Azumaya categories

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Abstract

We define the notions of Azumaya category and Brauer group in category theory enriched over some very general base category $\mathcal{V}$. We prove the equivalence of various definitions, in particular in terms of separable categories or progenerating bimodules. When $\mathcal{V}$ is the category of modules over a commutative ring R with unit, we recapture the classical notions of Azumaya algebra and Brauer group and provide an alternative, purely categorical treatment of those theories. But our theory applies as well to the cases of topological, metric or Banach modules, to the sheaves of such structures or graded such structures, and many other examples.

Introduction

Convention. *In this paper, all rings and algebras have a unit, but are not necessarily commutative. Except otherwise stated, all modules are right modules.*

If R is a commutative ring, two R-algebras A and B are Morita equivalent when there exist an A-B-bimodule M and a B-A-bimodule N such that the isomorphisms $M \otimes_B N \cong A$ and $N \otimes_A M \cong B$ hold. The tensor product induces the structure of a monoid on the Morita-equivalence classes of R-algebras, with the class of R as unit. An R-algebra A is Azumaya when its Morita equivalence class is invertible for the tensor product; the inverse of A is then the class of the dual algebra A^*. The invertible Morita equivalence classes of R-algebras, with the tensor product as multiplication, constitute thus an abelian group, called the Brauer group of the ring R.

We replace first the category Mod_R by an arbitrary complete and cocomplete symmetric monoidal closed category $\mathcal{V}$. That is, $\mathcal{V}$ is a category provided with an associative, commutative tensor product with unit, and an internal Hom-functor

$$\otimes\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}, \quad [-,-]\colon \mathcal{V}^* \times \mathcal{V} \longrightarrow \mathcal{V}$$

which yield natural isomorphisms

$$\big[A \otimes B, C\big] \cong \big[A, [B, C]\big]$$

for all $A, B, C \in \mathcal{V}$. Examples of such $\mathcal{V}$ are numerous: Mod_R of course, with $[M, N] = \mathsf{Lin}_R(M, N)$, but also all categories of sheaves or presheaves on a topological space (with the cartesian product as tensor product), the category of topological spaces itself (where $[A, B]$ is the set of continuous functions with the topology of pointwise convergence), and so on. But in the spirit of this paper, we are mainly interested in examples like the categories of locally convex, metric or Banach spaces, the categories of topological, metric or Banach modules and more generally, the categories of sheaves of such structures or graded such structures, and so on. All these examples enter the context of this paper and yield a theory of Azumaya categories and a corresponding Brauer group.

In this general context, the R-algebras are replaced by small $\mathcal{V}$-categories, that is, categories $\mathcal{A}$ with an arbitrary set of objects and whose sets $\mathcal{A}(A, B)$ of arrows are canonically provided with the structure of an object $\mathcal{A}(A, B) \in \mathcal{V}$. Algebras correspond to the case where the $\mathcal{V}$-category $\mathcal{A}$ has a single object. If $\mathcal{A}$, $\mathcal{B}$ are two such $\mathcal{V}$-categories, an $\mathcal{A}$-$\mathcal{B}$-bimodule is a $\mathcal{V}$-functor

$$\varphi: \mathcal{B}^* \otimes \mathcal{A} \longrightarrow \mathcal{V},$$

and this generalizes at once the classical notion of bimodule over rings or algebras. Just like bimodules over algebras compose, so do our $\mathcal{V}$-bimodules, yielding a corresponding notion of Morita equivalence for $\mathcal{V}$-categories. The tensor product induces the structure of a monoid on the Morita equivalence classes of small $\mathcal{V}$-categories, and a small $\mathcal{V}$-category is Azumaya when its Morita equivalence class is invertible. Those invertible classes constitute the categorical Brauer group of $\mathcal{V}$.

We generalize to the case of Azumaya categories the well-known equivalent definitions of Azumaya algebra, in terms of separability, centrality, projectivity and generators and develop a corresponding theory of the Brauer group. Of course, due to the great generality of the context in which we are working, the classical sophisticated techniques of ring and module theory cannot be used to achieve this. It is amazing to notice that classical arguments on adjunctions suffice to overcome the difficulty: this provides in particular a purely categorical treatment of the theory of Azumaya algebras over a ring, proving in some sense that this nice piece of mathematics is what some people would call "a special instance of general abstract nonsense".

Without any further assumption on $\mathcal{V}$, the categorical Brauer group needs not be a set, but just a class. Nevertheless, when the category $\mathcal{V}$ is locally presentable – an assumption which is satisfied in many cases of interest – the categorical Brauer group is actually a set. We also exhibit a condition on $\mathcal{V}$, in terms of categorical Cauchy completion, which forces the categorical Brauer group to be isomorphic to the one constructed using only algebras. As a consequence, our construction recaptures the classical Brauer group of the ring R in the very special case $\mathcal{V} = \mathsf{Mod}_R$.

Starting from the seventies, several categorical approaches to Azumaya algebras and the Brauer group have been proposed (see the bibliography in [23] for some references). Almost all these approaches deal with Azumaya monoids in a monoidal category $\mathcal{V}$. If one confines one's attention to monoids (i.e. one-object $\mathcal{V}$-categories), the basic results needed to develop the Azumaya-Brauer theory are quite simple: they reduce essentially to Morita theory in monoidal categories. Consequently, the conditions on $\mathcal{V}$ can be weaker than those assumed in our paper. For example, in the classical paper by Pareigis [20], the closedness of $\mathcal{V}$ is weakened, and in some recent papers by Van Oystaeyen and Zhang (see [22]) and by Alonso Alvarez et al. (see [1]), $\mathcal{V}$ is braided but not necessarily symmetric; however, some completeness and biclosedness requirements are needed on $\mathcal{V}$.

If one works with arbitrarily small $\mathcal{V}$-categories, and not only with $\mathcal{V}$-monoids, the basic ingredients of enriched category theory which are needed are much deeper, and they are presently available in the literature only when the base category $\mathcal{V}$ is symmetric and closed. For this reason, and as all the examples we have in mind fit into this context, we work in this first paper with such a good base $\mathcal{V}$. Anyway, let us observe that our construction of the categorical Brauer group of $\mathcal{V}$ as well as the equivalence of conditions (1) and (2) in theorem 3.4 certainly hold when $\mathcal{V}$ is only braided, because they depend only on the existence and the formal properties of the monoidal bicategory of $\mathcal{V}$-categories and $\mathcal{V}$-distributors (see also [8]). The direct ancestor of our paper and, at our knowledge, the only other paper in which $\mathcal{V}$-categories are used to develop an Azumaya-Brauer theory, is the memoir by Mitchell (see [18]), where the classical case $\mathcal{V} = \mathsf{Mod}_R$ is studied. Even in Mitchell's paper, general techniques of enriched category theory are neglected, and the main theorems are proved using ring and module-theoretical arguments.

The point of our paper is exactly that an extensive use of enriched category theory allows us to extend the theory to arbitrary small categories over a rather general base category $\mathcal{V}$ and to simplify proofs a lot.

