BIPULLBACKS AND CALCULUS OF FRACTIONS

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract. We prove that the class of weak equivalences between internal groupoids in a regular protomodular category is a bipullback congrence and, therefore, has a right calculus of fractions. As an application, we show that monoidal functors between internal groupoids in groups and homomorphisms of strict Lie 2-algebras are fractions of internal functors with respect to weak equivalences.

1. Introduction

It is well known that any monoidal category is monoidally equivalent to a strict one. This is not true for strong monoidal functors: not every strong monoidal functor is naturally isomorphic to a strict one (i.e., to a functor F such that the structural isomorphisms $FA \otimes FB \to F(A \otimes B)$ and $I \to FI$ are identities). An important example of this fact is given by Schreier theory of group extensions. In fact, let A and B be groups and write $D(A)$ for A seen as a discrete internal groupoid in the category *Grp* of groups, and $OUT(B)$ for the internal groupoid in *Grp* corresponding to the crossed module $B \to Aut(B)$ of inner automorphisms. Then internal (= strict) functors from $D(A)$ to $OUT(B)$ correspond to split extensions of A through B, whereas monoidal functors from $D(A)$ to $OUT(B)$ correspond to arbitrary extensions of A by B.

The previous example leads to the following question: what is the precise relation between the 2-category of internal groupoids and internal functors in *Grp* and the 2-category of internal groupoids in *Grp* and monoidal functors? The same question can be asked working internally to the category *Lie* of Lie K-algebras (for K a fixed field), replacing monoidal functors by homomorphisms of strict Lie 2-algebras (precise definitions are in Section 7).

A possible answer to the previous questions is suggested by the fact that if $F\colon \mathbb{C} \to \mathbb{D}$ is an internal functor in *Grp* which is a weak equivalence (i.e., full, faithful and essentially surjective on objects) then the quasi-inverse functor $F^{-1}\colon \mathbb{D} \to \mathbb{C}$ is no longer an internal functor, but it is still a monoidal one. More precisely, we prove that:

1. The 2-category of internal groupoids in *Grp* and monoidal functors is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Grp* with respect to weak equivalences.

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2. The 2-category of internal groupoids in *Lie* and homomorphisms is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Lie* with respect to weak equivalences.

The paper is organized as follows:

- In Section 2 we recall some basic facts on bicategories of fractions established by D. Pronk in [16]. We then revisit the right calculus of fractions for classes of 1-cells using bipullbacks.

- In Section 3 we show that, for a category $\mathcal{C}$ with finite limits, the 2-category *Grpd*($\mathcal{C}$) of internal groupoids and internal functors has bipullbacks. More precisely, we show that the standard homotopy pullback in *Grpd*($\mathcal{C}$) also satisfies the universal property of a bipullback.

- Using bipullbacks, we show in Section 4 that if $\mathcal{C}$ is regular, then the class of weak equivalences in *Grpd*($\mathcal{C}$) has a right calculus of fractions.

- In Section 5 we refine the previous result showing that if $\mathcal{C}$ is regular and protomodular, then weak equivalences satisfy the "$2 \Rightarrow 3$" property and therefore they are a bipullback congruence, a notion inspired by Bénabou's approach to categories of fractions (see [4]).

- In the last two sections we choose as base category $\mathcal{C}$ the category of groups (Section 6) and the category of Lie K-algebras (Section 7) and we prove the results announced above.

Since *Grp* and *Lie* are Mal'cev categories, internal categories coincide with internal groupoids (see [11]). This is the reason why we restrict our attention to internal groupoids.

Let me finish with some comments. The result established in Section 6 is not at all a surprise. In fact, if we work with isomorphism classes of internal functors, then Proposition 6.4 becomes a result on categories of fractions (not on 2-categories of fractions) quite easy to prove directly and also easy to deduce using the Quillen model structures studied in [13] and in [15]. So, in my opinion, what is interesting is not the result *per se* but the fact that the 2-categorical nature of its proof requires the use of bipullbacks, whereas other kinds of 2-dimensional limits (like homotopy pullbacks) are not convenient in this context (see the Introduction in [4] for some comments on bilimits). Concerning the analogous result for Lie algebras stated in Section 7, I think it is interesting for a completely different reason. The notion of monoidal functor is a well-established one, whereas the notion of homomorphism of Lie 2-algebras is much more recent, so Proposition 7.4 could help to understand the 2-dimensional theory of Lie algebras.

