\alpha} (e)^\rho + \frac{(e)^{\rho\mu}}{D} \alpha H^{1-\alpha} \right) - \gamma\, ne - \frac{\lambda}{2} (n)^2, \tag{1}$$

where

$$\begin{aligned}H &= \hat{n}^b(1, 1)(\hat{e}^b(1, 1))^\rho \\ D &= \hat{n}^b(1, 1)(\hat{e}^b(1, 1))^{\rho\mu}\end{aligned}$$

We have

$$\Omega(1,1,0) = \Psi(\hat{n}^a(1,1), \hat{e}^a(1,1))$$
$$\Omega(1,1,1) = \Psi(\hat{n}^b(1,1), \hat{e}^b(1,1)).$$

Note that $\Psi(n^i, e^i) = V(n^i, \hat{n}^b(1,1), e^i, \hat{e}^b(1,1), 0)$ so that

$$\begin{aligned}\max_{(n^i,e^i)\in[0,\bar{n}]\times[0,\bar{e}]} V(n^i, \hat{n}^b(1,1), e^i, \hat{e}^b(1,1), 0) &= \max_{(n^i,e^i)\in[0,\bar{n}]\times[0,\bar{e}]} \Psi(\hat{n}^i, \hat{e}^i)\\ &= V(\hat{n}^a(1,1), \hat{n}^b(1,1), \hat{e}^a(1,1), \hat{e}^b(1,1), 0)\\ &= \Psi(\hat{n}^a(1,1), \hat{e}^a(1,1)).\end{aligned}$$

We deduce that

$$\begin{aligned}&\Psi(\hat{n}^a(1,1), \hat{n}^a(1,1)) > \Psi(\hat{n}^b(1,1), \hat{e}^b(1,1)),\\ \Leftrightarrow &\Omega(1,1,0) > \Omega(1,1,1).\end{aligned}$$

Using a similar reasoning we can also deduce

$$\Omega(0,1,0) > \Omega(0,1,1).$$

(iii) Comparison between $\Omega(1,1,1)$ and $\Omega(1,0,1)$.

We have

$$\Omega(1,1,1) = \Psi(\hat{n}^b(1,1), \hat{e}^b(1,1))$$
$$\Omega(1,0,1) = \Psi(\hat{n}^b(1,1), \hat{e}^b(1,1)).$$

We deduce $\Omega(1,1,1) = \Omega(1,0,1)$.

(iv) Comparison between $\Omega(1,0,0)$ and $\Omega(1,1,1)$. We have

$$\Omega(1,0,0) = \beta\tau\hat{n}^a(1,0)\left((1-\alpha)H^{-\alpha}(\hat{e}^a(1,0))^\rho + \frac{(\hat{e}^a(1,0))^{\rho\mu}}{D}\alpha H^{1-\alpha}\right)$$
$$-\gamma\, \hat{n}^a(1,0)\hat{e}^a(1,0) - \frac{\lambda}{2}\left(\hat{n}^a(1,0)\right)^2,$$

where

$$H = \hat{n}^b(0,1)(\hat{e}^b(0,1))^\rho$$
$$D = \hat{n}^b(0,1)(\hat{e}^b(0,1))^{\rho\mu}$$

$$\Omega(1,1,1) = \beta\tau\hat{n}^b(1,1)\left((1-\alpha)H^{-\alpha}(\hat{e}^b(1,1))^\rho + \frac{(\hat{e}^b(1,1))^{\rho\mu}}{D}\alpha H^{1-\alpha}\right)$$

$$-\gamma\,\hat{n}^b(1,1)\hat{e}^b(1,1) - \frac{\lambda}{2}\left(\hat{n}^b(1,1)\right)^2,$$

where

$$H = \hat{n}^b(1,1)(\hat{e}^b(1,1))^\rho$$
$$D = \hat{n}^b(1,1)(\hat{e}^b(1,1))^{\rho\mu}$$

Note that $\forall d^b \in \{0,1\}$, the FOC for education at $x^b = 1$ gives

$$(1-\alpha)H^{-\alpha} = \frac{\gamma N^\alpha}{\rho\beta\tau}\,(\hat{e}^b(d^b,1))^{1-\rho}$$