1 Azumaya algebras and the Brauer group

This section is a crash course for category theorists, on the classical theory of Azumaya algebras and the Brauer group of a ring. The reader should consult the references [3], [10], [16], [19] of this paper for further details.

When A is a ring or a field, its center is the set of those elements $a \in A$ such that $ab = ba$ for every $b \in \mathcal{A}$. An algebra A on the ring R is central when the morphism

$$R \longrightarrow A, \quad r \mapsto r \cdot 1$$

induces an isomorphism between R and the center of A. Writing A^* for the dual algebra (A with the reversed multiplication), we shall use frequently the fact that A is a right $A \otimes_R A^*$-module for the multiplication $c(a \otimes b) = bca$.

A finite dimensional field extension $K \subseteq L$ is separable when the roots of the characteristic polynomial of every element $l \in L$ are all simple. Observe that

L is obviously a K-algebra. The separability property extends in the following way to algebras over a ring.

Proposition 1.1 *Let R be a commutative ring. For a R-algebra A, the following conditions are equivalent and define what is called a* separable *R-algebra.*

(1) A is projective as a $A \otimes_R A^*$-module.

(2) A is finitely generated projective as a $A \otimes_R A^*$-module.

(3) The multiplication

$$A \otimes_R A^* \longrightarrow A, \quad a \otimes b \mapsto ba$$

has a $A \otimes_R A^*$-linear section.

Proof (1)⇒(3) by $(A \otimes_R A^*)$-linearity and surjectivity of the multiplication; the other implications are obvious. ■

Consider now a field K. We are interested in the finite dimensional, central, separable skew field extensions of K, which for short, we call just "extensions". Given two such extensions, say $K \subseteq L$ and $K \subseteq M$, their tensor product $L \otimes_K L$ is no longer a field, but via the Wedderburn theorem (see [19]), is a matrix algebra $\mathsf{M}_n(N)$ on a uniquely determined extension $K \subseteq N$. Putting $N = L \star M$ yields the structure of an abelian group on the isomorphism classes of extensions, with the dual field L^* as inverse of L. This is the Brauer group of K. The various matrix algebras $\mathsf{M}_n(N)$ for the various extensions $K \subseteq N$ are the Azumaya K-algebras; they are exactly the central separable algebras over K.

Fix now a commutative ring R. Given two R-algebras A and B, their Morita equivalence as described in the introduction is equivalent to the categorical R-linear equivalence of the corresponding categories of modules. The following result is classical and generalizes the previous situation to the case of rings.

Theorem 1.2 *Let R be a commutative ring. For an R-algebra A, the following conditions are equivalent and define the notion of* Azumaya algebra *over the ring R.*

(1) There exists an R-algebra B such that $B \otimes_R A$ is Morita equivalent to R.

(2) $A^* \otimes_R A$ is Morita equivalent to R.

(3) A is a central, separable R-algebra.

(4) A is a finitely generated projective generator in the category of R-modules, and the canonical morphism

$$\sigma \colon A^* \otimes_R A \longrightarrow \mathsf{End}_R(A), \quad \sigma(a \otimes b)(c) = acb$$

is a R-linear isomorphism.

Moreover, in those conditions, A is a generator in the category of $A \otimes_R A^*$-modules. The Morita equivalence classes of Azumaya algebras constitute an abelian group with multiplication the tensor product over R; this is the Brauer group *of R.*

2 Generalized algebras and bimodules

This section is a crash course for ring theorists, on the basic notions of enriched category theory. See [6] and [14] for more details.

In this paper, we shall replace the category Mod_R of modules on a ring R by an arbitray category $\mathcal{V}$ admitting all projective and inductive limits and provided with two functors

$$- \otimes -: \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{V}, \quad [-,-]: \mathcal{V}^* \otimes \mathcal{V} \longrightarrow \mathcal{V},$$

where $\mathcal{V}^*$ indicates the dual category of $\mathcal{V}$ (all arrows are put in the reversed direction). The tensor product (up to canonical isomorphisms) is requested to be associative, commutaive and with unit I. Moreover the natural isomorphisms

$$\big[A \otimes B, C\big] \cong \big[A, [B, C]\big]$$

must hold for all objects A, B, C in $\mathcal{V}$. In the case $\mathcal{V} = \mathsf{Mod}_R$, the tensor product is the usual one over the ring R and $[B, C] = \mathsf{Lin}_R(C, D)$ is the module of R-linear mappings. For the sake of brevity, we shall refer to such a category as our "base" category $\mathcal{V}$.

Convention. *We fix a* base category $\mathcal{V}$ *with the properties we have just mentioned, that is, $\mathcal{V}$ is complete, cocomplete and symmetric monoidal closed.*

Numerous examples of such "base categories" can be found in [6] and [14], but in this paper we are mostly interested in examples like Banach spaces, locally convex spaces, modules on a commutative ring, topological modules on a commutative locally convex algebra, Banach modules on a commutative Banach algebra, sheaves of such things or graded such things, and so on. All these constitute good "base categories" for which we can generalize the theory of Azumaya algebras and construct a corresponding Brauer group.

Given a base category $\mathcal{V}$, a $\mathcal{V}$-category consists in a class $|\mathcal{A}|$ of objects, together with an object $\mathcal{A}(A, B) \in \mathcal{V}$ "of arrows from A to B" for every pair of objects. A composition law in $\mathcal{V}$

$$\gamma_{ABC}: \mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$$

is given for every triple of objects, and traditional axioms of associativity and existence of units $I \longrightarrow \mathcal{A}(A, A)$ are imposed. A $\mathcal{V}$-category is small when it has a set of objects. A $\mathcal{V}$-category with a single object $*$ is also called a $\mathcal{V}$-algebra; in the case $\mathcal{V} = \mathsf{Mod}_R$, this reduces exactly to giving a R-algebra

$A = \mathcal{A}(*,*)$. Observe that the base category $\mathcal{V}$ becomes itself a $\mathcal{V}$-category by putting $\mathcal{V}(V,W) = [V,W]$. The dual $\mathcal{V}$-category $\mathcal{A}^*$ has the same objects as $\mathcal{A}$, but $\mathcal{A}^*(A,B) = \mathcal{A}(B,A)$, with corresponding twisted composition law. Given two $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$, one construct easily a new $\mathcal{V}$-category $\mathcal{A} \otimes \mathcal{B}$ whose objects are the pairs (A,B) of objects in $\mathcal{A}$ and $\mathcal{B}$, while

$$(\mathcal{A} \otimes \mathcal{B})\big((A,B),(A',B')\big) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B').$$

This definition extends the classical notion of tensor product of algebras over a commutative ring.

A $\mathcal{V}$-functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ between $\mathcal{V}$-categories consists in a function mapping every object $A \in \mathcal{A}$ on an object $F(A) \in \mathcal{B}$, while the action of F on arrows is expressed by morphisms of $\mathcal{V}$

$$F_{A,A'}\colon \mathcal{A}(A,A') \longrightarrow \mathcal{B}\big(F(A),F(A')\big)$$

satisfying the traditional functoriality axioms. $\mathcal{V}$-functors are also called *covariant* $\mathcal{V}$-funtors. A *contravariant* $\mathcal{V}$-functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is just a covariant $\mathcal{V}$-functor $F\colon \mathcal{A}^* \longrightarrow \mathcal{B}$. A $\mathcal{V}$-functor between $\mathcal{V}$-algebras is also called a morphism of $\mathcal{V}$-algebras, extending so the classical situation for rings. The $\mathcal{V}$-functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is called *full and faithful* when all morphisms $F_{A,A'}$ are isomorphisms.

