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On the mathematical stability of stratified flow models with local turbulence closure schemes

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Abstract Occasionally, numerical simulations using local turbulence closure schemes to estimate vertical turbulent fluxes exhibit small-scale oscillations in space, causing the eddy coefficients to vary over several orders of magnitude on short distances. Theoretical developments suggest that these spurious oscillations are essentially due to the way the eddy coefficients depend on the vertical gradient of the model's variables. An instability criterion is derived based on the assumptions that the artefacts under study are due to the development of small-amplitude, small time- and space-scale perturbations of a smooth solution. The relevance of this criterion is demonstrated by applying it to a series a closure schemes, ranging from the Pacanowski–Philander formulas to the Mellor–Yamada level 2.5 model.

Keywords Marine modelling · Vertical mixing · Turbulence closure · Stability

1 Introduction

A number of present-day models of geophysical and environmental fluid flow have recourse to a Fourier–Fick parameterisation of the vertical fluxes of momentum, heat, and dissolved constituents: the flux of the relevant quantity is expressed as the product of its vertical gradient and a suitably defined eddy coefficient. The simplest closure assumption consists in assuming that the eddy coefficients are constant, an approach which was shown to be inappropriate for studying stratified marine and oceanic flows (e.g. Ruddick et al. 1995; Goosse et al. 1999). To consider flows with vertical density contrasts, parameterisations were suggested in which the eddy coefficients are functions of the Richardson number (Munk and Anderson 1948; Pacanowski and Philander 1981). Clearly, this option is better, but is not always able to properly capture the main processes governing the evolution of turbulent motions. This is why more complex models are often preferred, such as the k, $k-\varepsilon$ or Mellor–Yamada models (Mellor and Yamada 1982; Rodi 1987; Burchard 2002a).

The original Mellor–Yamada level 2.5 closure scheme (Yamada 1977) is known to be prone to instability. As mentioned by Mellor and Yamada (1982), "for some model simulations, a discontinuity in velocity could develop and persist" causing large-amplitude,

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quasi-steady, spatial oscillations in the eddy coefficients. Several remedies were suggested for addressing that problem: (1) applying a numerical filter (Mellor 2003), (2) modifying the turbulence model equations (Hassid and Galperin 1983; Galperin et al. 1988) and (3) modifying the numerical scheme (Girard and Delage 1990; Davies and Jones 1991; Burchard 2002b). The most popular of these remedies consists of using the quasi-equilibrium version of the stability functions (Galperin et al. 1988). So, although the possible occurrence of spurious oscillations and a series of remedies have been known for almost two decades, the cause of this problem has remained somewhat elusive (Deleersnijder and Luyten 1994; Burchard and Deleersnijder 2001; Deleersnijder and Burchard 2003). For instance, the related stability – or instability – criterion is yet to be established. It is even unclear whether the instability causing the spurious oscillations is of a mathematical or a numerical nature.

To our knowledge, the first investigation of the stability of turbulence closure schemes depending on the Richardson number was by Phillips (1972). He found that under some conditions such schemes could allow initially small disturbances in the density gradient to amplify. Kranenburg (1980, 1982) studied similar problems and performed a linear stability analysis of one- and two-dimensional disturbances in turbulent density-stratified shear flow. A stability criterion depending on the turbulent Prandtl number was derived by expanding the model variables about a reference state. Similar studies were performed by Brown and Pandolfo (1982), Davies (1983) and Girard and Delage (1990). They noticed that eddy parameterisations depending on the gradient of the model solution could allow instabilities due to the mathematical formulation of the model rather than on the details of its numerical discretisation. Common to all these studies is the use of rather complex mathematical developments to study the stability of rather simple turbulence models. The mathematical complexity of such methods probably prevented their application to more elaborate turbulence closures, like the Mellor–Yamada level 2.5 closure.

In this work, we present a general theoretical development suggesting that the spurious oscillations are due to the way the eddy coefficients depend on the gradient of the dependent variables. This leads to an instability criterion that is believed to be relevant for a large family of turbulence closures (Section 2). Then, in Section 3, this theory is applied in a cursory way to numerical simulations of the Kato and Phillips experiment (Kato and Phillips 1969) in which the eddy coefficients are successively obtained from a Richardson number-dependent expression and two versions of the Mellor–Yamada level 2.5 closure scheme.

2 Theory

As in many previous studies, it is assumed that a water column model is sufficient. Closure techniques commonly used in geophysical fluid flow studies are such that the eddy coefficients depend on the vertical gradient of the velocity and the buoyancy and, in some cases, on auxiliary variables, which satisfy partial differential equations. Therefore, the complete set of equations to be solved may be cast into the following generic form:

$$\frac{\partial \psi_m}{\partial t} = s_m + \frac{\partial}{\partial z}\left(\lambda_m \frac{\partial \psi_m}{\partial z}\right), \qquad m = 1, 2, \ldots, M, \quad (1)$$

where $\psi_m(t, z)$ is the m-th model variable and s_m and $\lambda_m > 0$ are the corresponding source/sink terms and eddy coefficient, respectively.

