Mathematical Social Sciences 59 (2010) 227–238

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

On the Golden Rule of capital accumulation under endogenous longevity[☆]

David de la Croix[a,*], Gregory Ponthiere[b]

[a] *Department of Economics and CORE, Université catholique de Louvain, Belgium*
[b] *PSE and Ecole Normale Supérieure (ENS), 48 Boulevard Jourdan, 75014 Paris, France*

ARTICLE INFO

Article history:
Received 17 February 2009
Received in revised form
4 June 2009
Accepted 10 July 2009
Available online 9 September 2009

JEL classification:
E13
E21
E22
I12

Keywords:
Golden Rule
Health
Longevity
OLG models

ABSTRACT

Health spending obviously increase with capital per worker. This paper derives the optimal accumulation policy in such a context. The optimal accumulation rule depends on whether health spending improve consumption enjoyment, and on whether the planner adheres to an instantaneous welfarist view or to a complete life view. First, when the only role of health is to enhance longevity, we show that the capital per worker maximizing steady-state consumption per head is inferior to the standard Golden Rule. Moreover, the capital per worker maximizing steady-state consumption per head, when consumption efficiency depends on the health status, tends to exceed the optimal capital level under purely longevity-enhancing spending. Finally, when the planner adheres to a complete life view, the capital per worker maximizing steady-state expected lifetime consumption per head exceeds the optimal capital per worker under the instantaneous view.

1. Introduction

Introduced by Phelps (1961), the Golden Rule of capital accumulation states the condition under which the stock of capital per worker maximizes steady-state consumption per head. In a simple model with no technological progress, the Golden Rule states that steady-state consumption per head is maximized when the marginal productivity of capital equals the sum of the population growth rate and the rate of depreciation of capital. It does not depend on preferences.

☆ We thank Andrew Clark, Daniel Cohen, Fabrice Etilé, André Masson, Pierre Pestieau, Mathieu Valdenaire and two anonymous referees for useful comments on this paper.
* Corresponding author.
E-mail addresses: david.delacroix@uclouvain.be (D. de la Croix), gregory.ponthiere@ens.fr (G. Ponthiere).

0165-4896/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.mathsocsci.2009.07.007

Whereas the Golden Rule has given rise to various studies in growth theory (see Diamond (1965) and Phelps (1965)), no attention has been paid so far to the Golden Rule of capital accumulation in an economy where some resources are spent on health and agents' lifetime is endogenous, as in Chakraborty's (2004) OLG model. Under such an endogenous (finite) time horizon, does the Golden Rule capital level remain the same, or, on the contrary, does the endogeneity of lifetime modify the Golden Rule?

Despite the increasing body of recent literature on the relation between economic growth and survival conditions, that question has not been discussed so far, as the literature focused mainly on purely *descriptive* issues (e.g. multiplicity of equilibria), and did not consider the normative question of the optimal capital accumulation in that context.[1]

Undoubtedly, studying the optimal capital accumulation in an economy where lifetime is endogenous consists of a most ambitious task, as the treatment of death, in welfare terms, remains problematic. Actually, the major difficulty concerns the definition of the utility level that should be assigned to death. However, given the growing body of recent literature with endogenous longevity, it is important, despite that difficulty, to study also the normative side of the growth/longevity relationship. This note, which aims at determining the Golden Rule capital level in the context of an economy with endogenous longevity, constitutes a first step in that direction. By focusing on consumption rather than utility, the present note will allow us to start the normative study of capital accumulation under endogenous longevity *without* having to make fragile postulates on the utility assigned to the death state.

In order to characterize the Golden Rule of capital accumulation, we use a two-period OLG model with physical capital based on Chakraborty (2004), where the probability of survival to the second period of life depends positively on the agent's health status, which is itself determined by some health expenditures. We then explore optimal accumulation policies in that context.[2]

Health spending obviously increase with capital per worker. Just like the two consumption levels of young and old individuals, health spending are not decided by the "planner" who determines the level of capital.[3] As a consequence, the capital accumulation rules we are going to derive here are no longer "first-best" as the original Golden Rule, but describe instead optimal accumulation policy in the second-best sense.

At this early stage of this study, it should be stressed that the relevancy of optimal accumulation rules in the context of endogenous health and longevity may be questioned on two distinct grounds, which, as we shall see, can serve as a basis for developing alternative policy concepts.

First, it may be argued that, as soon as *health* is a variable rather than a constant, focusing on the capital per worker maximizing steady-state consumption per head becomes a narrow objective. That criticism emphasizes an important limitation of optimal accumulation policy in the present context. However, as we shall see, it is nonetheless possible to account for that intuition, by expressing the consumption goal not in raw terms, but in efficiency units, under the assumption that agents tend, if they are healthy, to enjoy consumption to a larger extent than if they are not healthy. Note, however, that maximizing consumption in efficiency units is a better objective only to the extent that health spending improve the quality of each period lived. If health spending make people live longer but have no impact on the enjoyment of consumption (as in Chakraborty (2004)), then maximizing consumption per head is an adequate goal.

Second, it may also be argued that the usual objective of maximizing consumption per period becomes irrelevant once *longevity* is a variable. Obviously, whether agents live a short or a long life is generally not regarded as unimportant, and the usual Golden Rule, by focusing on consumption

1 Chakraborty (2004), Cervellati and Sunde (2005) and Chakraborty and Das (2005) are three highly cited examples. Other papers include Jones (2001), Tamura (2006) and de la Croix and Licandro (2007). A survey is in Boucekkine (2008).

2 Note that, in the present framework, we shall assume that individual productivity is fixed. A natural extension of the paper could consist of introducing productivity gains related to a better health.