A $\mathcal{V}$-natural transformation $\alpha\colon F \Longrightarrow G$ between $\mathcal{V}$-functors $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ is a family of morphisms $\alpha_A\colon I \longrightarrow \mathcal{B}\big(F(A),G(A)\big)$ for all $A \in \mathcal{A}$, with the traditional naturality condition. When $\mathcal{A}$ is small, the $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations from $\mathcal{A}$ to $\mathcal{B}$ constitute a $\mathcal{V}$-category which we denote simply by $[\mathcal{A},\mathcal{B}]$. Given two $\mathcal{V}$-functors F, G from $\mathcal{A}$ to $\mathcal{B}$, we use also the notation $[F,G]$ to indicate the object of $\mathcal{V}$-natural transformations from F to G. Given $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ with $\mathcal{A}$, $\mathcal{B}$ small, one gets an isomorphism of $\mathcal{V}$-categories

$$\big[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}\big] \cong \big[\mathcal{A}, [\mathcal{B},\mathcal{C}]\big].$$

Up to isomorphisms, the tensor product of $\mathcal{V}$-categories is associative and commutative and admits for unit the "unit" $\mathcal{V}$-category $\mathcal{I}$ with a single object $*$ and $\mathcal{I}(*,*) = I$ as object of morphisms. Observe that $\mathcal{I} = \mathcal{I}^*$.

For a small $\mathcal{A}$, the category $[\mathcal{A}^*,\mathcal{V}]$ is called the category of (right) modules on $\mathcal{A}$ and is simply written $\mathsf{Mod}_{\mathcal{A}}$; the corresponding notion of left module is obtained using functors on $\mathcal{A}$ instead of $\mathcal{A}^*$. When $\mathcal{V} = \mathsf{Mod}_R$ and $\mathcal{A}$ is a R-algebra, we recapture the usual notions of right and left A-module. In this special case, given two R-algebras A and B, an A-B-bimodule is just a right module on $B^* \otimes A$. In our general setting, for two small categories $\mathcal{A}$ and $\mathcal{B}$, we define the category $\mathsf{Bimod}_{\mathcal{A}\text{-}\mathcal{B}}$ of $\mathcal{A}$-$\mathcal{B}$-bimodules as the category of $(\mathcal{B}^* \otimes \mathcal{A})$-modules. An $\mathcal{A}$-$\mathcal{B}$-bimodule M is thus a $\mathcal{V}$-functor

$$M\colon \mathcal{B}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

which, via the isomorphism indicated above, yields corresponding $\mathcal{V}$-functors

$$\mathcal{A} \longrightarrow [\mathcal{B}^*, \mathcal{V}] = \mathsf{Mod}_{\mathcal{B}}, \quad \text{and} \quad \mathcal{B}^* \longrightarrow [\mathcal{A}, \mathcal{V}].$$

Observe also that the unit category $\mathcal{I}$ is a commutative $\mathcal{V}$-algebra, since $\mathcal{I} = \mathcal{I}^*$, and the category of $\mathcal{I}$-modules is just $\mathcal{V}$ itself. Since $\mathcal{I}$ is a unit for the tensor product of $\mathcal{V}$-categories, given a small $\mathcal{V}$-category $\mathcal{A}$, every right $\mathcal{A}$-module is a $\mathcal{I}$-$\mathcal{A}$-bimodule and every left $\mathcal{A}$-module is a $\mathcal{A}$-$\mathcal{I}$-bimodule.

Next, we extend the ordinary tensor product of bimodules over rings to our more general setting. Given small $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule M and a $\mathcal{B}$-$\mathcal{C}$-bimodule N,

$$M: \mathcal{B}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}, \quad N: \mathcal{C}^* \otimes \mathcal{B} \longrightarrow \mathcal{V},$$

we define their composite by the formula

$$N \odot M: \mathcal{C}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}, \quad (N \odot M)(C, A) = \operatorname{colim}_{\mathcal{B}} N(C, B) \otimes M(B, A).$$

Up to canonical isomorphisms, this composition is associative and admits as units the $\mathcal{B}$-$\mathcal{B}$-bimodules, still written $\mathcal{B}$,

$$\mathcal{B}: \mathcal{B}^* \otimes \mathcal{B} \longrightarrow \mathcal{V}, \quad (B, B') \mapsto \mathcal{B}(B, B').$$

This yields at once the notion of Morita equivalence.

Proposition 2.1 *Given small $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, the following conditions are equivalent and define the $\mathcal{V}$-Morita equivalence of $\mathcal{A}$ and $\mathcal{B}$.*

(1) The categories $\mathsf{Mod}_{\mathcal{A}}$ and $\mathsf{Mod}_{\mathcal{B}}$ are equivalent as $\mathcal{V}$-categories.

(2) There exist an $\mathcal{A}$-$\mathcal{B}$-bimodule M and a $\mathcal{B}$-$\mathcal{A}$-bimodule N, yielding isomorphisms $M \odot N \cong \mathcal{A}$ and $N \odot M \cong \mathcal{B}$.

It remains to recall the notion of adjoint bimodules.

Definition 2.2 *Given small $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{B}$-$\mathcal{A}$-bimodule N is right adjoint to the $\mathcal{A}$-$\mathcal{B}$-bimodule M when there exist $\mathcal{V}$-natural transformations*

$$\varepsilon: M \odot N \Longrightarrow \mathcal{B}, \quad \eta: \mathcal{A} \Longrightarrow N \odot M$$

making commutative the following triangles of $\mathcal{V}$-natural transformations

$$\begin{array}{ccc} M & \xrightarrow{\mathsf{id}_M \odot \eta} & M \odot N \odot M \\ & \diagdown\!\!\diagdown & \Big\downarrow {\scriptstyle \varepsilon \odot \mathsf{id}_M} \\ & & M \end{array} \qquad\qquad \begin{array}{ccc} N & \xrightarrow{\eta \odot \mathsf{id}_N} & N \odot M \odot N \\ & \diagdown\!\!\diagdown & \Big\downarrow {\scriptstyle \mathsf{id}_N \odot \varepsilon} \\ & & N \end{array}$$

The importance of the notion of adjoint bimodules is attested by the following proposition.

Proposition 2.3 *Let R be a commutative ring, A, B two R-algebras and M an A-B-bimodule. We consider the base category $\mathcal{V} = \mathsf{Mod}_R$, view A and B as one-object $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ and M as a $\mathcal{A}$-$\mathcal{B}$-bimodule. The following conditions are equivalent:*

(1) M is a finitely generated projective B-module;

(2) as an $\mathcal{A}$-$\mathcal{B}$-bimodule, M has a right adjoint;

This proposition suggests the corresponding notion of "small projective" bimodule over $\mathcal{V}$.