2.1 A simple example

As all eddy coefficients λ_m are strictly positive, it may be believed that Eq. 1 exhibits well-behaved solutions. This would most probably be true if the eddy coefficient were independent of the gradient of the solutions. To understand why this may cause problems when λ is dependent on the gradient of the solution, consider the simple "heat" equation

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left(\lambda \frac{\partial \psi}{\partial z}\right), \quad (2)$$

where the diffusivity $\lambda > 0$ is a function of $\psi_z = \frac{\partial \psi}{\partial z}$ only, i.e. $\lambda = \lambda(\psi_z)$. As the diffusivity is strictly positive, the solution ψ tends to remain bounded – for well-behaved boundary conditions – in the domain $z_1 \leq z \leq z_2$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{z_1}^{z_2} \psi^2 \,\mathrm{d}z = \left[2\lambda\psi\psi_z\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} 2\lambda(\psi_z)^2 \,\mathrm{d}z, \quad (3)$$

which is readily obtained from Eq. 2 by multiplication by 2ψ and subsequent integration. On the other hand, the gradient ψ_z is governed by the equation

$$\frac{\partial \psi_z}{\partial t} = \frac{\partial}{\partial z}\left(\tilde{\lambda} \frac{\partial \psi_z}{\partial z}\right),$$

where the "effective diffusivity" is

$$\tilde{\lambda} = \frac{\partial \lambda}{\partial \psi_z}\psi_z + \lambda. \quad (4)$$

As a result, the square of ψ_z also satisfies a relation similar to Eq. 3. However, since the effective diffusivity can now be negative, there is the potential for ψ_z to grow unbounded while ψ would remain bounded. If this were to occur, it is conceivable that the solution would exhibit finite-amplitude oscillations at arbitrarily small space scales. During the first stage of the development of the oscillations, it is possible that small-amplitude, small-scale spatial oscillations coexist with a finite-amplitude, large-scale solution. The appearance of these large wavenumber oscillations can be considered as an instability of the background large-scale state.

Such oscillations can easily be observed by numerically solving Eq. 2 with a diffusivity λ that depends on the gradient of the solution. As an example, the solution of Eq. 2 is computed in the domain $[-1, 1]$ for the following initial and boundary conditions:

$$\psi(t=0, z)=z, \quad \psi(t, z=-1)=-1, \quad \psi(t, z=1)=1.$$

In order to trigger instabilities, we assume that the diffusivity has the following unusual expression:

$$\lambda = \frac{1}{\left(\frac{\partial \psi}{\partial z}\right)^2},$$

which is always larger than zero. The corresponding effective diffusivity is thus

$$\tilde{\lambda} = \frac{-1}{\left(\frac{\partial \psi}{\partial z}\right)^2},$$

which is always smaller than zero. Therefore, when numerically computing the evolution of the solution, we can expect round-off errors in the initial condition to trigger instabilities. These instabilities will result in the gradient of the solution to grow unbounded. Figure 1 shows the unstable solution after 0.5 and 5 time units, as well as the initial condition $\psi = z$, which would have prevailed if the effective diffusivity $\tilde{\lambda}$ had not been negative. The grid resolution is equal to 0.01 length units. It is noteworthy that, while the gradient of the solution grows unbounded, the solution remains bounded. The numerical solution suggests that the grid-scale perturbations have the largest growth rate; this hypothesis is supported by the theoretical developments performed below.

2.2 General case

As long as the perturbation is small, linearisation of the equation is presumably legitimate. This suggests a method to investigate the stability of the solution of the system of Eq. 1 on which the present study is focused.

Let $\hat{\psi}_m(t, z)$ be a small-amplitude perturbation to the solution $\psi_m(t, z)$ of Eq. 1. By inserting $\psi_m + \hat{\psi}_m$ into Eq. 1 and subsequent linearisation, the perturbation vector may be seen to obey linear equations

$$\hat{\psi}_{m,t} = \sum_{n=1}^{M} \left[A_{mn}\hat{\psi}_n + (B_{mn} + C_{mn,z})\hat{\psi}_{n,z} + C_{mn}\hat{\psi}_{n,zz} \right] \tag{5}$$

where matrices A, B and C are based on the unperturbed solution:

$$A_{mn} = \frac{\partial s_m}{\partial \psi_n} + \left(\frac{\partial \lambda_m}{\partial \psi_n}\psi_{m,z} \right)_z,$$

$$B_{mn} = \frac{\partial s_m}{\partial \psi_{n,z}} + \frac{\partial \lambda_m}{\partial \psi_n}\psi_{m,z},$$

$$C_{mn} = \frac{\partial \lambda_m}{\partial \psi_{n,z}}\psi_{m,z} + \lambda_m \delta_{mn},$$

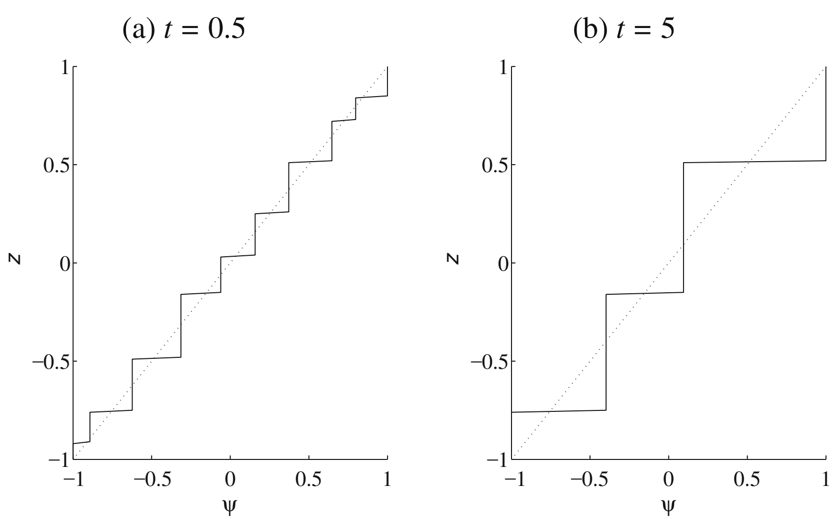

Fig. 1 Example of solution of the heat equation (Eq. 2) that exhibits instabilities when the diffusivity $\lambda = (\partial \psi/\partial z)^{-2}$ (*solid line*). In that case, the effective diffusivity is negative. The stable solution that would prevail for a well-behaved diffusivity depending only on the gradient of the solution is represented by a *dotted line*. Both solutions are shown after 0.5 and 5 time units