Notation: the composite of $f\colon A \to B$ and $g\colon B \to C$ is written $f \cdot g$ or fg.

Terminology: bicategory means bicategory with invertible 2-cells.

2. Bicategories of fractions

2.1 Categories of fractions have been introduced by P. Gabriel and M. Zisman in [14] (see also Ch. 5 in [5]). If $\mathcal{C}$ is a category and Σ a class of arrows in $\mathcal{C}$, the category of fractions of $\mathcal{C}$ with respect to Σ is a functor

$$P_\Sigma \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$$

universal among all functors $\mathcal{F} \colon \mathcal{C} \to \mathcal{A}$ such that $\mathcal{F}(s)$ is an isomorphism for all $s \in \Sigma$. This can be restated saying that for every category $\mathcal{A}$

$$P_\Sigma \cdot - \, : \, \mathit{Funct}(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \to \mathit{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$$

is an equivalence of categories, where $\mathit{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$ is the category of functors making the elements of Σ invertible. If the class Σ has a right calculus of fractions, then $\mathcal{C}[\Sigma^{-1}]$ has a quite simple description:

Proposition 2.2 *(Gabriel-Zisman) Assume that Σ satisfies the following conditions:*

CF1. Σ contains all identities;

CF2. Σ is closed under composition;

CF3. For every pair $f \colon A \to B \leftarrow C \colon g$ with $g \in \Sigma$ there exist $g' \colon P \to A$ and $f' \colon P \to C$ such that $g' \cdot f = f' \cdot g$ and $g' \in \Sigma$;

CF4. If a pair of parallel arrows is coequalized by an element of Σ, then it is also equalized by an element of Σ.

Then the objects of $\mathcal{C}[\Sigma^{-1}]$ are those of $\mathcal{C}$ and an arrow from A to B in $\mathcal{C}[\Sigma^{-1}]$ is a class of spans

$$A \xleftarrow{\;s\;} I \xrightarrow{\;f\;} B$$

with $s \in \Sigma$. Two spans (s, I, f) and (s', I', f') are equivalent if there exist arrows x, x' in $\mathcal{C}$ such that $x \cdot s = x' \cdot s' \in \Sigma$ and $x \cdot f = x' \cdot f'$.

The analogous problem for bicategories has been solved by D. Pronk in [16]. For an introduction to bicategories see [3] or Ch. 7 in [5] where 2-categories are also discussed.

Definition 2.3 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The bicategory of fractions of $\mathcal{B}$ with respect to Σ is a homomorphism of bicategories

$$P_\Sigma \colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F} \colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$. This can be restated saying that for every bicategory $\mathcal{A}$

$$P_\Sigma \cdot - \, : \, \mathit{Hom}(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \to \mathit{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where $\mathit{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$ is the bicategory of those homomorphisms $\mathcal{F}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

Definition 2.4 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The class Σ has a right calculus of fractions if the following conditions hold:

BF1. Σ contains all equivalences;

BF2. Σ is closed under composition;

BF3. For every pair $F\colon \mathbb{A} \to \mathbb{B} \leftarrow \mathbb{C}\colon G$ with $G \in \Sigma$ there exist $G'\colon \mathbb{P} \to \mathbb{A}$, $F'\colon \mathbb{P} \to \mathbb{C}$ and $\varphi\colon G' \cdot F \Rightarrow F' \cdot G$ with $G' \in \Sigma$;

BF4. For every $\alpha\colon F \cdot W \Rightarrow G \cdot W$ with $W \in \Sigma$ there exist $V \in \Sigma$ and $\beta\colon V \cdot F \Rightarrow V \cdot G$ such that $V \cdot \alpha = \beta \cdot W$, and for any other $V' \in \Sigma$ and $\beta'\colon V' \cdot F \Rightarrow V' \cdot G$ such that $V' \cdot \alpha = \beta' \cdot W$ there exist U, U' and $\varepsilon\colon U \cdot V \Rightarrow U' \cdot V'$ such that $U \cdot V \in \Sigma$ and

$$\begin{array}{ccc} U \cdot V \cdot F & \xrightarrow{U \cdot \beta} & U \cdot V \cdot G \\ {\scriptstyle \varepsilon \cdot F}\downarrow & & \downarrow{\scriptstyle \varepsilon \cdot G} \\ U' \cdot V' \cdot F & \xrightarrow[U' \cdot \beta']{} & U' \cdot V' \cdot G \end{array}$$

commutes;