Using that in the above equations, we obtain

$$\Omega(1,0,0) = \hat{n}^a(1,0)\frac{\gamma N^\alpha}{\rho(1-\alpha)}\left((1-\alpha)\;\hat{e}^b(0,1))^{(1-\rho)}(\hat{e}^a(1,0))^\rho + \alpha(\hat{e}^a(1,0))^{\rho\mu}\;\hat{e}^b(0,1))^{(1-\rho\mu)}\right)$$

$$-\gamma\,\hat{n}^a(1,0)\hat{e}^a(1,0) - \frac{\lambda}{2}\left(\hat{n}^a(1,0)\right)^2,$$

and

$$\Omega(1,1,1) = \hat{n}^b(1,1)\frac{\gamma N^\alpha}{\rho(1-\alpha)}\left((1-\alpha)\hat{e}^b(1,1))^{(1-\rho)}(\hat{e}^b(1,1))^\rho + \alpha(\hat{e}^b(1,1))^{\rho\mu}\hat{e}^b(1,1))^{(1-\rho\mu)}\right)$$

$$-\gamma\,\hat{n}^b(1,1)\hat{e}^b(1,1) - \frac{\lambda}{2}\left(\hat{n}^b(1,1)\right)^2,$$

Define the function $\Phi : [0,\bar{n}] \times [0,\bar{e}]^2 \to \mathbb{R}$ given by

$$\Phi(n,e,y) = n\frac{\gamma N^\alpha}{\rho(1-\alpha)}\left((1-\alpha)\;(y)^{(1-\rho)}(e)^\rho + \alpha(e)^{\rho\mu}\;(y)^{(1-\rho\mu)}\right) - \gamma\;ne - \frac{\lambda}{2}\,(n)^2\,.$$

Then we have

$$\Phi(\hat{n}^b(1,1),\hat{e}^b(1,1),\hat{e}^b(1,1)) = \Omega(1,1,1),$$

and

$$\Phi(\hat{n}^a(1,0),\hat{e}^a(1,0),\hat{e}^b(0,1)) = \Omega(1,0,0).$$

Note that when $\rho\mu \leq 1$, the sign of $\partial\Phi/\partial y$ is positive. When $\rho\mu > 1$, the sign of $\partial\Phi/\partial y$ is ambiguous. Let us consider the two cases.

First, suppose that $\partial\Phi/\partial y < 0$. Remind that $\hat{e}^b(1,1) > \hat{e}^b(0,1)$. Together these two conditions imply that $\forall(n,e) \in [0,\bar{n}] \times [0,\bar{e}]$

$$\Phi(n, e, \hat{e}^b(1,1)) < \Phi(n, e, \hat{e}^b(0,1)).$$

In particular, we have

$$\Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(1,1)) < \Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(0,1)).$$

But, we know that

$$\Phi(\hat{n}^a(1,0), \hat{e}^a(1,0), \hat{e}^b(0,1)) = \max_{(n,e)\in[0,\bar{n}]\times[0,\bar{e}]} \Phi(n, e, \hat{e}^b(0,1)) > \Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(0,1)).$$

We deduce that $\Omega(1,0,0) > \Omega(1,1,1)$.

Second, suppose that $\partial\Phi/\partial y > 0$. Using this condition and $\hat{e}^b(1,1) > \hat{e}^b(0,1)$ we obtain

$$\Phi(n, e, \hat{e}^b(1,1)) > \Phi(n, e, \hat{e}^b(0,1)).$$

Then, on the one hand, we have

$$\Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(1,1)) > \Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(0,1)).$$

But, on the other hand, we know that

$$\Phi(\hat{n}^a(1,0), \hat{e}^a(1,0), \hat{e}^b(0,1)) = \max_{(n,e)\in[0,\bar{n}]\times[0,\bar{e}]} \Phi(n, e, \hat{e}^b(0,1)) > \Phi(\hat{n}^b(1,1), \hat{e}^b(1,1), \hat{e}^b(0,1)).$$

In this case, the sign of $\Omega(1,0,0) - \Omega(1,1,1)$ is ambiguous.

(v) Comparison between $\Omega(1,0,0)$ and $\Omega(1,1,0)$. Using similar arguments (to the ones used in case iv) we can deduce that

$$\begin{aligned}
\frac{\partial\Phi}{\partial y} < 0 \quad &\Leftrightarrow \quad \Omega(1,0,0) > \Omega(1,1,0) \\
\frac{\partial\Phi}{\partial y} > 0 \quad &\Leftrightarrow \quad \Omega(1,0,0) < \Omega(1,1,0).
\end{aligned}$$

(vi) Comparison between $\Omega(0,0,0)$ and $\Omega(0,1,0)$. As well we can deduce the following.

$$\begin{aligned}
\frac{\partial\Phi}{\partial y} < 0 \quad &\Leftrightarrow \quad \Omega(0,0,0) > \Omega(0,1,0) \\
\frac{\partial\Phi}{\partial y} > 0 \quad &\Leftrightarrow \quad \Omega(0,0,0) < \Omega(0,1,0).
\end{aligned}$$

Step 2. The different outcomes of the game can be represented in the payoff matrix below.