where $\psi_{m,z}$ and $\psi_{m,zz}$ denote the first- and the second-order derivatives with respect to the vertical coordinate z of ψ_m; $\delta_{m,n}$ is the Kronecker delta, which is equal to unity if $m = n$, and is zero otherwise.

The system of partial differential equations (Eq. 5) is linear, but it is difficult to solve analytically as the coefficients $A_{m,n}$, $B_{m,n}$ and $C_{m,n}$ depend on the reference solution $\psi_m(t, z)$. However, previous studies suggest that growing perturbations tend to take the form of small-scale oscillations in space that can quickly develop in time (Deleersnijder and Luyten 1994; Burchard and Deleersnijder 2001). Hence, if we assume that the time and space scales of the perturbation $\hat{\psi}_m(t, z)$ are much smaller than those of the unperturbed solution $\psi_m(t, z)$, we can do a scale analysis and show that Eq. 5 is dominated by the terms exhibiting the highest-order derivatives in time and space. It may thus be simplified to

$$\hat{\psi}_{m,t} = \sum_{n=1}^{M} C_{mn} \hat{\psi}_{n,zz}, \tag{6}$$

where the matrix C may be regarded as locally constant. Therefore, it is legitimate to consider locally – in space and time – a solution of the form

$$\hat{\psi}_m(t, z) = \Re[\Psi_m e^{-\sigma t + ikz}] \tag{7}$$

where Ψ_m and σ are complex constants, while k is a real wavenumber. Then, substituting Eq. 7 into Eq. 6 yields the system of algebraic equations

$$\sum_{n=1}^{M} (C_{mn} - \mu\delta_{mn})\, \Psi_n = 0, \qquad m = 1, 2, \ldots, M,$$

where $\mu = \sigma/k^2$. As this system is homogeneous, for a non-trivial solution to exist, it is necessary that the determinant of the matrix $C - \mu I$ be zero, where I is the identity matrix. Obviously, the roots μ of this algebraic equation are the eigenvalues of the matrix C. If the real part of at least one eigenvalue is negative, then small-amplitude, small-scale perturbations will grow. So, to avoid this kind of instability, it is desirable that the real part of every eigenvalue of the matrix C be positive for any value of its components.

The previous analysis allows us to derive an instability criterion, which simply indicates that small-scale perturbations are locally unstable if there exists at least one eigenvalue of the stability matrix C whose real part is negative. It should be noted that this criterion is just a sufficient condition for *instability* but not a necessary one, as it only considers small-scale oscillations. Hence, it is not general enough to allow us to derive a sufficient *stability* condition. Nonetheless, since instabilities usually take the form of small-scale oscillations in space, the instability criterion derived here should have some utility, although it lacks generality.

As $\sigma^{-1} = \mu^{-1}k^{-2}$, the smaller the space scale, i.e. the larger the wavenumber k, the smaller the timescale of the associated perturbation. In other words, the shorter the space scale of a perturbation, the faster it is likely to increase or decrease. Thus, the fastest-growing perturbations that a given numerical model can represent are likely to be characterised by length scales comparable to the grid size. As the grid size is reduced, we should expect these instabilities to grow. This implies that, although instabilities are due to the mathematical formulation of the closure, their development, if any, depends crucially on the details of the numerical implementation. Any numerical study should thus investigate the sensitivity of the solution to the grid size.

3 Application to Kato–Phillips flows

The stress-driven penetration of a horizontally homogeneous, turbulent layer into a stratified fluid which is initially at rest is a common benchmark for turbulence closures of marine models (Burchard 2002a). This test case, based on the Kato and Phillips (1969) experiment, is used to help investigate the stability of various turbulence closure schemes. Let u and $b = -g(\rho - \rho_0)/\rho_0$ represent the mean horizontal velocity and the buoyancy, where g, ρ and ρ_0 are the gravitational acceleration, the water density and a reference value of the latter, respectively. If t and z denote time and the vertical coordinate, which is increasing upwards, the equations governing the evolution of the velocity and the buoyancy read

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(K_M \frac{\partial u}{\partial z}\right), \tag{8}$$

$$\frac{\partial b}{\partial z} = \frac{\partial}{\partial z}\left(K_H \frac{\partial b}{\partial z}\right), \tag{9}$$

where K_M is the eddy viscosity while K_H is the eddy diffusivity. The domain of interest is defined by $t \geq 0$ and $-\infty < z \leq 0$, where $z = 0$ is the sea surface. At the initial instant, the velocity is zero, i.e. $u(t = 0, z) = 0$, while the buoyancy is a linear function of the vertical coordinate, i.e. $b\,(t = 0, z) = N^2 z$, where N denotes the initial Brunt–Vaisala frequency. There is no buoyancy flux across the ocean surface, i.e. no heat and mass flux, which implies that $\left[K_H \frac{\partial b}{\partial z}\right]_{z=0} = 0$. A constant wind stress of amplitude τ is imposed by means of the boundary condition $\left[K_M \frac{\partial u}{\partial z}\right]_{z=0} = u_*^2$, where u_* is the friction velocity, $u_* = (\tau/\rho_0)^{1/2}$. As in Deleersnijder and Luyten (1994), the following forcing parameters are selected: $u_* = 10^{-2}$ m s^{-1} and $N = 10^{-2}$ s^{-1}.

