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Birkhoff's variety theorem in many sorts

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Abstract. For many-sorted algebras, Garrett Birkhoff's characterization of equational classes is proved to generalize in case of finitely many sorts. For infinitely sorted algebras, closure under directed unions needs to be added.

The classical result of Garrett Birkhoff [3] that equational classes of Σ-algebras are precisely those closed under products, subalgebras, and regular quotients (or homomorphic images) does not fully generalize to S-sorted signatures: it holds for finitely sorted algebras, but in case of infinitely many sorts, one has to add closedness under directed unions. Or one has to leave the realm of finitary logic and start working in the logic $\mathscr{L}_{\omega,\lambda}$ allowing quantifications of less than λ variables where $\lambda > \operatorname{card} S$. However, the "naive" generalization without directed unions appears rather persistently in books and papers. (See e.g. page 141 of [1], page 105 of [4], page 107 of [5] and page 248 of [6]). We first show a trivial counter-example to the "naive" generalization, and then prove the two generalizing theorems.

The failure of the "naive" generalization is caused by the role empty sorts of algebras are playing. If one would define a many-sorted algebra as a collection of non-empty sets with finitary many-sorted operations, no such problem with Birkhoff's variety theorem would appear. But such a definition would not work in applications. For example, when considering a graph (with multiple edges) as an algebra of two sorts (vertex, edge), we certainly do not want to exclude graphs without edges. (Thus the situation reminds us somewhat of the problem of many-sorted equational logic: empty sorts make equational logic more delicate, but we do not deal with this issue in our note.)

Recall that an S-sorted signature is a set Σ together with an arity function assigning to every symbol $\sigma \in \Sigma$ a pair $(s_1, \dots, s_n, s) \in S^* \times S$. By a Σ-algebra is then meant an S-sorted set $A = (A_s)_{s \in S}$ together with functions $\sigma^A \colon A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for every σ of arity $(s_1, \dots, s_n, s)$; if $n = 0$, then σ^A is an element of A_s. The category Σ-*Alg* of Σ-algebras has as morphisms the S-sorted functions preserving the operations in the expected sense. For every many-sorted set X (of variables) we denote by $F_\Sigma X$ the free Σ-algebra on X.

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Recall that X is called finite provided that $\coprod_{s\in S} X_s$ is a finite set. For an infinite cardinal λ, we call X λ-*presentable* provided that $\operatorname{card}\coprod_{s\in S} X_s < \lambda$.

Definition. An *equation* (in the finitary logic) is a formula $\forall\, X\colon t = t'$ where X is a finite S-sorted set and t, t' are elements of $F_\Sigma X$ of the same sort. A Σ-algebra A *satisfies* the equation provided that for every S-sorted function $f\colon X \to A$, the homomorphism $\bar{f}\colon F_\Sigma X \to A$ extending f merges t and t'. An *equational class* is a full subcategory of Σ-Alg specified by a set of equations.

Example (A class of Σ-algebras which is closed under products, subalgebras, and regular quotients in Σ-Alg, but is not equational)**.** Put $S = \mathbb{N}$ and $\Sigma = \emptyset$. Thus Σ-Alg $= \mathbf{Set}^{\mathbb{N}}$ is the category of sequences of sets. Let $\mathcal{A}$ be the full subcategory of all $(A_n)_{n\in\mathbb{N}}$ such that either $A_n = \emptyset$ for some $n \in \mathbb{N}$, or $\operatorname{card} A_n = 1$ for every $n \in \mathbb{N}$. This subcategory is clearly closed under products, subalgebras, and regular quotients.

However, $\mathcal{A}$ is not equational because then it would clearly be closed under directed unions. This is not the case: every object A of $\mathbf{Set}^{\mathbb{N}}$ is a directed union of objects A_i with almost all coordinates empty (and $A_i \in \mathcal{A}$).

The Birkhoff Variety Theorem. *For every set S of sorts and every S-sorted signature Σ, the equational classes of Σ-algebras are precisely the full subcategories of Σ-Alg closed under products, subalgebras, regular quotients, and directed unions.*

Proof. The verification of necessity is easy and standard. The only fact one needs to know is that regular epimorphisms are precisely the homomorphisms that are surjective in every sort (see e.g. [2], Corollary 3.5).