Definition 2.4 *Let $\mathcal{A}$, $\mathcal{B}$ be small $\mathcal{V}$-categories. An $\mathcal{A}$-$\mathcal{B}$-bimodule M is* small projective *when it admits a right adjoint. A $\mathcal{B}$-module is small projective when it is so as a $\mathcal{I}$-$\mathcal{B}$-bimodule.*

3 Azumaya $\mathcal{V}$-categories and $\mathcal{V}$-Brauer group

We want now to generalize theorem 1.2, replacing Mod_R by our base category $\mathcal{V}$. First, we must translate in this general context the various notions appearing in the statement of 1.2.

It is well-known that the center of an algebra A over a commutative ring R can equivalently be defined as the algebra of $A \otimes_R A^*$-linear endomorphisms of A. Viewed as a R-$(A \otimes_R A^*)$-bimodule, the algebra A corresponds to a Mod_R-functor

$$H_A\colon \mathcal{I} \longrightarrow \mathsf{Mod}_{A\otimes_R A^*}, \quad * \mapsto A$$

and the full and failthfulness of this functor means precisely that the canonical morphism

$$R = \mathcal{I}(*,*) \longrightarrow \mathsf{Lin}_{A\otimes_R A^*}\big(H_A(*), H_A(*)\big) = \mathsf{Lin}_{A\otimes_R A^*}(A, A)$$

is an isomorphisms, that is, the centrality of A.

Given a small $\mathcal{V}$-category $\mathcal{A}$, we shall refer to the functor

$$\mathcal{A}\colon \mathcal{A}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}, \quad (A, B) \mapsto \mathcal{A}(A, B)$$

as the "canonical functor $\mathcal{A}$". It is covariant on $\mathcal{A}^* \otimes \mathcal{A}$ and thus contravariant – i.e. a module – on $\mathcal{A} \otimes \mathcal{A}^*$.

Definition 3.1 *A small $\mathcal{V}$-category $\mathcal{A}$ is central when the corresponding $\mathcal{V}$-functor*

$$H_{\mathcal{A}}\colon \mathcal{I} \longrightarrow \mathsf{Mod}_{\mathcal{A}\otimes\mathcal{A}^*}, \quad * \mapsto \mathcal{A}$$

is full and faithful.

Next we generalize the notion of generator. Consider two algebras A and B on a commutative ring R. In the category Mod_B of modules over B, there are many equivalent ways to express the fact that a module M is a generator (see [5]). We choose the following characterization: $M \in \mathsf{Mod}_B$ is a generator when, for every morphism $f\colon X \longrightarrow Y$ in Mod_B, f is an isomorphism iff

$$\mathsf{Lin}_B(M,f)\colon \mathsf{Lin}_B(M,X) \longrightarrow \mathsf{Lin}_B(M,Y), \quad g \mapsto g \circ f$$

is bijective. When M turns out to be an A-B-bimodule, the left-A-module structure of M induces corresponding structures of right-A-modules on $\mathsf{Lin}_B(M,X)$ and $\mathsf{Lin}_B(M,Y)$; moreover, $\mathsf{Lin}_B(M,f)$ is A-linear for those structures. Therefore the A-B-bimodule M is a generator in Mod_B when, for every $f\colon X \longrightarrow Y$ in Mod_B,

$$\big(\ \mathsf{Lin}_B(M,f) \text{ is an } A\text{-isomorphism}\ \big) \Rightarrow \ \big(\ f \text{ is a } B\text{-isomorphism}\ \big).$$

Definition 3.2 *Let $\mathcal{A}$, $\mathcal{B}$ be small $\mathcal{V}$-categories. An $\mathcal{A}$-$\mathcal{B}$-bimodule M is called a* strong generator *when the $\mathcal{V}$-functor*

$$[M,-]\colon \mathsf{Mod}_{\mathcal{B}} \longrightarrow \mathsf{Mod}_{\mathcal{A}}, \quad [M,F](A) = \mathsf{Mod}_{\mathcal{B}}\big(M(-,A),F\big) = \big[M(-,A),F\big]$$

reflects isomorphisms. A $\mathcal{B}$-module M is a strong generator when it is so as an $\mathcal{I}$-$\mathcal{B}$-bimodule.

It remains to generalize definition 1.1, but this is easy in view of definition 2.4.

Definition 3.3 *A small $\mathcal{V}$-category $\mathcal{A}$ is separable when the canonical functor $\mathcal{A}$, viewed as a $\mathcal{A} \otimes \mathcal{A}^*$-module, is small projective.*

We are now ready to state the main theorem of this paper:

Theorem 3.4 *For a small $\mathcal{V}$-category A, the following conditions are equivalent and define the notion of* Azumaya $\mathcal{V}$-category*.*

(1) There exists a $\mathcal{V}$-category $\mathcal{B}$ such that $\mathcal{B} \otimes \mathcal{A}$ is Morita equivalent to $\mathcal{I}$.

(2) $\mathcal{A}^* \otimes \mathcal{A}$ is Morita equivalent to $\mathcal{I}$.

(3) $\mathcal{A}$ is a central, separable $\mathcal{V}$-category and the canonical functor $\mathcal{A}$, viewed as a $(\mathcal{A} \otimes \mathcal{A}^*)$-module, is a strong generator.

(4) the canonical functor $\mathcal{A}$ is full and faithful and, viewed as a $(\mathcal{A}^* \otimes \mathcal{A})$-$\mathcal{I}$-bimodule, is a small projective strong generator.

In those conditions, the category $\mathcal{B}$ in condition (1) is itself Morita equivalent to $\mathcal{A}^*$. The Morita equivalence classes of Azumaya categories constitute a (possibly large) abelian group with multiplication induced by the tensor product of $\mathcal{V}$-categories; this is the $\mathcal{V}$-Brauer group*. The inverse of the class of $\mathcal{A}$ is the class of $\mathcal{A}^*$.*

This theorem will be proved in the next section, but some comments are necessary right now.

In the case where $\mathcal{V} = \mathsf{Mod}_R$ and $\mathcal{A}$ is an R-algebra A, the functor $\mathcal{A}$ is given by

$$\mathcal{A}\colon \mathcal{A}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}, \quad (*,*) \mapsto \mathcal{A}(*,*) = A.$$

Its full and faithfulness means that the following mapping is an isomorphism:

$$A^* \otimes_R A = \big(\mathcal{A}^* \otimes \mathcal{A}\big)\big((*,*),(*,*)\big) \longrightarrow \mathcal{V}\big(\mathcal{A}(*,*),\mathcal{A}(*,*)\big) = \mathsf{Lin}_R(A,A).$$

This is precisely one of the requirements in condition 4 of theorem 3.4.

In the classical case of Azumaya algebras over a ring, a central separable algebra A is necessarily a generator in the category of $A \otimes A^*$-modules : this is the well-known Auslander –Goldman theorem (see for example [10], [16], [19]). The problem remains open to decide if this property generalizes to the context of theorem 3.4.