Equations 8 and 9 can be included in the system Eq. 1 by setting $\psi_1 = u$ and $\psi_2 = b$ so that the auxiliary variables, if any, correspond to $m > 2$. For all the closure models considered below, we will take $s_1 = s_2 = 0$, $\lambda_1 = K_M$ and $\lambda_2 = K_H$. To close the equations introduced above, it is still necessary to parameterise the eddy coefficients K_M and K_H.

3.1 Eddy coefficients depending only on *Ri*

To illustrate the theoretical developments above, a class of simple closure models is first considered, in which the eddy viscosity and eddy diffusivity depend only on the Richardson number, $Ri = b_z/(u_z)^2$. In this case, there is no auxiliary variable: only Eqs. 8 and 9 are to be dealt with, so that $M = 2$. Then, matrix C is

$$C = \begin{pmatrix} K_M - 2RiK'_M & K'_M/u_z \\ -2u_z Ri^2 K'_H & K_H + RiK'_H \end{pmatrix}$$

with $K'_M = \mathrm{d}K_M/\mathrm{d}Ri$ and $K'_H = \mathrm{d}K_H/\mathrm{d}Ri$. It should be noted that the eigenvalues of C depend only on the Richardson number and not on u_z. As a result, the instability criterion can be formulated in terms of Ri only.

For the standard versions of the Munk and Anderson (1948) and Pacanowski and Philander (1981) parameterisations, the eigenvalues of C are positive for any values of Ri. Let us for instance consider the Pacanowski–Philander closure:

$$K_M = \frac{K_M^0}{(1 + \beta_M Ri)^{\alpha_M}} + K_M^*,$$

$$K_H = \frac{K_M}{(1 + \beta_M Ri)^{\alpha_H}} + K_H^*,$$

where the constant $K_M^0 \gg K_M^*$ can be interpreted as the neutral value of the eddy viscosity, i.e. the value prevailing when there is no stratification. The constants K_M^* and K_H^* are the values of the viscosity and diffusivity that are prevailing for strong stratifications and are about one order of magnitude larger than molecular values. Finally, the coefficients α_M, α_H and β_M are usually taken to be equal to 2, 1 and 5, respectively. The corresponding eigenvalues of C are real and positive, so that no spurious eigenvalues should arise. This is in agreement with the marine science literature, in which no such problem with the Pacanowski–Philander closure model has ever been reported. This is also confirmed by the numerical results displayed in Fig. 2.

However, by slightly modifying these parameterisations, one of the eigenvalues of C can be made negative. We could for instance modify the parameter α_H of the Pacanowski–Philander closure by setting it equal to 5, instead of 1. In that case, for a range of values of the Richardson number, one eigenvalue of the matrix C is real and negative. Thus, it is not surprising that the eddy coefficients are somewhat jittery and that they are more so if the grid size is reduced (Fig. 2). It should be noted that the same behaviour is observed for the Munk and Anderson (1948) and Blanke and Delecluse (1993) closure schemes. These schemes are stable in their original formulation but can be rendered unstable by changing the value of some coefficients.

3.2 Mellor–Yamada level 2.5 model

In the Mellor and Yamada level 2.5 closure scheme (Yamada 1977; Mellor and Yamada 1982), the eddy viscosity and eddy diffusivity are parameterised as follows:

$$K_M = lqS_M,$$

$$K_H = lqS_H,$$

where l and q are, respectively, an appropriate length scale, called "turbulence macroscale", and a velocity scale, derived from the turbulence kinetic energy (TKE) $q^2/2$. These additional variables are found by solving the following closure equations:

$$\frac{\partial q^2}{\partial t} = 2(P_s + P_b - \varepsilon) + \frac{\partial}{\partial z}\left(K_q \frac{\partial q^2}{\partial z}\right), \tag{10}$$

$$\frac{\partial q^2 l}{\partial t} = E_1 l P_s + E_1 l P_b - l\varepsilon W + \frac{\partial}{\partial z}\left(K_q \frac{\partial q^2 l}{\partial z}\right), \tag{11}$$

where $P_s = K_M M^2$ is the shear production of TKE, $P_b = K_H N^2$ is the rate of conversion of TKE into potential energy, $\varepsilon = \frac{q^3}{B_1 l}$ represents the viscous dissipation of TKE and $W = 1 + E_2\left(\frac{l}{\kappa|z|}\right)^2$ is the so-called wall-proximity function. The eddy diffusivity of the turbulence model equations is $K_q = 0.2lq$. The remaining constants take the following values: $E_1 = 1.8$, $E_2 = 1.33$ and $B_1 = 16.6$. Finally, the stability functions S_M