3 Note that assuming that individual health expenditures are not optimally chosen constitutes an important assumption, which affects our results (see *infra*). However, assuming, alternatively, that the social planner could choose individual health spending but not individual consumptions at the two periods of life would be rather hard to justify.

per period of life, may thus miss an important aspect of the picture. However, it should be stressed that, under pure longevity-enhancing health spending, consumption remains, even in the presence of longevity, the *unique* determinant of *temporal* welfare, that is, of the welfare associated to a particular period of life. This property comes from the singular nature of longevity as a dimension of welfare: its influence can be acknowledged only if a lifetime perspective is adopted (i.e. the complete life view). However, from an *instantaneous* welfarist point of view, the only piece of information that matters is the consumption per period, as studied by the Golden Rule. Naturally, one is not forced to adopt such an instantaneous point of view, but it is far from obvious that one can reject *a priori* a study of the Golden Rule of capital on the mere basis of the variability of longevity: a chance must be given to both the complete life view and the instantaneous, intensity view. This is the approach adopted throughout this paper.

The rest of this note is organized as follows. Section 2 presents the model. Section 3 derives optimal capital accumulation in an economy with endogenous longevity. Section 4 characterizes an alternative definition of the objective, which consists of maximizing steady-state consumption per head measured in efficiency units (and thus depending on the health level of agents). Section 5 explores a second alternative which relies on the complete life view rather than the intensity view. Section 6 concludes.

2. The model

Let us consider an OLG model with the same population structure and technology as in the model studied by Chakraborty (2004). Time is discrete and goes from 0 to infinity; households live at best for two periods.

Demography. The size of the cohort born at t is L_t. It grows over time at a constant, exogenous rate n ($n > -1$):

$$L_{t+1} = (1+n)L_t. \tag{1}$$

All agents of a cohort t live the first period of life for sure, but only a proportion π_{t+1} of that cohort will enjoy a second period of life. Hence, life expectancy at birth for the cohort born at time t is equal to $1 + \pi_{t+1}$. The proportion of survivors π_{t+1} $(0 < \pi_{t+1} < 1)$ depends positively on the amount of health expenditures per worker h_t:

$$\pi_{t+1} = \pi(h_t) \tag{2}$$

with $\pi(h) \in [0, 1]$ for all $h > 0$, and $\pi'(h) > 0$. We also assume that $\lim_{h_t \to \infty} \pi(h_t) = 1$, which gives an upper bound to the life expectancy $1 + \pi$.

As in Chakraborty, first-period agents supply their labour inelastically, while second-period agents are retired.

Technology. Firms at time t produce some output Y_t according to the following production function: $Y_t = F(K_t, L_t)$ where Y_t denotes the total output, K_t the total capital stock, and L_t denotes the labour force. $F(.)$ is a positively-valued production function, increasing, and strictly concave with respect to capital. Capital depreciates at a constant rate $\delta(0 \leq \delta \leq 1)$. Under constant returns to scale, production can be rewritten as:

$$y_t = f(k_t) \tag{3}$$

where y_t denotes the output per worker, and k_t the capital stock per worker, while $f(.) = F(k, 1)$ is the production function in its intensive form. Under the above assumptions on $F(.)$, we have, for all $k > 0, f(k) > 0, f'(k) > 0$ and $f''(k) < 0$ (de la Croix and Michel, 2002).

The marginal productivity of capital is equal to $f'(k)$. The marginal productivity of labour is given by the function

$$\omega(k) = f(k) - kf'(k).$$

It can be shown that the marginal productivity of labour $\omega(k)$ satisfies $\omega(k) \geq 0$ and $\omega'(k) = -kf''(k) > 0$.

Health spending. Total health spending are: $H_t = L_t h_t$. We assume that health spending are possibly a function of capital per worker through the function: $h_t = h(k_t)$. We consider three alternative assumptions concerning this function.

A1 Health spending per worker are constant: $h_t = \bar{h}$.

A2 Health spending per worker are a constant fraction τ of the marginal productivity of labour:

$$h_t = \tau\omega(k_t). \tag{4}$$

This is the assumption made by Chakraborty (2004).

A3 Health spending per worker are a constant fraction θ of output per worker:

$$h_t = \theta f(k_t). \tag{5}$$

3. Optimal accumulation policy

Consider a stationary environment in which the variables k, h and π are constant over time and all the aggregate variables, production Y_t, consumption C_t, investment I_t, health spending H_t, and capital K_t grow at the constant rate n. Let us derive the level of capital per worker k maximizing steady-state consumption per head. The feasibility constraint imposes that investment I_t is equal to production $F(K_t, L_t)$ minus consumption C_t minus health spending H_t: $I_t = K_{t+1} - (1-\delta)K_t = F(K_t, L_t) - C_t - H_t$ so that total consumption C_t is equal to: $C_t = F(K_t, L_t) - K_{t+1} - H_t + (1-\delta)K_t$. Thus, consumption per worker, equal to $C_t/L_t = c_t$, can be written as:

$$c_t = f(k_t) - k_t(\delta + n) - h(k_t). \tag{6}$$

Consumption per head $C_t/(L_t + \pi_t L_{t-1})$ where $\pi_t = \pi(h(k)) = \pi$ is related to consumption per worker through the following identity:

$$\frac{C_t}{L_t + \pi L_{t-1}} = \frac{c_t}{\left(1 + \frac{\pi}{1+n}\right)} = \frac{1+n}{1+n+\pi}\, c_t$$

given that $L_t = (1+n)L_{t-1}$. Consumption per head corresponds to consumption per worker c_t, multiplied by $(1+n)/(1+n+\pi)$. Note that this latter factor depends on π, and, thus, under **A2** and **A3**, on capital per worker. Hence, contrary to what prevails in standard OLG models with exogenous longevity, the capital level maximizing consumption per head does not necessarily coincide with the one maximizing consumption per worker.