BF5. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

If the class Σ has a right calculus of fractions, the bicategory $\mathcal{B}[\Sigma^{-1}]$ can be described in a way similar to that recalled in Proposition 2.2. Here we do not give full details because what we will use in Sections 6 and 7 is the following useful result:

Proposition 2.5 *(Pronk) Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$ which has a right calculus of fractions. Consider a homomorphism of bicategories $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$ and let $\widehat{\mathcal{F}}\colon \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$ be its extension. Then $\widehat{\mathcal{F}}$ is a biequivalence provided that $\mathcal{F}$ satisfies the following conditions:*

EF1. $\mathcal{F}$ is surjective up to equivalence on objects;

EF2. $\mathcal{F}$ is full and faithful on 2-cells;

EF3. For every 1-cell F in $\mathcal{A}$ there exist 1-cells G and W in $\mathcal{B}$ with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$.

(In [16] it is stated that conditions EF1-EF3 are also necessary for $\widehat{\mathcal{F}}$ being a biequivalence. This is not true, as proved by M. Dupont in [12].)

2.6 Recall that a diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in a bicategory $\mathcal{B}$ is a bipullback of F and G if for any other diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{K} & \mathbb{C} \\ {\scriptstyle H}\big\downarrow & \overset{\psi}{\Nearrow} & \big\downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

there exists a fill-in, that is a triple $(L\colon \mathbb{X} \to \mathbb{P},\ \alpha\colon L \cdot G' \Rightarrow H,\ \beta\colon L \cdot F' \Rightarrow K)$ such that

$$\begin{array}{ccc} L \cdot G' \cdot F & \xrightarrow{L\cdot\varphi} & L \cdot F' \cdot G \\ {\scriptstyle \alpha\cdot F}\big\downarrow & & \big\downarrow{\scriptstyle \beta\cdot G} \\ H \cdot F & \xrightarrow[\psi]{} & K \cdot G \end{array}$$

commutes, and for any other fill-in (L', α', β') there exists a unique $\lambda\colon L' \Rightarrow L$ such that

$$\begin{array}{ccc} L' \cdot G' & \xrightarrow{\lambda\cdot G'} & L \cdot G' \\ & {\scriptstyle \alpha'}\searrow \quad \swarrow{\scriptstyle \alpha} & \\ & H & \end{array} \qquad \begin{array}{ccc} L' \cdot F' & \xrightarrow{\lambda\cdot F'} & L \cdot F' \\ & {\scriptstyle \beta'}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & K & \end{array}$$

commute.

Remark 2.7 1. Bipullbacks are determined uniquely up to equivalence.

2. A 1-cell $W : \mathbb{B} \to \mathbb{A}$ is called full and faithful if for every $\mathbb{X}$ the hom-functor

$$\mathcal{B}(\mathbb{X}, W)\colon \mathcal{B}(\mathbb{X}, \mathbb{B}) \to \mathcal{B}(\mathbb{X}, \mathbb{A})$$

is full and faithful in the usual sense. Consider now the following diagrams, the first one being a bipullback,

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\ {\scriptstyle W_1}\big\downarrow & \overset{w}{\Nearrow} & \big\downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array} \qquad \begin{array}{ccc} \mathbb{B} & \xrightarrow{id} & \mathbb{B} \\ {\scriptstyle id}\big\downarrow & \overset{W}{\Nearrow} & \big\downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array}$$

Let $(D_W : \mathbb{B} \to \mathbb{K}, \delta_1 \colon D_W \cdot W_1 \Rightarrow id, \delta_2 \colon D_W \cdot W_2 \Rightarrow id)$ be the fill-in of the second diagram through the first one. Then W is full and faithful iff the second diagram is a bipullback iff the diagonal D_W is an equivalence.

Proposition 2.8 *Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. Assume that Σ satisfies the following conditions:*

BP1. Σ contains all equivalences;

BP2. Σ is closed under composition;

BP3. Σ is stable under bipullbacks;

BP4. If W is in Σ, then the diagonal D_W is in Σ;

BP5. If $\alpha \colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

Then Σ has a right calculus of fractions.