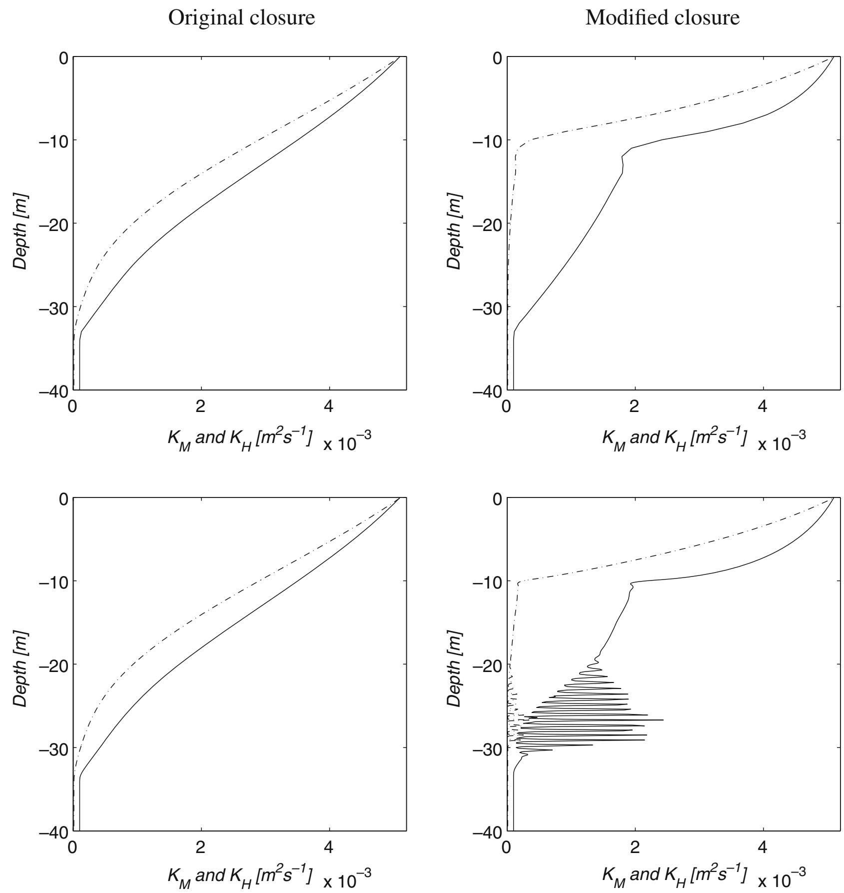

Fig. 2 Eddy viscosity (*dashed lines*) and eddy diffusivity (*dashed-dot lines*) as simulated after 30 h for a Kato–Phillips flow using the Pacanowski and Philander (1981) closure scheme. Grid sizes of 1 m (*top*) and 0.1 m (*bottom*) have been selected. *Left panels* depict results obtained with the standard values of the coefficient α_H, while the *right panels* correspond to $\alpha_H = 5$, in lieu of 1. The instabilities of the modified equations are clearly sensitive to the grid size

and S_H depend on the following dimensionless measures of the current shear and stratification:

$$G_M = \frac{l^2 u_z^2}{q^2},$$

$$G_H = -\frac{l^2 b_z}{q^2}.$$

The two additional partial differential Eqs. 10 and 11 can be included in the system Eq. 1 by setting $\psi_3 = q^2$ and $\psi_4 = q^2 l$. The eddy diffusivities related to the auxiliary variables are $\lambda_3 = \lambda_4 = K_q = 0.2lq$. Then, it is readily seen that matrix C may be expressed as

$$C = lq \begin{pmatrix} 2\frac{\partial S_M}{\partial G_M} G_M + S_M & \frac{\partial S_M}{\partial G_H}\frac{G_M u_z}{b_z} & 0 & 0 \\ 2\frac{\partial S_H}{\partial G_M}\frac{G_M b_z}{u_z} & \frac{\partial S_H}{\partial G_H} G_H + S_H & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{pmatrix}$$

There are two identical eigenvalues that are equal to $0.2lq$. The latter are obviously real and positive and can therefore be disregarded. The other two eigenvalues are the eigenvalues of the 2×2 matrix C_r:

$$C_r = lq \begin{pmatrix} 2\frac{\partial S_M}{\partial G_M} G_M + S_M & \frac{\partial S_M}{\partial G_H}\frac{G_M u_z}{b_z} \\ 2\frac{\partial S_H}{\partial G_M}\frac{G_M b_z}{u_z} & \frac{\partial S_H}{\partial G_H} G_H + S_H \end{pmatrix}.$$

Luckily, the eigenvalues of C_r depend on G_M and G_H only and not on u_z and b_z, so that we can evaluate a instability condition in a two-dimensional parameter space.

The expression of the stability functions S_M and S_H plays a crucial role on the stability of the whole turbulence closure. In the original expression of the Mellor–Yamada closure scheme, the stability functions were found to lead to oscillations in the eddy coefficients profiles. This problem led to the introduction of an alternative set of stability functions, namely the quasi-equilibrium stability functions of Galperin

Table 1 Coefficients of the stability functions for the original and quasi-equilibrium versions of the Mellor–Yamada (M-Y) level 2.5 turbulence closures