It follows that consumption per head at the steady-state, denoted by $\phi(k)$, can be written as:

$$\phi(k) = [f(k) - k(\delta + n) - h(k)]\,\frac{1+n}{1+n+\pi(h(k))}. \tag{7}$$

In order to discuss the conditions necessary and sufficient for the existence of a Golden Rule capital level, let us first differentiate consumption per head $\phi(k)$ with respect to capital:

$$\phi'(k) = \left[f'(k) - (\delta + n) - h'(k)\right] - \frac{[f(k) - k(\delta+n) - h(k)]\,\pi'(h(k))h'(k)}{1+n+\pi(h(k))} \tag{8}$$

where $h'(k) = 0$ under **A1**, $h'(k) = \tau\omega'(k) = -\tau k f''(k) > 0$ under **A2**, and $h'(k) = \theta f'(k) > 0$ under **A3**.

As de la Croix and Michel (2002) argued, the expression $\phi'(k) = 0$ defines an *interior* Golden Rule capital level only if $\phi(k)$ is neither always decreasing in k (implying that the capital level maximizing $\phi(k)$ is 0), nor always increasing in k (implying that the capital maximizing $\phi(k)$ is infinite). The interiority of the solution requires the following condition, which guarantees that $\phi'(k)$ is positive when k tends to 0, but negative when it tends to $+\infty$.

Proposition 1. *Assume that* $\{n, \delta, f(k), \pi(h)\}$ *satisfy:*

$$\lim_{k\to 0+} f'(k) > \delta + n + \lim_{k\to 0+}\left(\frac{\pi'(h(k))\,[f(k)-h(k)]}{1+n+\pi(h(k))}+1\right)h'(k)$$
$$\lim_{k\to +\infty} f'(k) < \delta + n.$$

Then, there exists a capital per worker $k^\star$ *maximizing consumption per head in* $\mathbb{R}_+$. *Such a level satisfies* $\phi'(k^\star) = 0$:

$$f'(k^\star) = \delta + n + h'(k^\star)\left(1 + \frac{\pi'(h(k^\star))(f(k^\star) - k^\star(\delta+n) - h(k^\star))}{1+n+\pi(h(k^\star))}\right). \tag{9}$$

Proof. The conditions $\lim_{k\to 0+}\phi'(k) > 0$ and $\lim_{k\to +\infty}\phi'(k) < 0$ are sufficient to obtain an interior maximum. The first limit can be written as:

$$\lim_{k\to 0+}\phi'(k) = \lim_{k\to 0+}\left[f'(k) - (\delta+n) - h'(k) - \frac{\pi'(h(k))h'(k)\,[f(k)-h(k)]}{1+n+\pi(h(k))}\right].$$

The condition $\lim_{k\to 0+}\phi'(k) > 0$ can be rewritten as:

$$\lim_{k\to 0+} f'(k) > \delta + n + \lim_{k\to 0+}\left(\frac{\pi'(h(k))\,[f(k)-h(k)]}{1+n+\pi(h(k))}+1\right)h'(k)$$

which is the condition in the Proposition. Regarding the second condition, we have:

$$\lim_{k\to\infty}\phi'(k) = \lim_{k\to\infty}\left[f'(k) - (\delta+n) - h'(k) - \frac{\pi'(h(k))h'(k)\,[f(k)-h(k)]}{1+n+\pi(h(k))}\right]$$
$$\lim_{k\to\infty}\phi'(k) = \lim_{k\to\infty}\left[f'(k) - (\delta+n) - h'(k)\left(1 + \frac{\pi'(h(k))\,[f(k)-h(k)]}{1+n+\pi(h(k))}\right)\right].$$

Given that $h'(k) \geq 0$, the condition $\lim_{k\to\infty}\phi'(k) < 0$ is always true when $\lim_{k\to\infty} f'(k) - (\delta+n) < 0$.

Hence, under the two conditions of the Proposition, the function $\phi(k)$ reaches a maximum between 0 and $+\infty$. Given that the function $\phi(k)$ is continuous, this maximum $k^\star$ satisfies $\phi'(k^\star) = 0$. ■

To better understand the Proposition it is useful to look at its implications for the three cases **A1**–**A3**. With the case in which health spending are constant, the condition of the proposition would collapse to

$$\lim_{k\to +\infty} f'(k) < \delta + n < \lim_{k\to 0+} f'(k)$$

which is assumption A5 in de la Croix and Michel (2002). Moreover, Eq. (8) would simplify into:

$$\phi'(k) = f'(k) - (\delta + n)$$

and the Golden Rule $\bar{k}_{GR}$ satisfies the usual condition

$$f'(\bar{k}_{GR}) = \delta + n. \tag{10}$$

The Golden rule capital level with exogenous longevity is independent from the postulated level of the probability of survival π. However, it is important to stress that the level of consumption per head for a given capital level is not independent from the level of π. Although it is for the same level of capital per worker that steady-state consumption per head is maximized, the level of the consumption profile is higher the lower π is. The intuition behind this is that π, by increasing the population size, reduces consumption per head per period of life for a given level of k. Thus, although π does not affect the Golden Rule capital level, it does influence the level of consumption per head under each capital level.