		Individual a	
		$d^a = 0$	$d^a = 1$
Individual b	$d^b = 0$	$\Omega(0,0,1) > 0, \Omega(0,0,0) > 0$	$\Omega(0,1,1) > 0, \Omega(1,0,0) - \kappa$
	$d^b = 1$	$\Omega(1,0,1) - \kappa, \Omega(0,1,0) > 0$	$\Omega(1,1,1) - \kappa, \Omega(1,1,0) - \kappa$

Let us look at the values of these payoffs. To do so, we define

$$\begin{aligned}
&\tilde{\kappa}_1 \equiv \Omega(1,1,0) - \Omega(0,1,0),\\
&\tilde{\kappa}_2 \equiv \Omega(1,1,1) - \Omega(0,1,1) = \Omega(1,1,1) - \Omega(0,0,0) = \Omega(1,0,1) - \Omega(0,0,0),\\
&\tilde{\kappa}_3 \equiv \Omega(1,0,0) - \Omega(0,0,0).
\end{aligned}$$

From (i) we deduce that $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 0$, $\tilde{\kappa}_3 > 0$. From (ii) we deduce that the sign of $\tilde{\kappa}_1 - \tilde{\kappa}_2$ is ambiguous. From (iii) and (iv) we deduce that the sign of $\tilde{\kappa}_2 - \tilde{\kappa}_3$ is ambiguous. Finally, from (v) and (vi) we deduce that the sign of $\tilde{\kappa}_1 - \tilde{\kappa}_3$ is ambiguous. Hence, depending on the model's parameters, there are six possibilities for the ranking of the thresholds. Let us focus on the case $\tilde{\kappa}_2 < \min\{\tilde{\kappa}_3, \tilde{\kappa}_1\}$.

a. Suppose that $\kappa > \max\{\tilde{\kappa}_3, \tilde{\kappa}_1\}$. Then $\Omega(1,0,0) - \kappa < \Omega(0,0,0)$, $\Omega(1,1,0) - \kappa < \Omega(0,1,0)$, $\Omega(1,0,1) - \kappa < \Omega(0,0,1)$ and $\Omega(1,1,1) - \kappa < \Omega(0,1,1)$. We deduce that there exists only one equilibrium of this game: $(d_t^{a\star}, d_t^{b\star}, n_t^{a\star}, n_t^{b\star}, e_t^{a\star}, e_t^{b\star}) = (0, 0, \hat{n}^a(0,0), \hat{n}^b(0,0), \hat{e}^a(0,0), \hat{e}^b(0,0))$.

b. Suppose that $\kappa < \tilde{\kappa}_2$. Then $\Omega(1,0,0) - \kappa > \Omega(0,0,0)$, $\Omega(1,1,0) - \kappa > \Omega(0,1,0)$, $\Omega(1,0,1) - \kappa > \Omega(0,0,1)$ and $\Omega(1,1,1) - \kappa > \Omega(0,1,1)$. We deduce that there exists only one equilibrium of this game: $(d_t^{a\star}, d_t^{b\star}, n_t^{a\star}, n_t^{b\star}, e_t^{a\star}, e_t^{b\star}) = (1, 1, \hat{n}^a(1,1), \hat{n}^b(1,1), \hat{e}^a(1,1), \hat{e}^b(1,1))$.

c. Suppose that $\tilde{\kappa}_3 > \kappa > \tilde{\kappa}_2$. We know that $\Omega(1,0,1) - \kappa < \Omega(0,0,1)$ and $\Omega(1,1,1) - \kappa < \Omega(0,1,1)$ so that $d_t^{b\star} = 0$. Furthermore, we know that $\Omega(1,0,0) - \kappa > \Omega(0,0,0)$ so that $d_t^{a\star} = 1$.

C.7 Proof of Corollary 1

Suppose that $\tilde{\kappa}_2 < \min\{\tilde{\kappa}_1, \tilde{\kappa}_3\}$, from Proposition 5 we know that $\forall \kappa \in (\tilde{\kappa}_2, \tilde{\kappa}_3)$ the equilibrium is given by $(d_t^{a\star}, d_t^{b\star}, n_t^{a\star}, n_t^{b\star}, e_t^{a\star}, e_t^{b\star}) = (1, 0, \hat{n}^a(1,0), \hat{n}^b(0,1), \hat{e}^a(1,0), \hat{e}^b(0,1))$.