4 Proof of theorem 3.4

In this section, we refer freely to the classical techniques and notions of category theory. We write $\mathsf{Bimod}_{\mathcal{V}}$ for the bicategory of small $\mathcal{V}$-categories, $\mathcal{V}$-bimodules and $\mathcal{V}$-natural transformations between them. For the sake of brevity, we shall omit the prefix $\mathcal{V}$- when no confusion can occur. We shall also frequently use the classical terminology for 2-categories in the context of the bicategory $\mathsf{Bimod}_{\mathcal{V}}$, where everything must clearly be understood "up to isomorphism". For example, mutually inverse arrows in $\mathsf{Bimod}_{\mathcal{V}}$ is another way of saying "Morita equivalence". Such an attitude can be justified by the fact that every bicategory is bi-equivalent to the 2-category obtained by choosing as arrows the 2-isomorphism classes of 1-arrows.

We recall first some results which are part of the "folklore".

Lemma 4.1 *In a 2-category $\mathcal{A}$, suppose an arrow $f\colon A \longrightarrow B$ has an inverse $g\colon B \longrightarrow A$, that is, there are isomorphisms $f \circ g \cong 1_B$ and $g \circ f \cong 1_A$. In those conditions, the canonical 2-cells of every adjunction involving f are necessarily isomorphisms.*

Proof The problem occurs entirely in the full sub-2-category generated by A and B, so there is no restriction in assuming $\mathcal{A}$ to be small. Viewing $\mathcal{A}$ as a category enriched in the category Cat of small categories, we consider the corresponding Cat-Yoneda embedding

$$Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^*, \mathsf{Cat}].$$

which reduces the question to proving the theorem in the 2-category $[\mathcal{A}^*, \mathsf{Cat}]$. This reduces further the problem to proving the theorem pointwise in Cat, where it is a well-known fact about equivalences (see [5], section 3.4). ■

Proposition 4.2 *The bicategory* $\mathsf{Bimod}_{\mathcal{V}}$ *is compact. This means that it is monoidal, as a bicategory, and every object of* $\mathsf{Bimod}_{\mathcal{V}}$ *has an adjoint, when these objects are considered as the arrows of a tricategory with a single formal object, the composition of arrows in the tricategory being the tensor product of objects in* $\mathsf{Bimod}_{\mathcal{V}}$*. The adjoint of* $\mathcal{A}$ *is* $\mathcal{A}^*$ *and the 2-cells of the adjunction are the two bimodules corresponding to the canonical functor* $\mathcal{A}$*.*

Proof We refer to [7], [11] and [15] for the notions of compact bicategory or tricategory. The tensor product of $\mathcal{V}$-categories extends in a straightforward way to bimodules, that is, given an $\mathcal{A}$-$\mathcal{B}$-bimodule M and a $\mathcal{C}$-$\mathcal{D}$-bimodule N, one gets at once an $(\mathcal{A}\otimes\mathcal{C})$-$(\mathcal{B}\otimes\mathcal{D})$-bimodule $M\otimes N$. This makes the bicategory $\mathsf{Bimod}_{\mathcal{V}}$ a monoidal one (see [8]).

The identity arrow on the unique object of the tricategoy is the unit category $\mathcal{I}$. The canonical functor $\mathcal{A}: \mathcal{A}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}$ can be seen both as a $(\mathcal{A}^* \otimes \mathcal{A})$-$\mathcal{I}$-bimodule and a $\mathcal{I}$-$(\mathcal{A} \otimes \mathcal{A}^*)$-bimodule. For those bimodules, seen as 2-cells in the tricategory, the triangular identities for adjunction both reduce – up to permutation of the factors – to the identity

$$\mathcal{A}(A, A') \cong \int^{A_1, A_2, A_3} \mathcal{A}(A, A_1) \otimes \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_2, A_3) \otimes \mathcal{A}(A_3, A'),$$

which holds for every $\mathcal{V}$-category $\mathcal{A}$ (see [14]). ■

Corollary 4.3 *Conditions (1) and (2) in theorem 3.4 are equivalent.*

Proof (2)⇒(1) is obvious. Conversely, it is a classical argument to verify that given a Morita equivalence in $\mathsf{Bimod}_{\mathcal{V}}$, the natural isomorphisms can be modified to get an adjoint equivalence in $\mathsf{Bimod}_{\mathcal{V}}$ (the proof of 4.1 yields this conclusion as well). Applying 4.2 and the "uniqueness" of the adjoint, we conclude that in $\mathsf{Bimod}_{\mathcal{V}}$, $\mathcal{B}$ is equivalent to $\mathcal{A}^*$. ■

The reader should be well aware that equivalence in $\mathsf{Bimod}_{\mathcal{V}}$ reduces to Morita equivalence as $\mathcal{V}$-categories, not at all to an equivalence as $\mathcal{V}$-categories.

Now, let us make even more precise condition (2) of theorem 3.4.

Proposition 4.4 *For a* $\mathcal{V}$*-category* $\mathcal{A}$*, the following conditions are equivalent.*

(1) $\mathcal{A}^* \otimes \mathcal{A}$ *is Morita equivalent to the unit category* $\mathcal{I}$*.*

(2) The $\mathcal{I}$-$(\mathcal{A} \otimes \mathcal{A}^*)$*-bimodule induced by the canonical functor* $\mathcal{A}$ *is invertible in* $\mathsf{Bimod}_{\mathcal{V}}$*.*

(3) The $(\mathcal{A}^* \otimes \mathcal{A})$-$\mathcal{I}$*-bimodule induced by the canonical functor* $\mathcal{A}$ *is invertible in* $\mathsf{Bimod}_{\mathcal{V}}$*.*

Proof Proposition 4.2 implies at once the existence of a functorial involution

$$\mathsf{Bimod}_{\mathcal{V}} \longrightarrow \mathsf{Bimod}_{\mathcal{V}}, \quad \mathcal{A} \to \mathcal{A}^*, \quad (\varphi\colon \mathcal{A} \to \mathcal{B}) \mapsto (\varphi^*\colon \mathcal{B}^* \to \mathcal{A}^*)$$

where φ^* is the bimodule defined as follows:

$$\mathcal{B}^* \cong \mathcal{B}^* \otimes \mathcal{I} \xrightarrow{1 \otimes \mathcal{A}} \mathcal{B}^* \otimes \mathcal{A} \otimes \mathcal{A}^* \xrightarrow{1 \otimes \varphi \otimes 1} \mathcal{B}^* \otimes \mathcal{B} \otimes \mathcal{A}^* \xrightarrow{\mathcal{B} \otimes 1} \mathcal{I} \otimes \mathcal{A}^* \cong \mathcal{A}^*.$$

This involution interchanges the bimodules in conditions (2) and (3), from which the equivalence of those conditions.

Conditions (2) or (3) imply at once condition (1). Conversely, going back to the adjunction described in 4.2, condition (1) means that the adjoint arrows $\mathcal{A}$, $\mathcal{A}^*$ of the tricategory are such that there exist isomorphisms $\mathcal{A}^* \otimes \mathcal{A} \cong \mathsf{id}$ and $\mathcal{A} \otimes \mathcal{A}^* \cong \mathsf{id}$. By 4.1, the natural 2-cells of the canonical adjunction described in 4.2 are invertible; this means precisely conditions (2) and (3). ■

The conclusion of the proof of theorem 3.4 will follow from two applications of the next proposition.