To prove the sufficiency, let $\mathcal{A}$ be a full subcategory of Σ-Alg closed as above. Then $\mathcal{A}$ is a full reflective subcategory such that for every Σ-algebra A, the reflection $r_A\colon A \to R(A)$ is a regular epimorphism. We prove that $\mathcal{A}$ is presented by all equations $\forall\, X : t = t'$ where X is a finite S-sorted set and $t, t' \in (F_\Sigma X)_s$ are merged by $(r_{F_\Sigma X})_s$. Every algebra of $\mathcal{A}$ clearly satisfies all these equations. Conversely, if A in Σ-Alg satisfies them, we prove $A \in \mathcal{A}$. We can assume that A is finitely generated (since $\mathcal{A}$ is closed under directed unions and A is a directed union of finitely generated subalgebras). That is, there exists a finite S-sorted subset $i\colon X \hookrightarrow A$ such that the corresponding homomorphism $\bar{i}\colon F_\Sigma X \to A$ is a regular epimorphism. Observe that $\bar{i}$ factorizes through $r_{F_\Sigma X}$: this follows from the fact that whenever $r_{F_\Sigma X}$ merges two elements t, t' (of the same sort s), then so does $\bar{i}$. Indeed, A satisfies $\forall\, X : t = t'$ which implies $\bar{i}_s(t) = \bar{i}_s(t')$. Consequently, we have an S-sorted function $e\colon R(F_\Sigma X) \to A$ with $\bar{i} = e \cdot r_{F_\Sigma X}$. Since $r_{F_\Sigma X}$ is a regular epimorphism, the fact that $\bar{i}$ is a homomorphism implies that e is one, too. And since $\bar{i}$ is a regular epimorphism, so is e. This proves $A \in \mathcal{A}$. □

Definition. Let λ be the least infinite cardinal larger than $\operatorname{card} S$. By a λ-*ary equation* is meant a formula $\forall\, X : t = t'$ where X is a λ-presentable S-sorted set and t, t' are elements of $F_\Sigma X$ of the same sort. Satisfaction and λ-ary equational classes are defined analogously to the finitary case above.

Theorem. *For every set S of sorts and every S-sorted signature Σ, the λ-ary equational classes of Σ-algebras are precisely those closed under products, subalgebras, and regular quotients.*

Proof. Given a full subcategory $\mathcal{A}$ of Σ-Alg closed under products, subalgebras, and regular quotients, we prove that $\mathcal{A}$ is closed under λ-directed unions. This is sufficient: the proof is as above, using that every algebra A is a λ-directed union of λ-generated subalgebras. Let A be a Σ-algebra and A_i $(i \in I)$ a λ-directed collection of subalgebras with

$$A = \bigcup_{i \in I} A_i \quad \text{and} \quad A_i \in \mathcal{A} \quad \text{for all } i \in I.$$

For every sort $s \in S$ with $A_s \neq \emptyset$, choose $i_0 \in I$ so that $(A_{i_0})_s \neq \emptyset$. The choice of i_0 can be made independently of s (since I is λ-directed and $\operatorname{card} S < \lambda$). The algebra $\prod_{i \geq i_0} A_i$ lies in $\mathcal{A}$. Consider the S-sorted subset B of this product which in sort s with $A_s \neq \emptyset$ consists of precisely those tuples $(x_i)_{i \geq i_0}$ (where $x_i \in (A_i)_s$) which are eventually constant. That is, there exists $j \in I$ with $x_i = x_j$ for all $i \geq j$. It is easy to see that B carries a subalgebra of $\prod_{i \geq i_0} A_i$. We have $B \in \mathcal{A}$. And this proves $A \in \mathcal{A}$ because we have a regular quotient $e \colon B \to A$ assigning to every eventually constant tuple the constant value. In fact, since for every sort s with $A_s \neq \emptyset$ we have $B_s \neq \emptyset$, it is easy to see, due to our choice of i_0, that e_s is surjective. $\square$

Corollary. *In case of finitely many sorts, the Birkhoff variety theorem holds without the requirement of closure under directed unions.*

Remark. For more on equational classes of many-sorted algebras, the reader may consult [2].

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