To simplify the notations, here let us denote $\hat{e}^b(0,1) \equiv e^b$, $\hat{e}^a(1,0) \equiv e^a$, $\hat{n}^b(0,1) \equiv n^b$ and

$\hat{n}^a(1,0) \equiv n^a$. The FOC give us

$$\begin{aligned}
(e^a)^{1-\rho} &= \frac{\rho\beta\tau N^{-\alpha}}{\gamma}(n^b)^{-\alpha}(e^b)^{-\alpha\rho}\left((1-\alpha)+\alpha\mu\frac{(e^a)^{\rho(\mu-1)}}{(e^b)^{\rho(\mu-1)}}\right),\\
(e^b)^{1-\rho} &= \frac{\rho\beta\tau N^{-\alpha}}{\gamma}(n^b)^{-\alpha}(e^b)^{-\alpha\rho}(1-\alpha),\\
n^a &= \frac{\beta\tau N^{-\alpha}}{\lambda}(e^a)^{\rho}(n^b)^{-\alpha}(e^b)^{-\alpha\rho}\left((1-\alpha)(1-\rho)-\alpha\frac{(e^a)^{\rho(\mu-1)}}{(e^b)^{\rho(\mu-1)}}(\rho\mu-1)\right),\\
n^b &= \frac{\beta\tau N^{-\alpha}}{\lambda}(e^b)^{\rho}(n^b)^{-\alpha}(e^b)^{-\alpha\rho}\left((1-\alpha)(1-\rho)+\alpha\right).
\end{aligned}$$

First, we easily deduce $e^a > e^b$. Second, $n^a < n^b$ is equivalent to

$$\hat{\Gamma}(e^a/e^b) < 0,$$

where the function $\hat{\Gamma} : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\hat{\Gamma}(x,\alpha) = x^{\rho}\left((1-\alpha)(1-\rho)-\alpha x^{\rho(\mu-1)}(\rho\mu-1)\right)-(1-\alpha)(1-\rho)-\alpha.$$

One has

$$\hat{\Gamma}'(x) = \rho x^{\rho-1}\left((1-\alpha)(1-\rho)-\alpha x^{\rho(\mu-1)}(\mu\rho-1)\right)-\alpha x^{\rho}\rho(\mu-1)x^{\rho(\mu-1)-1}(\rho\mu-1).$$

Suppose that $\rho\mu > 1$. Then, the function $\hat{\Gamma}$ reaches a maximum at $x_m = \left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right)^{\frac{1}{\rho(\mu-1)}}$. A sufficient condition for $\hat{\Gamma}(e^{a0}/e^{a1}) < 0$ is $\hat{\Gamma}(x_m) < 0$ which is equivalent to $\hat{\Lambda}(\mu) < 0$ where $\hat{\Lambda} : [1,+\infty) \to \mathbb{R}$ is given by

$$\hat{\Lambda}(\mu) = (1-\alpha)(1-\rho)\left(\left(\frac{(1-\alpha)(1-\rho)}{\alpha(\rho\mu-1)\mu}\right)^{\frac{1}{(\mu-1)}}(1-\frac{1}{\mu})-1.\right)$$

First, suppose that $(1-\alpha)(1-\rho)/\left(\alpha(\rho\mu-1)\mu\right) < 1$ which is equivalent to $\mu > \mu_h$ (defined in proof of Proposition 4). In that case, one finds $\hat{\Gamma}(x_m) < 0$. Second, suppose that $1/\rho < \mu < \mu_h$. We have $\hat{\Lambda}'(\mu) = \Lambda'(\mu) < 0$.

Since $\hat{\Lambda}(1/\rho) = +\infty$, $\hat{\Lambda}(\mu_h) < 0$ and $\hat{\Lambda}$ is continuous, we deduce that there exists a unique $\mu^{**} \in [1/\rho, \mu_h]$ such that $\mu \geq \mu^*$ is equivalent to $\Lambda(\mu) \leq 0$ which implies $n^a < n^b$. Furthermore we know that $\forall \mu \in [1,+\infty)$, $\hat{\Lambda}(\mu) < \Lambda(\mu)$ which implies $\hat{\Lambda}(\mu^*) < \Lambda(\mu^*) = 0$. We deduce that $\mu^{**} < \mu^*$.