Proposition 4.5 *Consider two small $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ and a $\mathcal{A}$-$\mathcal{B}$-bimodule φ. The following conditions are equivalent.*

(1) The bimodule φ is invertible in $\mathsf{Bimod}_{\mathcal{V}}$.

(2) When φ is viewed as a functor $\varphi\colon \mathcal{A} \longrightarrow [\mathcal{B}^*, \mathcal{V}]$, its left Kan extension $\mathsf{Lan}_Y \varphi$ along the Yoneda embedding is an equivalence of categories.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;Y\;} & [\mathcal{A}^*, \mathcal{V}] \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \mathsf{Lan}_Y \varphi} \\ & & [\mathcal{B}^*, \mathcal{V}] \end{array}$$

(3) a. The bimodule φ is a strong generator;
b. the bimodule φ has a right adjoint;
c. the functor $\varphi\colon \mathcal{A} \longrightarrow [\mathcal{B}^*, \mathcal{V}]$ is full and faithful.

Proof It is well known (see [14]) that the composition of bimodules corresponds to the ordinary composition of the corresponding Kan extensions $\mathsf{Lan}_Y \varphi$, from which the equivalence of (1) and (2).

The functor $\mathsf{Lan}_Y \varphi$ admits as a right adjoint the functor

$$[\varphi, -]\colon [\mathcal{B}^*, \mathcal{V}] \longrightarrow [\mathcal{A}^*, \mathcal{V}], \quad [\varphi, F](A) = \big[\varphi(-, A), F\big]$$

as follows from the pointwise definition of Kan extensions and the definition of weighted colimits (see [14]):

$$\big[(\mathsf{Lan}_Y\varphi)(G), F\big] \cong \big[G * \varphi, F\big] \cong \big[G, [\varphi, F]\big].$$

Observe that this functor $[\varphi, -]$ is precisely that appearing in definition 3.2.

When $\mathsf{Lan}_Y\varphi$ is an equivalence, so is its adjoint $[\varphi, -]$, from which condition (3).a follows at once. Moreover $\mathsf{Lan}_Y\varphi$ is an equivalence precisely when the bimodule φ is an equivalence in the bicategory $\mathsf{Bimod}_{\mathcal{V}}$, from which condition (3).b. Next since Y is full and faithful, the triangle in the statement commutes, and when $\mathsf{Lan}_Y\varphi$ is itself full and faithful as an equivalence of categories, it follows that the composite φ is full and faithful, yielding condition (3).c.

Now assume condition (3). By condition (3).c we can identify $\mathcal{A}$ with a full subcategory of $[\mathcal{B}^*, \mathcal{V}]$. By conditions (3).b, the objects of this subcategory are small projective in $[\mathcal{B}^*, \mathcal{V}]$ and by condition (3).a, they constitute a strong generator. Applying 5.26 in [14], we conclude that the functor $[\varphi, -]$ is an equivalence of categories, thus also its left adjoint $\mathsf{Lan}_Y\varphi$. ■

Corollary 4.6 *Conditions (2) and (3) in theorem 3.4 are equivalent.*

Proof It suffices to consider condition (2) in 4.4 and apply proposition 4.5 to the canonical functor $\mathcal{A}$ viewed as a $\mathcal{I}$-$(\mathcal{A} \otimes \mathcal{A}^*)$-bimodule. Condition (3).b is the separability of $\mathcal{A}$ and condition (3).c, the centrality of $\mathcal{A}$. ■

Corollary 4.7 *Conditions (2) and (4) in theorem 3.4 are equivalent.*

Proof It suffices to consider condition (3) in 4.4 and apply proposition 4.5 to the canonical functor $\mathcal{A}$ viewed as a $(\mathcal{A}^* \otimes \mathcal{A})$-$\mathcal{I}$-bimodule. ■

5 The $\mathcal{V}$-Brauer groups

Since the bicategory $\mathsf{Bimod}_{\mathcal{V}}$ is monoidal (see 4.2), the tensor product induces the structure of an abelian monoid on the Morita equivalence classes of small $\mathcal{V}$-categories. The invertible elements of this monoid constitute thus an abelian group. There is no reason a priori for the elements of this Brauer group to constitute a set, but this will nevertheless be true in a special case of interest (see proposition 5.2 and its corollary).

Definition 5.1 *Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category. The $\mathcal{V}$-Brauer group is the – possibly large – abelian group of invertible Morita equivalence classes of small $\mathcal{V}$-categories, for the multiplication induced by the tensor product of $\mathcal{V}$-categories.*

Proposition 5.2 *When the Cauchy completion of the unit $\mathcal{V}$-category $\mathcal{I}$ is small (up to equivalence), the $\mathcal{V}$-Brauer group is a set.*

Proof By condition (4) in theorem 3.4, the canonical functor

$$\mathcal{A}\colon \mathcal{A}^* \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

is full and faithful. But by the same condition, the corresponding $(\mathcal{A}^* \otimes \mathcal{A})$-$\mathcal{I}$-bimodule has a right adjoint; thus each object $\mathcal{A}(A,B)$ lies in the Cauchy completion $\overline{\mathcal{I}}$ of $\mathcal{I}$ (see [14]). Therefore $\mathcal{A}^* \otimes \mathcal{A}$ is equivalent – via the canonical functor $\mathcal{A}$ – to a full subcategory of $\overline{\mathcal{I}}$, thus to a small category, by assumption.

Without any restriction, we suppose $\overline{\mathcal{I}}$ to be small. Let us choose a regular cardinal α stricly bigger than the number of arrows in $\overline{\mathcal{I}}$. By what has just been proved, we can choose an α-family of objects (A,B) in $\mathcal{A}^* \otimes \mathcal{A}$ representative of all its isomorphism classes. The full subcategory $\mathcal{B}$ of $\mathcal{A}$ generated by all these objects A, B has still less than α objects and $\mathcal{B}^* \otimes \mathcal{A}$ contains a fortiori a representative of each isomorphism class in $\mathcal{A}^* \otimes \mathcal{A}$, thus is equivalent to $\mathcal{A}^* \otimes \mathcal{A}$. Therefore $\mathcal{B}^* \otimes \mathcal{A}$ is equivalent to $\mathcal{A}^* \otimes \mathcal{A}$, thus Morita equivalent to $\mathcal{I}$, and by theorem 3.4, $\mathcal{B}^*$ is Morita equivalent to $\mathcal{A}^*$ and is in particular an Azumaya category. Thus $\mathcal{A}$ is Morita equivalent to $\mathcal{B}$, which is an Azumaya category with less than α objects.