Under **A2** or **A3**, the interiority of the optimal capital level requires a stronger condition regarding the level of $\lim_{k\to 0+} f'(k)$. The intuition behind the additional term in the condition is that the interiority of the optimal capital requires also, in the context of endogenous health spending and longevity, that a small increase of capital in the neighborhood of 0 does not lead to an explosion of the population through a rise of the survival probability, in which case the optimal capital level would be zero. In other words, $\pi'(h(0+))$ should be small enough.

Corollary 1. *The optimal capital level under* **A2** *or* **A3** *is lower than under* **A1**.

Proof. Compare (9) to (10). Under **A2** or **A3** the right hand side of (9) is larger than the right hand side of (10) for any $k^\star > 0$ because $h' > 0$ and because net production $f(k^\star) - k^\star(\delta + n) - h(k^\star)$ is positive thanks to the limit conditions in Proposition 1. This implies that the marginal productivity of capital must here be strictly larger than its level under **A1**. Thus, it follows from $f''(k) < 0$ for all $k > 0$ that the optimal capital level must be smaller under **A2** or **A3** than under **A1**. ■

The intuition behind that result goes as follows. In a Chakraborty-type economy, raising capital per worker tends also to increase, through $h(k)$, the proportion π of survivors in the cohort, and, thus, the size of the retirees in the total population (unlike what happens in economies where longevity is exogenous). Hence, under that additional effect, the level of k maximizing steady-state consumption per head must be inferior to its level under exogenous longevity.[4]

Note that, as far as Proposition 1 and its corollary are concerned, the postulate according to which individual health expenditures do not constitute a choice variable for the social planner plays a crucial role. Actually, if health spending were directly chosen by the social planner, then these would not be automatically related to capital per worker as these are here. As a consequence, the traditional Golden Rule would still prevail, but now there would be two optimality conditions (one for the capital stock, and one for the health spending) instead of one, so that the traditional Golden Rule would still prevail, but would no longer be sufficient for optimality.[5]

Finally, it should be stressed that the condition stated in Proposition 1 is not necessarily satisfied by a *unique* capital level. Clearly, several capital levels may satisfy that condition, and lead to local maxima of steady-state consumption, unlike what prevails in the standard model, where steady-state consumption exhibits only a single maximum in capital per worker. More precisely, only in case **A1**, we have $\phi''(k) = f''(k)$, which is negative, so that $\phi(k)$ is concave for all values of k. Under **A2** or **A3**, the second order derivatives $\phi''(k)$ depends on the second order derivatives of the survival function π. Moreover, under **A2**, $\phi''(k)$ also depends on the third order derivative of the production function via ω'', for which we have no reasonable assumption to impose. It is therefore not necessarily negative, contrary to what prevails when health spending are exogenous.

An example illustrates this point. Take a Cobb-Douglas production function $f(k) = 10k^{1/3}$. Assume **A2** with a tax rate $\tau = 0.15$, full depreciation of capital $\delta = 1$ and constant population $n = 0$. The Golden Rule is $\bar{k}_{\mathrm{GR}} = 6.086$. Consider two different cases for the survival function $\pi(h)$. One concave, $\pi(h) = h/(1+h)$, as in Chakraborty (2004), and one logistic $\pi(h) = 0.0001/(0.0001 - \exp(-10h))$. The left panel of Fig. 1 shows that net production is globally concave in the first example and the first order condition $\phi'(k^\star) = 0$ gives a global maximum with $k^\star = 4.42$. In the right panel, net production is concave–convex–concave. We have two local maxima, 0.428 and 5.182, and the one with the smallest k is the global maximum.

4. The health-adjusted optimal capital

When interpreting the above results, one may argue that the maximization of consumption per head per period of life is not an adequate goal in the present context, in which individual health and longevity are endogenous. True, in Chakraborty (2004), health spending are purely longevity-enhancing spending, which do not contribute to the quality of life-periods, but only to the length or quantity of life. Hence, on that basis, health spending cannot be treated as part of the consumption to

[4] Note that health expenditures have also a direct effect on the optimal capital, regardless of the reactivity of the probability of survival to health spending (i.e. even if $\pi(h)$ is flat).

[5] Another way to express this is to say that the present analysis is at the second-best level, as this presupposes that the social planner cannot choose health spending directly.

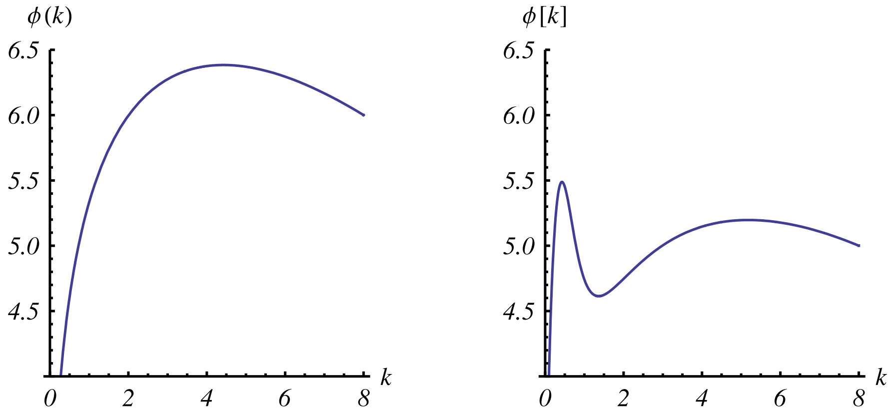

Fig. 1. Net production $\phi(k)$ with a concave survival function $\pi(h)$ (left panel) and with a logistic one (right panel).

be maximized. However, nothing forces us to adhere here to such a restrictive view of health spending. Actually, it is likely that health spending contribute not only to raise longevity (i.e. the quantity of life), but, also, to improve its quality.