D Numerical example

Assume $\alpha = 0.25$, $\lambda = 0.2$, $\rho = 0.5$, $\gamma = 0.3$. Figure 2 shows how fertility and education of group a change when the share of the group varies, in the case $\mu = 1$. This illustrates the results of Proposition 3. Figure 3 shows the same variables in the case $\mu = 3$. This illustrates the results of Proposition 4.

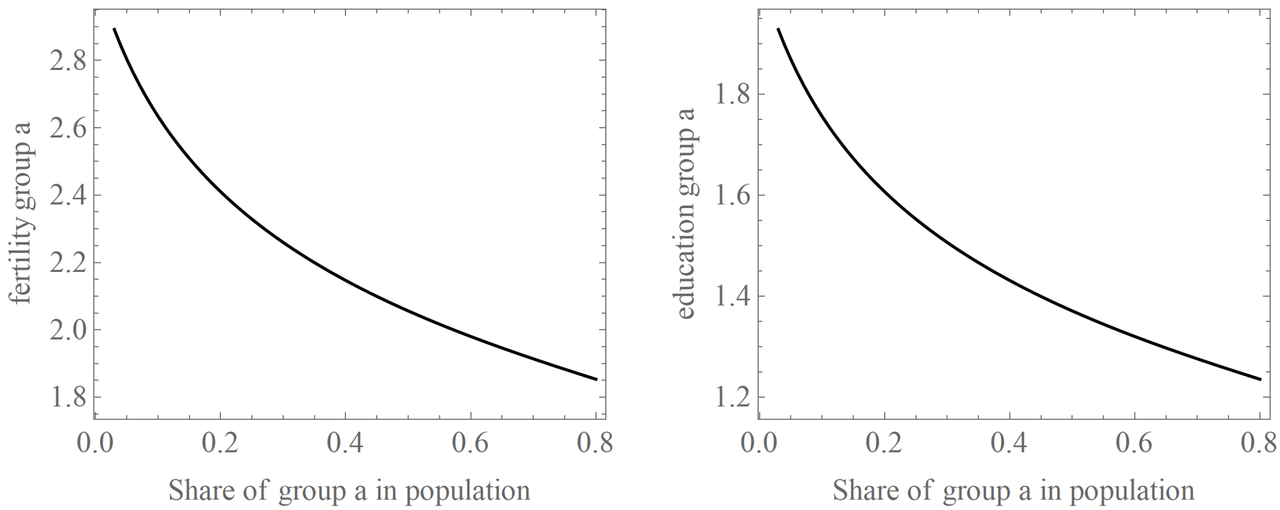


Figure 2: Numerical example: Case $\mu = 1$

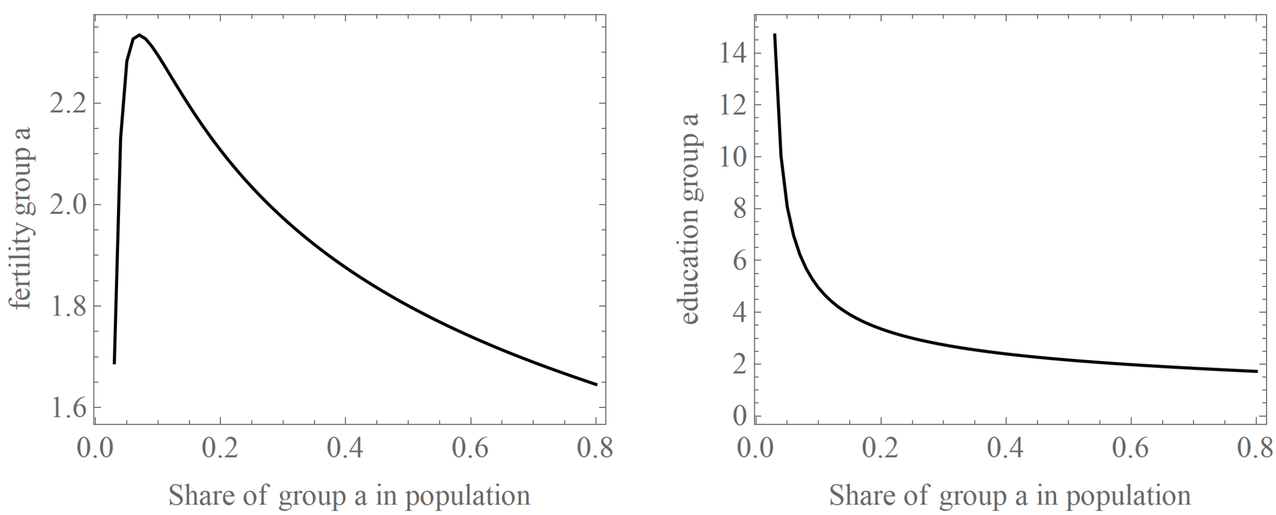
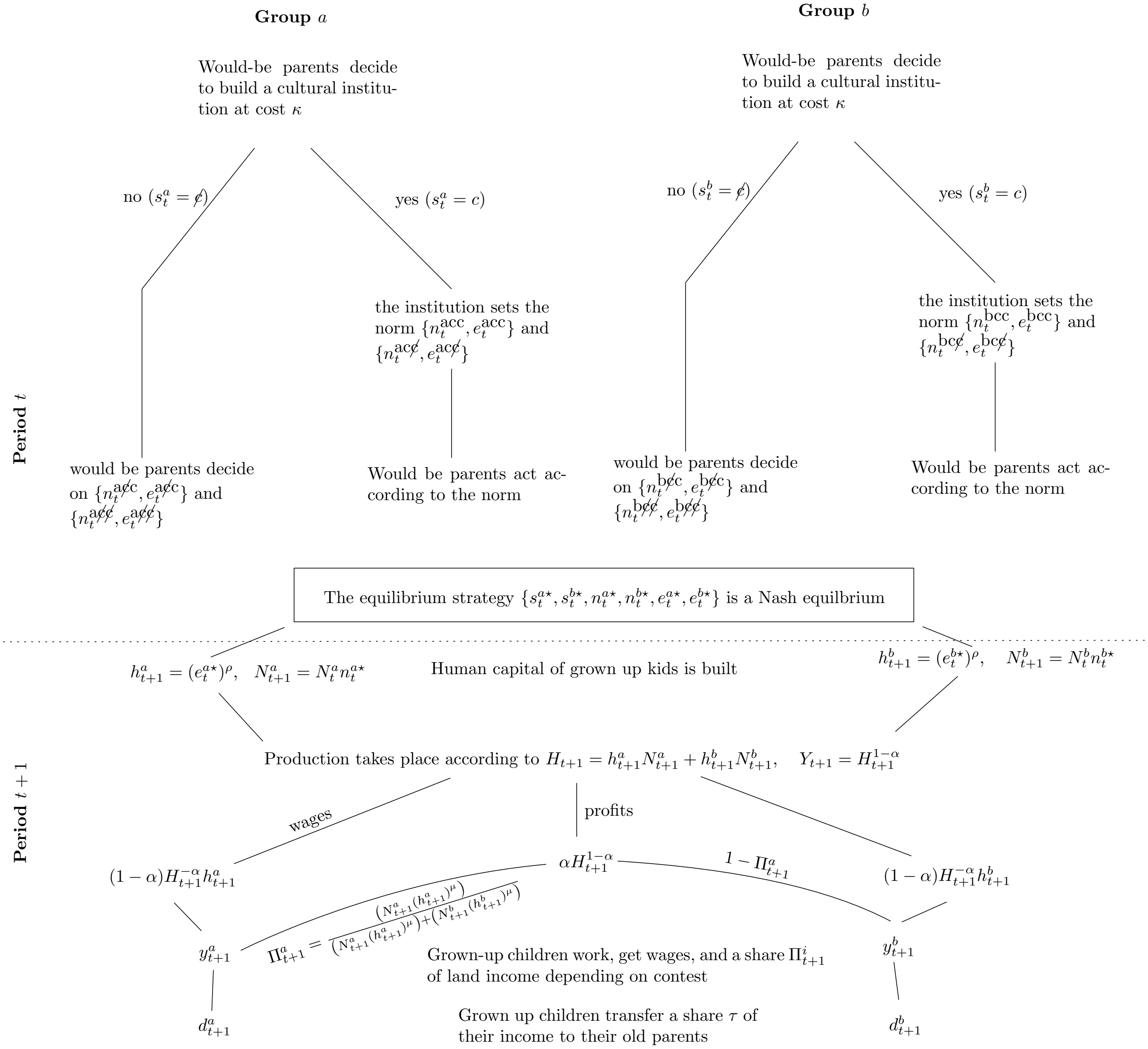


Figure 3: Numerical example: Case $\mu = 3 (> 1/\rho)$

E The full game



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