The conclusion follows at once. Each Azumaya category is Morita equivalent to an Azumaya category $\mathcal{B}$ with less than α objects. Moreover each object of arrows $\mathcal{B}(A,B)$ lies in $\overline{\mathcal{I}}$ by condition (4) in theorem 3.4. Finally since $\overline{\mathcal{I}}$ is stable under tensor product and contains the unit I, the structure of a $\mathcal{V}$-category on $\mathcal{B}$ (composition, units) is given by arrows in $\overline{\mathcal{I}}$. By the choice of α, it follows at once that the number on non equivalent such categories $\mathcal{B}$ is less than α. The corresponding number of Morita equivalence classes is a even smaller, proving that the $\mathcal{V}$-Brauer group is a set with cardinality less than α. ■

Corollary 5.3 *When the base category $\mathcal{V}$ is locally presentable, the $\mathcal{V}$-Brauer group is small.*

Proof By [12] and proposition 5.2. ■

We recall that a $\mathcal{V}$-algebra is a one-object $\mathcal{V}$-category. We call it an Azumaya $\mathcal{V}$-algebra when it is Azumaya, as a $\mathcal{V}$-category. Since the dual of a one-object category and the tensor product of one-object categories are again one-object categories, it follows at once that the classes of Azumaya $\mathcal{V}$-algebras constitute a subgroup of the $\mathcal{V}$-Brauer group. We conclude this paper with a result and various examples showing that in many cases of interest, this subgroup coincides with the whole $\mathcal{V}$-Brauer group.

Proposition 5.4 *Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category. Suppose the following assumption holds.*

> *If $\mathcal{A}$, $\mathcal{B}$ are Cauchy complete $\mathcal{V}$-categories*
> *Then every object in the Cauchy completion of $\mathcal{A} \otimes \mathcal{B}$ is a retract of an object in $\mathcal{A} \otimes \mathcal{B}$.*

In these conditions, every separable $\mathcal{V}$-category is Morita equivalent to a separable $\mathcal{V}$-algebra. In particular, every Azumaya $\mathcal{V}$-category is equivalent to an Azumaya $\mathcal{V}$-algebra.

Proof Let $\mathcal{A}$ be a separable $\mathcal{V}$-category. This category $\mathcal{A}$ is Morita equivalent to its (possibly large) Cauchy completion; therefore we shall assume at once that $\mathcal{A}$ is Cauchy complete separable, but not necessarily small. Observe that this makes perfect sense, since the category of modules on the Cauchy completion of a small category is always legitimate, as equivalent to the category of modules on the original category. We refer to [14] for the theory of Cauchy completion and adjoint bimodules.

By condition (3) in theorem 3.4, the canonical functor $\mathcal{A}$ viewed as a $\mathcal{I}$-$(\mathcal{A}\otimes\mathcal{A}^*)$-bimodule has a right adjoint, thus the functor $\mathcal{A}$ belongs to the Cauchy completion of $\mathcal{A}\otimes\mathcal{A}^*$. Since $\mathcal{A}$ is Cauchy complete, so is $\mathcal{A}^*$ and by the assumption in the statement, this implies that the canonical functor $\mathcal{A}$ is a retract of a representable functor on $\mathcal{A}^*\otimes\mathcal{A}$. In other words, there are an object $(U,V)\in\mathcal{A}^*\otimes\mathcal{A}$ and natural retractions

$$\mathcal{A}(A,B)\underset{\varepsilon_{AB}}{\overset{\varphi_{AB}}{\rightleftarrows}}\mathcal{A}(A,U)\otimes\mathcal{A}(V,B),\quad \varphi_{AB}\circ\varepsilon_{AB}=\mathsf{id}.$$

We shall prove that $\mathcal{A}$ is Morita equivalent to the full subcategory $\langle V\rangle\subseteq\mathcal{A}$ generated by the single object V, the Morita equivalence being constituted of the two canonical bimodules induced by the inclusion functor $\langle V\rangle\subseteq\mathcal{A}$.

- the $\langle V\rangle$-$\mathcal{A}$-bimodule α given by $\alpha(A,V)=\mathcal{A}(A,V)$;
- the $\mathcal{A}$-$\langle V\rangle$-bimodule β given by $\beta(V,A)=\mathcal{A}(V,A)$.

The relation $\beta\otimes\alpha\cong\mathsf{id}$ is easily checked:

$$\begin{aligned}(\beta\otimes\alpha)(V,V) &= \int^{A\in\mathcal{A}}\beta(V,A)\otimes\alpha(A,V)\\ &= \int^{A\in\mathcal{A}}\mathcal{A}(V,A)\otimes\mathcal{A}(A,V)\\ &= \mathcal{A}(V,V).\end{aligned}$$

The converse implication consists in proving that

$$\begin{aligned}(\alpha\otimes\beta)(A,B) &= \int^{V\in\langle V\rangle}\alpha(A,V)\otimes\beta(V,B)\\ &= \int^{V\in\langle V\rangle}\mathcal{A}(A,V)\otimes\mathcal{A}(V,B)\end{aligned}$$

is isomorphic to $\mathcal{A}(A,B)$. The coend we have to compute is given by the following coequalizer diagram

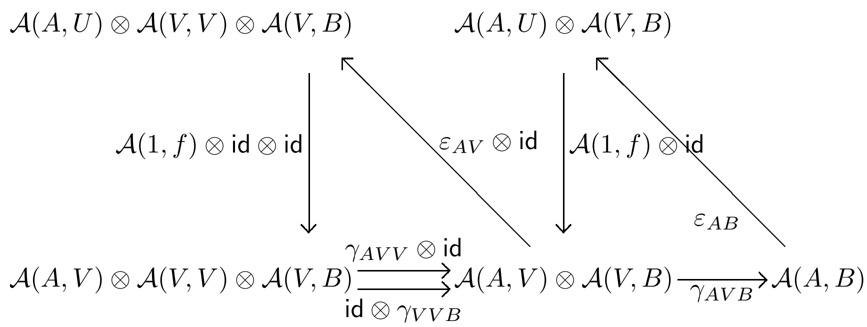
$$\mathcal{A}(A,V)\otimes\mathcal{A}(V,V)\otimes\mathcal{A}(V,B)\underset{\mathsf{id}\otimes\gamma_{VVB}}{\overset{\gamma_{AVV}\otimes\mathsf{id}}{\rightrightarrows}}\mathcal{A}(A,V)\otimes\mathcal{A}(V,B)\twoheadrightarrow\int^{V\in\langle V\rangle}\mathcal{A}(A,V)\otimes\mathcal{A}(V,B)$$

where the γ_{XYZ} are the "composition" morphisms of $\mathcal{A}$.

The composite

$$I \cong I \otimes I \xrightarrow{h_U \otimes h_V} \mathcal{A}(U,U) \otimes \mathcal{A}(V,V) \xrightarrow{\varphi_{UV}} \mathcal{A}(U,V),$$

where h_U, h_V are the unit arrows, defines a morphism $f\colon U \longrightarrow V$ in $\mathcal{A}$. Long but routine computation shows that the following diagram is an absolute coequalizer:



One concludes the proof by comparing the two coequalizers. ■

Let us conclude this paper with some examples indicating that the assumption in proposition 5.4 is widely satisfied.