If that critique is correct, then the above analysis, which concentrates on maximizing steady-state consumption per head, cannot serve as an adequate goal, and should be complemented by some concerns for the impact of health spending on the quality of life-periods. That intuition can be formalized simply by assuming that the level of health spending affects directly the efficiency units of consumption. Clearly, when one is healthy, one enjoys consumption more than when one is sickly, so that consumption, expressed in efficiency units, depends on the health level. Following that intuition, let us suppose that the level of health spending per worker h_t augments the efficiency units of consumption as follows:

$$\tilde{c}_t = g(h_t)c_t \tag{11}$$

with $g' \geq 0$ and $g'' < 0$. The function $g(h_t)$ captures the effect of health spending in the first period on the enjoyment of consumption.[6] For analytical convenience, we assume that the function $g(h_t)$ is bounded from below and from above: $\lim_{h_t \to 0+} g(h_t) = \check{g} > 0$ and $\lim_{h_t \to \infty} g(h_t) = \hat{g} > 0$, and that $\lim_{h_t \to \infty} g'(h_t) = 0$.

Let us now derive the level of capital per worker maximizing steady-state consumption per head expressed in efficiency units.

Steady-state consumption per head in efficiency units, denoted by $\varphi(k)$, can be written as

$$\varphi(k) = g(h(k))\phi(k) = [f(k) - k(\delta + n) - h(k)]\, g(h(k)) \frac{1+n}{1+n+\pi(h(k))}. \tag{12}$$

The existence of a (finite positive) level of k maximizing $\varphi(k)$ is guaranteed provided $\varphi'(k)$ is positive for low capital levels and negative for high ones. Under the assumptions made on the function $g(h)$, those conditions coincide with the conditions guaranteeing $\phi'(k)$ is positive when k tends to 0, but negative when it tends to $+\infty$.

Proposition 2. *Assume that* $\{n, \delta, f(k), \pi(h)\}$ *satisfy:*

$$\lim_{k \to 0+} f'(k) > \delta + n + \lim_{k \to 0+} \left(\frac{\pi'(h(k))\,[f(k) - h(k)]}{1+n+\pi(h(k))} + 1 \right) h'(k)$$
$$\lim_{k \to +\infty} f'(k) < \delta + n.$$

[6] For simplicity, we assume that health spending when being young affect the efficiency of consumption identically at all ages.

Then, there exists a capital per worker $k^{\star}_{\mathrm{H}}$ maximizing consumption per head in efficiency units in $\mathbb{R}_{+}$. Such a level satisfies $\varphi'(k^{\star}_{\mathrm{H}}) = 0$:

$$f'(k^{\star}_{\mathrm{H}}) = \delta + n + h'(k^{\star}_{\mathrm{H}}) \left(1 + \frac{\pi'(h(k^{\star}_{\mathrm{H}}))\ (f(k^{\star}_{\mathrm{H}}) - k^{\star}(\delta + n) - h(k^{\star}_{\mathrm{H}}))}{1 + n + \pi(h(k^{\star}_{\mathrm{H}}))} - \frac{g'(h(k^{\star}_{\mathrm{H}}))(f(k^{\star}_{\mathrm{H}}) - k^{\star}(\delta + n) - h(k^{\star}_{\mathrm{H}}))}{g(h(k^{\star}_{\mathrm{H}}))} \right). \tag{13}$$

Proof. Under the assumptions $\lim_{h \to 0+} g(h) = \check{g} > 0$, $\lim_{h \to \infty} g(h) = \hat{g} > 0$, and $\lim_{h \to \infty} g'(h) = 0$, the conditions $\lim_{k \to 0+} \phi'(k) > 0$ and $\lim_{k \to +\infty} \phi'(k) < 0$ are sufficient to obtain an interior maximum. Hence, under the two conditions of the Proposition, the function $\varphi(k)$ reaches a maximum between 0 and $+\infty$. Given that the function $\varphi(k)$ is continuous, this maximum satisfies $\varphi'(k^{\star}_{\mathrm{H}}) = 0$. ■

To interpret the health-adjusted optimal capital $k^{\star}_{\mathrm{H}}$, let us first focus on the special case where the enjoyment of consumption is not affected by health status, and, thus, by health spending. In that special case, we have $g'(k^{\star}_{\mathrm{H}}) = 0$, so that the health-adjusted optimal capital collapses to optimal capital derived in Proposition 1.

However, in the general case where $g'(k^{\star}_{\mathrm{H}}) > 0$, the health-adjusted optimal capital is likely to differ from the previously defined optimal capital.

Corollary 2. *Under* **A1** *we have that $k^{\star}_{\mathrm{H}} = k^{\star}$.*
Under **A2** *or* **A3** *we have that $k^{\star}_{\mathrm{H}} > k^{\star}$.*

Proof. Actually, contrasting the Propositions 1 and 2 suggests that $k^{\star}_{\mathrm{H}}$ exceeds $k^{\star}$, as the additional term in the condition of Proposition 2 always leads to a lower RHS under **A2** or **A3**, so that, under those assumptions, only a larger capital per worker can meet the health-adjusted optimal policy. ■

The intuition behind that result goes as follows. In the present setting, gains in the efficiency of consumption thanks to a better health play as a kind of technological progress, which allows a higher quality of life to be reached. But the important thing is that this kind of technological progress is here endogenous, as this depends on capital accumulation through health spending. That additional effect of capital accumulation plays in favor of a higher level of capital per worker.