	Original M-Y closure		Quasi-equilibrium M-Y closure	
	$X = M$	$X = H$	$X = M$	$X = H$
n_X^{00}	0.6992	0.7400	0.3933	0.4939
n_X^{10}	−9.3395	−4.5341	−3.0858	0.0
n_X^{01}	0.0	0.9019	0.0	0.0
d_X^{10}	−36.7188	−36.7188	−40.8036	−34.6764
d_X^{01}	5.0784	5.0784	0.0	0.0
d_X^{20}	187.4409	187.4409	212.4693	0.0
d_X^{11}	−88.8395	−88.8395	0.0	0.0

et al. (1988). Both sets of stability functions can be expressed as:

$$S_X = \frac{n_X^{00} + n_X^{10} G_H + n_X^{01} G_M}{1 + d_X^{10} G_H + d_X^{01} G_M + d_X^{20} G_H^2 + d_X^{11} G_H G_M},$$

where $X = H$ or M. The values of these coefficients for the original Mellor–Yamada level 2.5 closure and for the quasi-equilibrium version of Galperin et al. (1988) are given in Table 1. For the original Mellor–Yamada level 2.5 closure, it was found that regions of exceedingly high shear could develop (Mellor and Yamada 1982; Hassid and Galperin 1983). The following redefinition of G_M was therefore introduced (Mellor and Yamada 1982):

$$G_M = \min\left(\frac{l^2 M^2}{q^2}, 0.825 - 25.0 G_H\right), \tag{12}$$

which amounts to constraining the growth of G_M and, hence, the subsequent reduction of S_M and K_M. When using the quasi-equilibrium model, the stability functions do not depend on M^2. It is, however, necessary to impose the following minor constraining conditions:

$$l^2 \leq \frac{0.28 q^2}{\max(0, N^2)}, \tag{13}$$

$$-0.28 \leq G_H \leq 0.0233. \tag{14}$$

Constraint Eq. 13 has been used by several authors to take into account the fact that stable stratification strongly limits the size of the turbulent eddies and, thus, the magnitude of l. This implies that G_H must satisfy the first part of condition Eq. 14. The second part of that condition expresses the need for G_M to be positive when the phenomena producing turbulent kinetic energy balance the dissipative effects (Galperin et al. 1988).

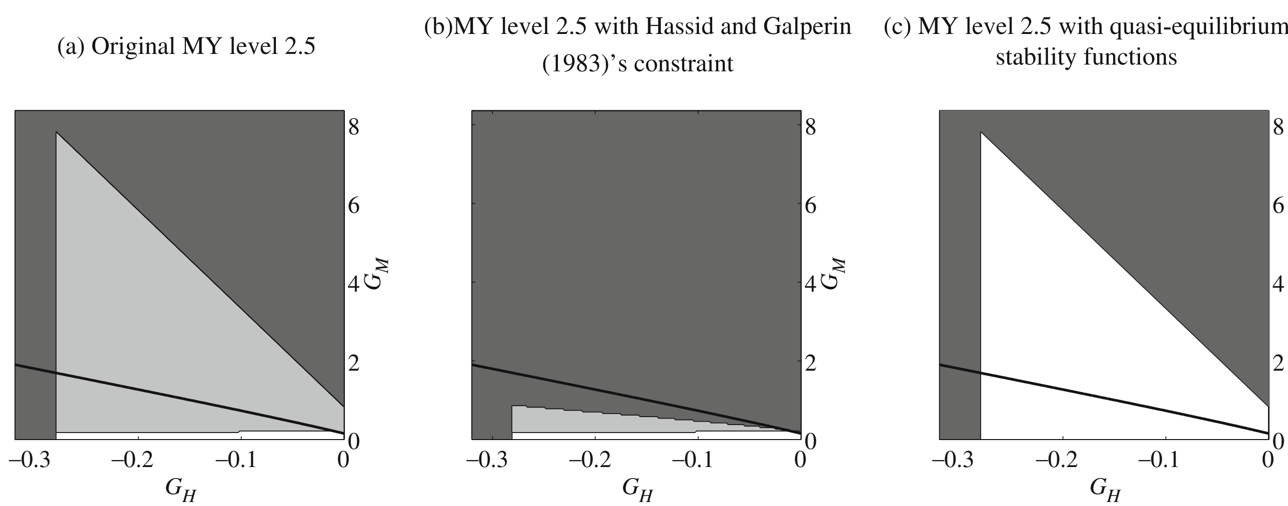

Fig. 3 Stability regions of the original Mellor–Yamada level 2.5 closure scheme (**a**), Hassid and Galperin (1983) version (**b**) and Galperin et al. (1988) quasi-equilibrium version (**c**). The area in *light grey* is where at least one of the eigenvalues of the matrix C has a negative real part. The stable and forbidden regions are represented in *white* and *dark grey*, respectively. The *solid black curve* shows the local equilibrium state (production = dissipation). For the original Mellor–Yamada level 2.5 closure, that curve lies almost entirely in the unstable region

The quasi-equilibrium version of the Mellor–Yamada level 2.5 closure has been introduced primarily to avoid the spurious oscillations observed in the eddy coefficients profiles obtained with the original scheme. Mellor (2003) argued that the spurious oscillations may also be prevented by applying a low pass filter to G_M. This suggestion is of interest since the theoretical developments above indicate that the smallest-scale perturbations are those which are likely to be the most unstable. However, as may be seen in Deleersnijder and Burchard (2003), Mellor's low-pass filter is not always sufficient to prevent spurious oscillations from arising. In addition, it must be kept in mind that applying such a filter amounts to a grid-dependent, noticeable modification of the equations to be solved.