Proof. Clearly BP3 implies BF3. We have to show that BF4 holds. Consider the following diagrams, the first one being a bipullback,

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\ {\scriptstyle W_1}\downarrow & \overset{w}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array} \qquad \begin{array}{ccc} \mathbb{B} & \xrightarrow{id} & \mathbb{B} \\ {\scriptstyle id}\downarrow & \overset{W}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{B} \\ {\scriptstyle F}\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array}$$

Let $(D_W \colon \mathbb{B} \to \mathbb{K}, \delta_1, \delta_2)$ be the fill-in of the second diagram through the first one, and $(H \colon \mathbb{C} \to \mathbb{K}, \alpha_1, \alpha_2)$ the fill-in of the third diagram through the first one. Consider also the bipullback

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{V} & \mathbb{C} \\ {\scriptstyle L}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle H} \\ \mathbb{B} & \xrightarrow[D_W]{} & \mathbb{K} \end{array}$$

and define $\beta \colon V \cdot F \Rightarrow V \cdot G$ as follows

$$VF \xrightarrow{V\alpha_1^{-1}} VHW_1 \xrightarrow{\varphi^{-1}W_1} LD_WW_1 \xrightarrow{L\delta_1} L \xrightarrow{L\delta_2^{-1}} LD_WW_2 \xrightarrow{\varphi W_2} VHW_2 \xrightarrow{V\alpha_2} VG$$

Observe that since $W \in \Sigma$, then $D_W \in \Sigma$ by BP4, and then $V \in \Sigma$ by BP3. Moreover, the condition $V \cdot \alpha = \beta \cdot W$ follows from the fill-in condition on $(D_W, \delta_1, \delta_2)$ and (H, α_1, α_2).

Let $\beta' \colon V' \cdot F \Rightarrow V' \cdot G$ be such that $V' \in \Sigma$ and $V' \cdot \alpha = \beta' \cdot W$. We obtain two fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V' \cdot F} & \mathbb{B} \\ {\scriptstyle V' \cdot F}\downarrow & \overset{V'FW}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle W} \\ \mathbb{B} & \xrightarrow[W]{} & \mathbb{A} \end{array}$$

through the bipullback $(\mathbb{K}, W_1, W_2, w)$: the first one is

$$(\, \mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{F} \mathbb{B} \xrightarrow{D_W} \mathbb{K}\, , V' \cdot F \cdot \delta_1, V' \cdot F \cdot \delta_2)$$

and the second one is

$$(\, \mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{H} \mathbb{K}\, , V' \cdot \alpha_1,\ V'HW_2 \xrightarrow{V'\alpha_2} V'G \xrightarrow{(\beta')^{-1}} V'F\,)$$

By the universal property of $(\mathbb{K}, W_1, W_1, w)$, there exists a unique $\beta^* \colon V' \cdot F \cdot D_W \Rightarrow V' \cdot H$ such that

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_1 & \xrightarrow{\beta^* \cdot W_1} & V' \cdot H \cdot W_1 \\ & {\searrow}^{V' \cdot F \cdot \delta_1} \quad {\swarrow}_{V' \cdot \alpha_1} & \\ & V' \cdot F & \end{array} \qquad \begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_2 & \xrightarrow{\beta^* \cdot W_2} & V' \cdot H \cdot W_2 \\ {\scriptstyle V' \cdot F \cdot \delta_2} \downarrow & & \downarrow {\scriptstyle V' \cdot \alpha_2} \\ V' \cdot F & \xrightarrow[\beta']{} & V' \cdot G \end{array}$$

commute. Let $(U \colon \mathbb{D}' \to \mathbb{D}, \eta \colon U \cdot L \Rightarrow V' \cdot F, \varepsilon \colon U \cdot V \Rightarrow V')$ be the fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V'} & \mathbb{C} \\ {\scriptstyle V' \cdot F} \downarrow & \overset{\beta^*}{\nearrow\!\!\!\nearrow} & \downarrow {\scriptstyle H} \\ \mathbb{B} & \xrightarrow[D_W]{} & \mathbb{K} \end{array}$$

through the bipullback $(\mathbb{D}, L, V, \varphi)$. If we choose $U' = id$, we have $\varepsilon \colon U \cdot V \Rightarrow U' \cdot V'$. Since $V' \in \Sigma$, then also $U' \cdot V'$ and $U \cdot V$ are in Σ because of BP1, BP2 and BP5. It remains to check the compatibility of ε, β and β' as in BF4, but this is just a diagram chasing. ∎