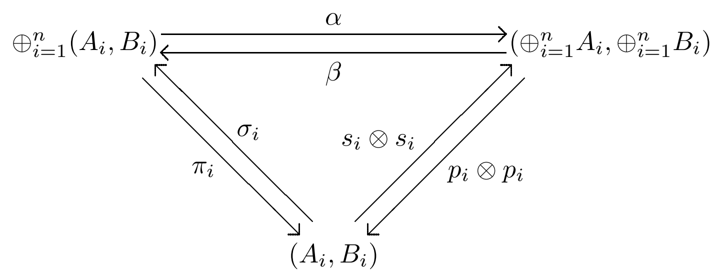
Example 5.5 *A complete and cocomplete cartesian closed category $\mathcal{V}$ satisfies the assumption in proposition 5.4*

Proof The tensor product of two $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ is their cartesian product $\mathcal{A} \times \mathcal{B}$. It is proved in [21] that a $\mathcal{V}$-category is Cauchy complete iff it admits all absolute weighted colimits. If $\mathcal{A}$ and $\mathcal{B}$ are such, then $\mathcal{A} \times \mathcal{B}$ admits all absolute weighted colimits, since these are computed componentwise. Therefore, $\mathcal{A} \otimes \mathcal{B}$ is at once Cauchy complete and the assumption in proposition 5.4 becomes redundant. ■

Example 5.6 *The category $\mathcal{V}$ of modules on a commutative ring R with unit satisfies the assumption in proposition 5.4*

Proof Let $\mathcal{A}$, $\mathcal{B}$ be Cauchy complete $\mathcal{V}$-categories. We know (see for example [12]) that an object in the Cauchy completion $\overline{\mathcal{A} \otimes \mathcal{B}}$ of $\mathcal{A} \otimes \mathcal{B}$ is a retract of a finite biproduct of objects in $\mathcal{A} \otimes \mathcal{B}$. So it suffices to prove that every finite biproduct in $\overline{\mathcal{A} \otimes \mathcal{B}}$ of objects lying in $\mathcal{A} \otimes \mathcal{B}$ is a retract of an object in $\mathcal{A} \otimes \mathcal{B}$.

Consider $\oplus_{i=1}^n (A_i, B_i) \in \overline{\mathcal{A} \otimes \mathcal{B}}$, with $(A_i, B_i) \in \mathcal{A} \otimes \mathcal{B}$ for each index i. By Cauchy completeness of $\mathcal{A}$ and $\mathcal{B}$, $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \in \mathcal{A} \otimes \mathcal{B}$. In $\overline{\mathcal{A} \otimes \mathcal{B}}$, it remains to consider the diagram



where p_i, s_i, π_i, σ_i are the canonical morphisms of the biproducts and α, β are the corresponding factorizations. An obvious diagram chasing yields $\alpha \circ \beta = \mathsf{id}$, which concludes the proof. ■

Corollary 5.7 *Let $\mathcal{V}$ be the category of modules on a commutative ring R with unit. The $\mathcal{V}$-Brauer group is isomorphic to the Brauer group of R.*

Proof By 5.6 and 5.4. ■

Example 5.8 *The category $\mathcal{V}$ of $\bigvee$-lattices satisfies the assumption of proposition 5.4.*

Proof The proof is analogous to that of example 5.6, working with infinite biproducts (see [13] and [12]). ■

Example 5.9 *If G is a commutative monoid, the category of G-sets provided with the tensor product*

$$A \otimes B = A \times B / \sim, \quad \textit{where} \quad (ag, b) \sim (a, bg)$$

for all $a \in A$, $b \in B$, $g \in G$, satisfies the conditions of proposition 5.4.

Proof Since in the category of G-sets, colimits are computed as in the category of sets, the explicit formula given in section 2 for the composite of a $\mathcal{A}$-$\mathcal{B}$-bimodule M and a $\mathcal{B}$-$\mathcal{C}$-bimodule N becomes

$$(N \odot M)(C, A) \cong \frac{\coprod_B N(C, B) \otimes M(B, A)}{\approx}$$

for A in $\mathcal{A}$, B in $\mathcal{B}$ and C in $\mathcal{C}$, with $\approx$ generated by

$$y \otimes M(f \otimes 1_A)(x) \approx N(1_C \otimes f)(y) \otimes x$$

for

$$y \otimes f \otimes x \in N(C, B'') \otimes \mathcal{B}(B'', B') \otimes M(B', A)$$

and B', B'' varying in $\mathcal{B}$.

Starting from this, one can prove that a $\mathcal{I}$-$\mathcal{A}$-bimodule M has a right adjoint if and only if, seen as a functor $\mathcal{A}^* \longrightarrow G\text{-}\mathsf{Set}$, it is a retract of a representable functor. The proof runs, up to straightforward modifications, as in the Set-based case (see [5]) and we omit it. This implies at once that the tensor product of two Cauchy complete $\mathcal{V}$-categories is Cauchy complete, which makes redundant the assumption in proposition 5.4. ■

Example 5.10 *Let $\mathcal{V}$ be the poset $[0,\infty]$ of extended positive reals, viewed as a category where $r \geq s$ induces an arrow $r \longrightarrow s$. This category $\mathcal{V}$ is complete and cocomplete and is symmetric monoidal closed when defining*

$$a \otimes b = a + b, \quad [b,c] = \max\{b - a, 0\}$$

(see [17]). This category satisfies the assumption in propositin 5.4.

Proof In this case a small $\mathcal{V}$-category is a set X provided with a "distance"

$$d\colon X \times X \longrightarrow [0,\infty]$$

satisfying the axioms

$$d(x,y) + d(y,z) \geq d(x,z), \quad 0 = d(x,x).$$

A sequence $(a_n)_{n\in\mathbb{N}}$ in (X,d) is a Cauchy sequence when

$$\forall \varepsilon > 0 \ \ \exists n \in \mathbb{N} \ \ \forall k,l > n \ \ d(x_k, x_l) < \varepsilon$$

and it converges to $x \in X$ when

$$\forall \varepsilon > 0 \ \ \exists n \in \mathbb{N} \ \ \forall k > n \ \ d(x, x_n) < \varepsilon \text{ and } d(x_n, x) < \varepsilon.$$

A $\mathcal{V}$-category is categorically Cauchy complete precisely when every Cauchy sequence converges uniquely (see [17] and [9]).

The tensor product of two $\mathcal{V}$-categories is

$$(A,d') \otimes (B,d'') = (A \times B, d), \quad d\big((a,b),(a',b')\big) = d'(a,a') + d''(b,b').$$

It is now routine to verify that the tensor product of two Cauchy complete categories is still Cauchy complete, which makes redundant the assumption in proposition 5.4.

Notice moreover that the unit (I,d) of the tensor product of $\mathcal{V}$-categories is the singleton $\{*\}$ with the obvious distance $d(*,*) = 0$; this (I,d) is obviously Cauchy complete. By condition (4) in theorem 3.4, every Azumaya category (A,d') is such that d' takes values in the Cauchy completion of (I,d), proving that $d'(a,a') = 0$ for all $a,a' \in A$; moreover (A,d') cannot be empty, due to its generating properties. Therefore every Azumaya category is Morita equivalent to the singleton and the Brauer group is reduced to the zero group. ■

To conclude this paper, let us mention that presently, we do not know any example where the assumption in proposition 5.4 is not satisfied. This will be investigated further in forthcoming papers, together with the applications of the present theory in contexts like Banach, metric or locally convex modules.

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