Another remedy was suggested, apparently independently, by Hassid and Galperin (1983), Canuto et al. (2001) and Burchard and Deleersnijder (2001), which consists in using the original closure with condition 12 and the addition of a limitation on G_M so that an increase in the shear cannot be associated with a decrease in the momentum flux. This implies that G_M should satisfy the condition

$$\frac{\partial}{\partial |u_z|}|K_M u_z| \geq 0. \tag{15}$$

The latter is equivalent to

$$\frac{\partial (S_M G_M^{1/2})}{\partial G_M} \geq 0.$$

As

$$C_{11} = \frac{\partial K_M}{\partial u_z} u_z + K_M = 2qlG_M^{1/2}\frac{\partial (S_M G_M^{1/2})}{\partial G_M},$$

enforcing condition 15 amounts to requiring that the element C_{11} of the stability matrix C must remain positive. Unfortunately, this is not sufficient to guarantee that its eigenvalues are always positive, which is

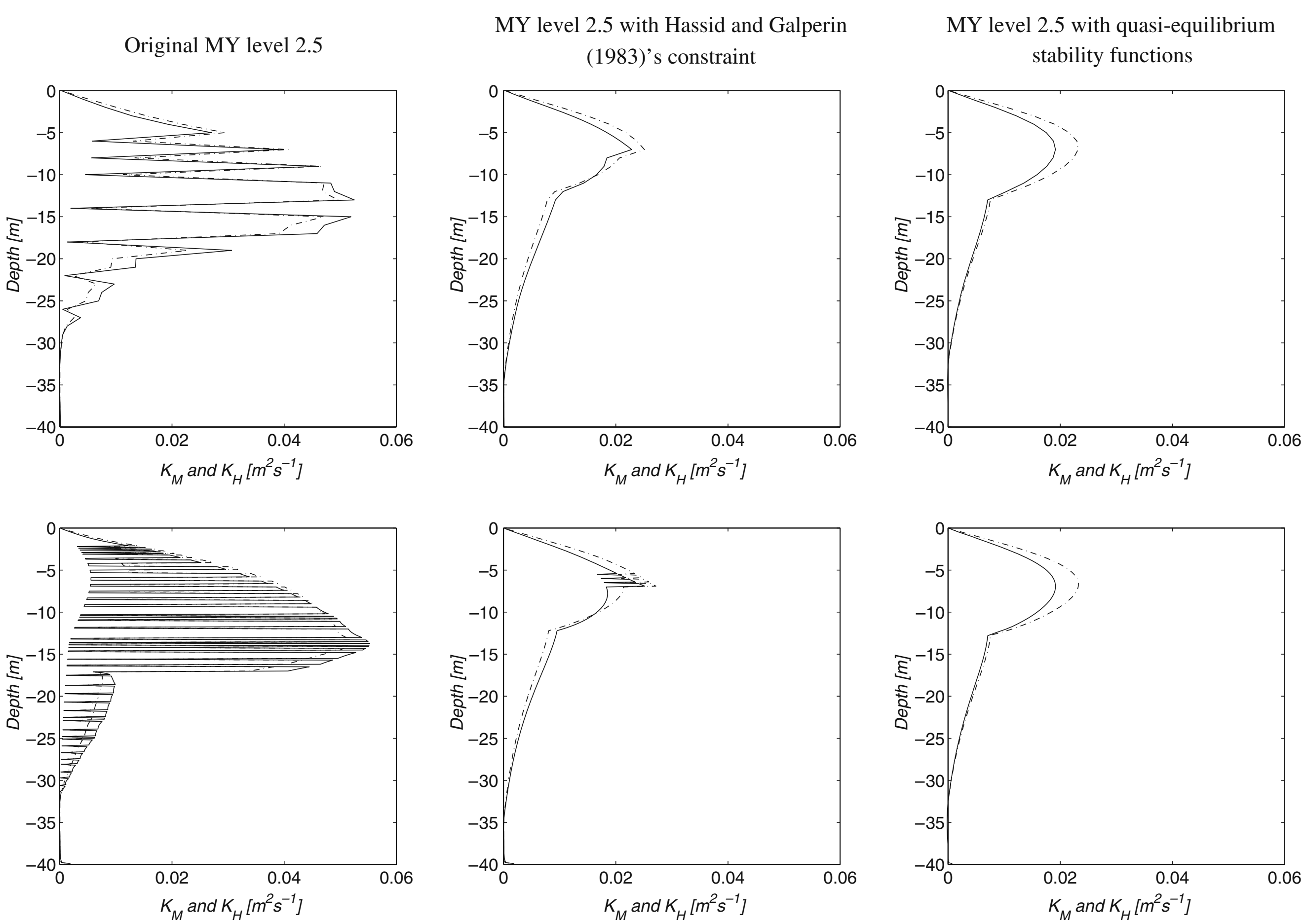

Fig. 4 Eddy viscosity (*dashed lines*) and eddy diffusivity (*dash–dot lines*) as simulated after 30 h for a Kato–Phillips flow using the original Mellor–Yamada level 2.5 closure scheme (*left*), the Mellor–Yamada level 2.5 closure with Hassid and Galperin (1983) constraint (*middle*) and quasi-equilibrium version of Galperin et al. (1988) (*right*). The grid size is set to 1 m (*top*) and 0.1 m (*bottom*)