Thus, shifting from the goal of maximizing steady-state consumption per head to steady-state health-adjusted consumption per head tends to raise the target level of capital. As capital accumulation makes consumption more efficient, this is an additional reason for accumulating more, to an extent that depends on the function $g(h)$. Note, however, that whether the health-adjusted optimal capital level exceeds the standard Golden Rule capital level under exogenous longevity (i.e. under **A1**) is ambiguous, as the introduction of health tends to affect both the number of retirees and the efficiency of consumption.

5. Lifetime optimal capital

Besides the modification introduced above, there is another reason why the maximization of (possibly health-adjusted) consumption per head per period of life is not an adequate goal. Actually, that goal only captures the intensity of life's goodness (i.e. in per period terms), but not the goodness of life *as a whole*. More precisely, it can be argued that the standard Golden Rule ceases to be an appropriate goal once longevity becomes a variable, that is, once there appears some trade-offs between consumption and longevity. The introduction of longevity, by making lifetime welfare dependent on two – rather than one – variables, would thus make the standard Golden Rule – focusing on a single dimension of welfare – inadequate.

Although that criticism of the relevancy of the notions of optimal capital defined above in the context of endogenous longevity is certainly appealing, it is far from clear that this suffices to make them irrelevant. Paying an exclusive attention to the intensity of life's goodness may still be defendable, even in a context where longevity is variable. This defendability comes from the singular nature of longevity as a dimension of welfare. It is only through the passage of time that longevity, unlike consumption, takes its value. But a social planner may want to maximize the level of welfare *per period lived*, and such an objective, which does not take longevity into account, does not seem implausible at all. This would consist of an intensity view of welfare, in contrast with (more standard) complete view of welfare. It is not obvious that such an intensity view of welfare can be *a priori* regarded as more or less plausible than the complete view.

Having stressed this, it remains true that the intuitive appeal of the complete view of welfare invites the development of an alternative concept of optimal capital, which would incorporate longevity achievements. This is the task of the present section.

Under the assumption of endogenous lifetime, a plausible – possibly more adequate – goal may be the maximization not of consumption per head, but of *expected lifetime* consumption per head. Such an objective has the virtue to take longevity into account, but without having to rely on assumptions on preferences. Let us now derive the capital level maximizing expected lifetime consumption per head, defined as the consumption per head multiplied by life expectancy:

$$\frac{C_t}{L_t + \pi_t L_{t-1}}(1+\pi_t) = \frac{(1+n)(1+\pi_t)}{1+n+\pi_t} c_t. \tag{14}$$

Expected lifetime consumption per head at the steady-state, denoted by $\psi(k)$, is:

$$\psi(k) \equiv c\frac{(1+n)(1+\pi)}{1+n+\pi} = [f(k) - k(\delta+n) - h(k)]\frac{(1+n)(1+\pi)}{1+n+\pi}. \tag{15}$$

The existence of a (finite positive) level of k maximizing $\psi(k)$ would be guaranteed under the conditions insuring that $\psi'(k)$ is positive for low capital levels but negative for high ones. Those conditions coincide with the ones implying $\phi'(k) > 0$ for k tending towards 0 and $\phi'(k) < 0$ for k tending towards $+\infty$.

Proposition 3. *Under the conditions of Proposition 1, there exists a capital per worker maximizing the expected lifetime consumption per head at the steady-state. That lifetime optimal capital level, denoted by $k_L^\star$, is such that:*

$$f'(k_L^\star) = \delta + n + h'(k_L^\star)\left(1 - \frac{n\pi'(h(k_L^\star))(f(k_L^\star) - k_L^\star(\delta+n) - h(k_L^\star))}{(1+\pi(h(k_L^\star)))(1+n+\pi(h(k_L^\star)))}\right). \tag{16}$$

Corollary 3. *Under* **A1** *we have that $k_L^\star = k^\star$.*
Under **A2** *or* **A3** *we have that $k_L^\star > k^\star$.*

Proof. Under **A1**, $h'(k) = 0$, (16) is equivalent to (10) which implies that $k_L^\star = \bar{k}_{GR}$.

Under **A2** or **A3**, different cases should be distinguished, depending on the sign of n. Under $n = 0$, the RHS of (16) is smaller than the RHS of (9), so that the lifetime optimal capital level must exceed the optimal capital level $k^\star$. Moreover, given that the RHS of (16) is $\delta + h'(h) > \delta$, it follows that the lifetime optimal capital is here lower than under exogenous health spending.

Under $n > 0$, the RHS of (16) is now smaller than $\delta + n + h'(k)$, from which one can see that $k_L^\star$ must necessarily exceed the optimal capital level $k^\star$. However, $k_L^\star$ may or may not exceed the Golden Rule capital under exogenous longevity $\bar{k}_{GR}$.

A similar reasoning could be applied to the case $-1 < n < 0$. The difference between the RHS of (16) and the one of (9) is a factor $-n/(1+\pi)$. That factor is, in the case $-1 < n < 0$, positive but lower than 1. Hence we also have $k_L^\star > k^\star$ in this case. ■

Hence, when we take care of lifetime consumption rather than instantaneous consumption, the optimal stock of capital is larger.

For completeness, one can also derive the health-adjusted lifetime optimal capital, which states the condition under which capital per worker maximizes the expected lifetime consumption per head, expressed in efficiency units. Such a concept would incorporate simultaneously the two shortcomings of the optimal capital rule in the context of endogenous health: an ignorance of the impact of health on the *quality* of life-periods, and on the *quantity* of life-periods.