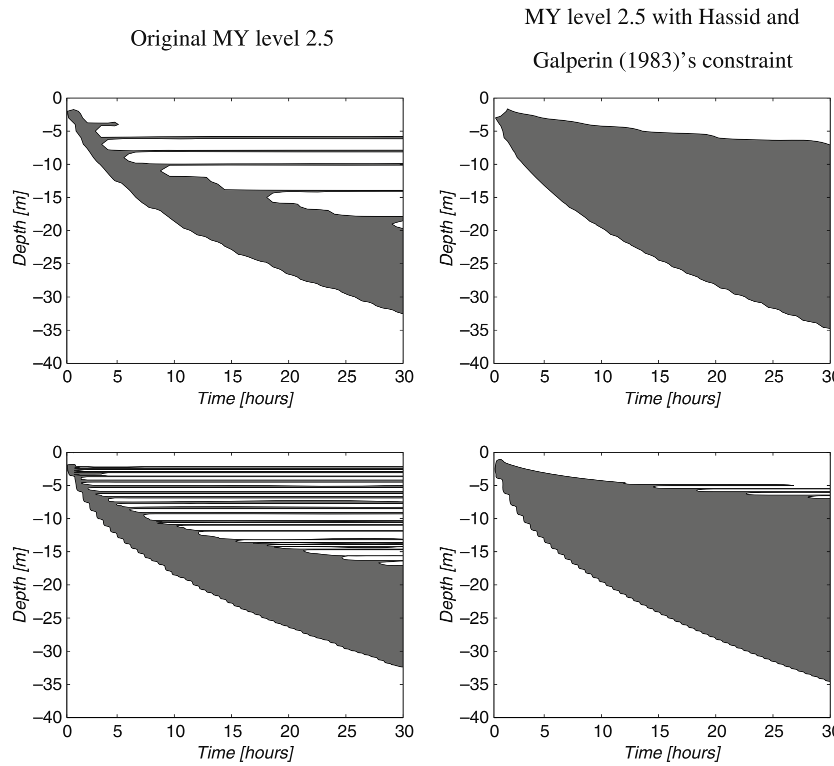

Fig. 5 Space-time region (in *black*) where at least one of the eigenvalues of the matrix C is negative for the Kato–Phillips test case simulated using the original Mellor–Yamada level 2.5 closure scheme (*left*) and the Mellor–Yamada level 2.5 closure with Hassid and Galperin (1983) constraint (*right*). In each case, the bottom of the unstable region corresponds to the position of the pycnocline. Note that the matrix C obtained with the quasi-equilibrium closure of Galperin et al. (1988) does not have any negative eigenvalues. The grid size is set to 1 m (*top*) and 0.1 m (*bottom*)

why this limitation has not been found entirely satisfactory in the numerical simulations of Burchard and Deleersnijder (2001).

Figure 3 shows the stability region of the original Mellor–Yamada closure and the alternative closures suggested by Hassid and Galperin (1983) and Galperin et al. (1988). It can be seen that the original closure is unstable (Fig. 3a) in almost the entire region satisfying the constraints on G_M and G_H. In particular, the local equilibrium curve defined by

$$P_s + P_b + \varepsilon = 0 \Leftrightarrow S_M G_M + S_H G_H = \frac{1}{B_1}$$

lies almost entirely in the unstable region. The remedy suggested by Hassid and Galperin (1983) mainly amounts to increasing the size of the forbidden zone without any significant change to the stable region (Fig. 3b). The local equilibrium curve is still almost entirely outside the stable region. Finally, there is no unstable region when the quasi-equilibrium stability functions of Galperin et al. (1988) are used (Fig. 3c). In that case, the forbidden zone is the same as for the original Mellor–Yamada closure and the local equilibrium curve lies entirely in the stable region for admissible values of G_M and G_H.

The eddy coefficient profiles obtained after 30 h for a Kato–Philips flow are shown in Fig. 4 for grid sizes of 1 and 0.1 m. The model equations have been discretised with a centered finite difference scheme similar to the one used by Deleersnijder and Luyten (1994). Results for the original and quasi-equilibrium versions of the Mellor–Yamada closure are similar to those shown by Burchard and Deleersnijder (2001) and clearly illustrate the problems of the original version of the closure. As expected from the eigenvalues analysis, the remedy suggested by Hassid and Galperin (1983) does not totally remove the instabilities, though it reduces their extent. The unstable behaviour of the original Mellor–Yamada level 2.5 closure and the constrained version of Hassid and Galperin (1983) is confirmed by Fig. 5, which shows the space-time region where at least one eigenvalue of the matrix C is negative for the Kato–Phillips test-case. It can be seen that, for both the original closure and the modified version due to Hassid and Galperin (1983), there is an unstable region whose extent grows over time. The bottom of the unstable region corresponds to the position of the pycnocline.

4 Conclusions

The present study does not imply criticism of either the Mellor–Yamada hierachy of turbulence models (Mellor and Yamada 1974, 1982) or the so-called Mellor–Yamada level 2.5 closure scheme (Yamada 1977). The latter was used for illustration purposes only. We are convinced that the physical underpinning of the original level 2.5 model was perfectly tenable. Nonetheless, a mathematical model, however elaborate it may be, is no substitute to reality. Therefore, physical considerations cannot be sufficient to establish it; mathematical aspects, especially the stability of the solutions, must also be taken into account. Of course, this philosophy is far from novel, and the present article is just another illustration thereof.

Though strong simplifications are needed to derive the instability criterion based on the eigenvalues of the stability matrix C defined in Section 2.2, numerical experiments point to its relevance. It is believed that the present theory may be applied to all local turbulence closure schemes used in atmospheric and oceanic modelling. Whether or not it could be of use for studying the stability of non-local approaches is still an open question.

The lower bound of the space scale of the perturbations that can be represented in a model is set by the grid size. Therefore, the occurrence of spurious oscillations is strongly influenced by the details of the discretisations. The larger the grid size, the lesser the validity of the scale separation hypothesis that is the key assumption to our approach and the smaller the growth rate of the fastest-growing perturbations. Therefore, numerical details do matter and a numerical stability theory is needed, but it is unlikely to be easy to establish.

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