Expected lifetime consumption per head at the steady-state, if expressed in efficiency units, is denoted by:

$$\zeta(k) \equiv \tilde{c}\frac{(1+n)(1+\pi)}{1+n+\pi} = g(h(k))\left[f(k) - k(\delta+n) - h(k)\right]\frac{(1+n)(1+\pi)}{1+n+\pi}. \tag{17}$$

The existence of a (finite positive) level of k maximizing $\zeta(k)$ would be guaranteed provided $\zeta'(k)$ is positive for low capital levels but negative for high ones. Under the postulated function g(h), those conditions coincide with the ones implying $\phi'(k) > 0$ for k tending towards 0 and $\phi'(k) < 0$ for k tending towards $+\infty$.

Proposition 4. *Under the conditions of Proposition 1, there exists a capital per worker maximizing the expected lifetime health-adjusted consumption per head at the steady-state. That lifetime health-adjusted optimal capital level, denoted by $k^\star_{\mathrm{LH}}$, is such that:*

$$\begin{aligned} f'(k^\star_{\mathrm{LH}}) = \delta + n + h'(k^\star_{\mathrm{LH}})\Bigg(1 &- \frac{n\pi'(h(k^\star_{\mathrm{LH}}))(f(k^\star_{\mathrm{LH}}) - k^\star_{\mathrm{LH}}(\delta+n) - h(k^\star_{\mathrm{LH}}))}{(1+\pi(h(k^\star_{\mathrm{LH}})))(1+n+\pi(h(k^\star_{\mathrm{LH}})))} \\ &- \frac{g'(h(k^\star_{\mathrm{LH}}))(f(k^\star_{\mathrm{LH}}) - k^\star_{\mathrm{LH}}(\delta+n) - h(k^\star_{\mathrm{LH}}))}{g(h(k^\star_{\mathrm{LH}}))}\Bigg). \end{aligned} \tag{18}$$

Proof. Under the assumptions $\lim_{h\to 0+} g(h) = \check{g} > 0$, $\lim_{h\to\infty} g(h) = \hat{g} > 0$, and $\lim_{h\to\infty} g'(h) = 0$, the conditions $\lim_{k\to 0+}\phi'(k) > 0$ and $\lim_{k\to+\infty}\phi'(k) < 0$ are sufficient to obtain an interior maximum. Hence, under the two conditions of the Proposition, the function $\zeta(k)$ reaches a maximum between 0 and $+\infty$. Given that the function $\zeta(k)$ is continuous, this maximum satisfies $\zeta'(k^\star_{\mathrm{H}}) = 0$. ■

Here again, assuming that health spending do not affect the efficiency of consumption (i.e. $g'(h) = 0$) would lead us back to the lifetime optimal capital. However, under the general case $g'(h) > 0$, $k^\star_{\mathrm{LH}}$ is likely to differ from $k^\star_{\mathrm{L}}$.

Corollary 4. *Under* **A1** *we have that $k^\star_{\mathrm{LH}} = k^\star_{\mathrm{L}} = k^\star_{\mathrm{H}} = k^\star$.*
Under **A2** *or* **A3** *we have that $k^\star_{\mathrm{LH}} > k^\star_{\mathrm{L}} > k^\star$ and $k^\star_{\mathrm{LH}} > k^\star_{\mathrm{H}} > k^\star$.*

Proof. Actually, we necessarily have, under **A2** or **A3**, $k^\star_{\mathrm{LH}} > k^\star_{\mathrm{L}}$, as the inequality $n\pi'(k)g(k) + (1+\pi(k))g'(k)(1+n+\pi(k)) > 0$ always holds under **A2** or **A3**. Regarding the inequality $k^\star_{\mathrm{LH}} > k^\star_{\mathrm{H}}$, this follows from merely contrasting Propositions 2 and 4 under **A2** or **A3**. ■

Thus, in comparison with the lifetime optimal capital, the lifetime health-adjusted optimal capital is always larger, which is in line with the previous findings in Section 4, where the introduction of consumption efficiency concerns, playing like an endogenous technological progress, made capital accumulation more desirable *ceteris paribus*. Moreover, given that the extension of the time horizon to the agent's expected lifetime had already, under **A2** or **A3**, the effect to raise the optimal capital level, it does not come as a surprise that the same extension yields, in terms of health-adjusted consumption, the same effect, so that $k^\star_{\mathrm{LH}} > k^\star_{\mathrm{H}}$.

6. Concluding remarks

Endogenizing health spending and longevity does not leave the optimal capital accumulation unchanged. Clearly, if the goal is the maximization of steady-state consumption per head, the optimal capital level is inferior to its level under exogenous longevity (that is, the Golden Rule), as raising k increases the elderly population size through a higher survival to the second period.

Accounting for the effect of health status on the enjoyment of consumption through the measurement of consumption in efficiency units tends, on the contrary, to raise the target capital level, as shown by the derivation of the health-adjusted optimal capital, which points to a higher capital level than the one under purely longevity-enhancing health spending. In that modified framework, capital accumulation acts, through health spending, as a source of consumption efficiency gains, which supports more capital accumulation.

Given that one may not be fully satisfied with the goal of maximization of (possibly health-adjusted) consumption per head (as this leaves longevity aside), we also characterize the lifetime optimal capital level, which maximizes the expected lifetime consumption per head. The lifetime optimal capital level is superior to the standard optimal capital level under endogenous longevity.

It should be stressed that this study does not rely on assumptions on preferences. That independence from preferences can be regarded as either a weakness or as a strength of the present study. True, the best social objective consists ideally of the Golden Age, i.e. the capital per worker maximizing steady-state *lifetime utility*. However, its definition is not trivial, as this requires to deal with some necessary assumptions on the utility of death, unlike what was needed in the study of the optimal and lifetime optimal capital levels.[7] In the light of the difficulty to fix some level to the utility of death, avoiding assumptions of that kind by focusing on consumption may well be a virtue.

But despite its focus on consumption rather than utility, the present study allowed us to highlight some problematic issues that will still have to be dealt with when considering the Golden Age problem (i.e. taking utilities into account). For instance, the question of the adequate time horizon to be taken into account (i.e. instantaneous view against complete life view) will remain, as well as the question of the treatment of health spending by the planner (i.e. a choice variable or not).

In sum, this note constitutes only a first step in the normative analysis of the relation between capital accumulation and survival conditions. The examination of the Golden Age under various assumptions on the utility of death is on our research agenda.

Acknowledgements

De la Croix acknowledges the financial support of the Belgian French speaking community (Grant ARC 0 "Sustainability") and the Belgian Federal Government (Grant PAI P6/07 "Economic Policy and Finance in the Global Economy: Equilibrium Analysis and Social Evaluation"). This research is part of the ARC project 09/14-018 on "sustainability" (French speaking community of Belgium).

Appendix. Summary of the important rules

Exogenous longevity

Golden Rule $\bar{k}_{\mathrm{GR}}$:

$$f'(\bar{k}_{\mathrm{GR}}) = \delta + n.$$

Endogenous longevity

Optimal capital $k^\star$:

$$f'(k^\star) = \delta + n + h'(k^\star)\left(1 + \frac{\pi'(h(k^\star))(f(k^\star) - k^\star(\delta+n) - h(k^\star))}{1+n+\pi(h(k^\star))}\right).$$

Health-adjusted optimal capital $k^\star_{\mathrm{H}}$:

$$f'(k^\star_{\mathrm{H}}) = \delta + n + h'(k^\star_{\mathrm{H}})\left(1 + \frac{\pi'(h(k^\star_{\mathrm{H}}))\,(f(k^\star_{\mathrm{H}}) - k^\star(\delta+n) - h(k^\star_{\mathrm{H}}))}{1+n+\pi(h(k^\star_{\mathrm{H}}))} - \frac{g'(h(k^\star_{\mathrm{H}}))(f(k^\star_{\mathrm{H}}) - k^\star(\delta+n) - h(k^\star_{\mathrm{H}}))}{g(h(k^\star_{\mathrm{H}}))}\right).$$

Lifetime optimal capital $k^\star_{\mathrm{L}}$:

$$f'(k^\star_{\mathrm{L}}) = \delta + n + h'(k^\star_{\mathrm{L}})\left(1 - \frac{n\pi'(h(k^\star_{\mathrm{L}}))(f(k^\star_{\mathrm{L}}) - k^\star_{\mathrm{L}}(\delta+n) - h(k^\star_{\mathrm{L}}))}{(1+\pi(h(k^\star_{\mathrm{L}})))(1+n+\pi(h(k^\star_{\mathrm{L}})))}\right).$$

7 Note that other, more empirical papers, such as Murphy and Topel (2006) and Hall and Jones (2007), are also making assumptions on the utility of death, as this is a necessary requirement for the monetization of longevity gains.

Health-adjusted lifetime optimal capital $k^{\star}_{\mathrm{LH}}$:

$$f'(k^{\star}_{\mathrm{LH}}) = \delta + n + h'(k^{\star}_{\mathrm{LH}}) \left(1 - \frac{n\pi'(h(k^{\star}_{\mathrm{LH}}))(f(k^{\star}_{\mathrm{LH}}) - k^{\star}_{\mathrm{LH}}(\delta + n) - h(k^{\star}_{\mathrm{LH}}))}{(1 + \pi(h(k^{\star}_{\mathrm{LH}})))(1 + n + \pi(h(k^{\star}_{\mathrm{LH}})))} - \frac{g'(h(k^{\star}_{\mathrm{LH}}))(f(k^{\star}_{\mathrm{LH}}) - k^{\star}_{\mathrm{LH}}(\delta + n) - h(k^{\star}_{\mathrm{LH}}))}{g(h(k^{\star}_{\mathrm{LH}}))}\right).$$

References

Boucekkine, R., 2008. Growth economics of epidemics: A review of the theory. Mathematical Population Studies 15, 1–26.
Cervellati, M., Sunde, U., 2005. Human capital formation, life expectancy, and the process of development. American Economic Review 95, 1653–1672.
Chakraborty, S., 2004. Endogenous lifetime and economic growth. Journal of Economic Theory 116, 119–137.
Chakraborty, S., Das, M., 2005. Mortality, human capital and persistent inequality. Journal of Economic Growth 10, 159–192.
de la Croix, D., Licandro, O., 2007. The child is father of the man — implications for the demographic transition, CORE Discussion paper 2007/72.
de la Croix, D., Michel, P., 2002. A Theory of Economic Growth. Dynamics and Policy in Overlapping Generations. Cambridge University Press, Cambridge.
Diamond, P., 1965. National debt in a neoclassical growth model. American Economic Review 55 (5), 1126–1150.
Hall, R., Jones, C., 2007. The value of life and the rise in health spending. Quarterly Journal of Economics 122 (1), 39–72.
Jones, C., 2001. Was an industrial revolution inevitable? Economic growth over the very long run. Advances in Macroeconomics 1 (2), 1–45.
Murphy, K., Topel, R., 2006. The value of health and longevity. Journal of Political Economy 114 (5), 871–904.
Phelps, E., 1961. The golden rule of accumulation: A fable for growthmen. American Economic Review 51 (4), 638–643.
Phelps, E., 1965. Second essay on the golden rule of accumulation. American Economic Review 55 (4), 793–814.
Tamura, R., 2006. Human capital and economic development. Journal of Development Economics 79 (1), 26–72.