3. Bipullbacks in *Grpd*($\mathcal{C}$)

The aim of this section is to prove the following result:

Proposition 3.1 *Let $\mathcal{C}$ be a category with finite limits, and let Grpd($\mathcal{C}$) be the 2-category of internal groupoids, internal functors and internal natural transformations in $\mathcal{C}$. The 2-category Grpd($\mathcal{C}$) has bipullbacks.*

3.2 Let us fix notation (details can be found in Ch. 7 of [5] or in Appendix 3 of [7]):

- An internal groupoid $\mathbb{C}$ is represented by

$$C_1 \times_{c,d} C_1 \xrightarrow{m} C_1 \underset{c}{\overset{d}{\underset{\xleftarrow{e}}{\rightrightarrows}}} C_0 \qquad C_1 \xrightarrow{i} C_1$$

  where the following diagram is a pullback

$$\begin{array}{ccc} C_1 \times_{c,d} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

- An internal functor $F \colon \mathbb{C} \to \mathbb{D}$ is represented by

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle d} \downarrow\downarrow {\scriptstyle c} & & {\scriptstyle d} \downarrow\downarrow {\scriptstyle c} \\ C_0 & \xrightarrow[F_0]{} & D_0 \end{array}$$

- An internal natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{C} \to \mathbb{D}$ is represented by

$$\begin{array}{ccc} C_1 & \overset{F_1}{\underset{G_1}{\rightrightarrows}} & D_1 \\ d \downarrow\downarrow c & \nearrow^{\alpha} & d \downarrow\downarrow c \\ C_0 & \overset{F_0}{\underset{G_0}{\rightrightarrows}} & D_0 \end{array}$$

3.3 It is helpful to start recalling that in *Grpd*(*Set*) bipullbacks are comma-squares. With the notations of 2.6:

- an object in $\mathbb{P}$ is a triple $(a_0 \in A_0, b_1\colon F_0(a_0) \to G_0(c_0), c_0 \in C_0)$,
- an arrow from (a_0, b_1, c_0) to (a'_0, b'_1, c'_0) is a pair of arrows $(a_1\colon a_0 \to a'_0, c_1\colon c_0 \to c'_0)$ such that $F_1(a_1) \cdot b'_1 = b_1 \cdot G_1(c_1)$,
- $G'\colon \mathbb{P} \to \mathbb{A}$ and $F'\colon \mathbb{P} \to \mathbb{C}$ are the obvious projections, and $\varphi(a_0, b_1, c_0) = b_1$,
- $L_0(x_0) = (H_0(x_0), \psi(x_0), K_0(x_0))$, $L_1(x_1) = (H_1(x_1), K_1(x_1))$, $\alpha = id$ and $\beta = id$,
- $\lambda(x_0) = (\alpha'(x_0), \beta'(x_0))$.

3.4 The description of bipullbacks in *Grpd*(*Set*) recalled in 3.3 indicates that the first step to obtain bipullbacks in *Grpd*($\mathcal{C}$) is to construct from an internal groupoid $\mathbb{B}$ a new internal groupoid $\vec{\mathbb{B}}$ whose objects are arrows in $\mathbb{B}$ and whose arrows are commutative squares in $\mathbb{B}$. The construction of $\vec{\mathbb{B}}$ is quite standard:

$$\vec{\mathbb{B}} = \left(\vec{B}_1 \times_{\vec{c},\vec{d}} \vec{B}_1 \xrightarrow{\vec{m}} \vec{B}_1 \overset{\vec{d}}{\underset{\vec{c}}{\rightrightarrows}} B_1 \; (\vec{e}\colon B_1 \to \vec{B}_1) \qquad \vec{B}_1 \xrightarrow{\vec{i}} \vec{B}_1 \right)$$

- $\vec{B}_1$ is defined by the following pullback

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{m_2} & B_1 \times_{c,d} B_1 \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m} \\ B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

- $\vec{d} = m_1 \cdot \pi_1$ and $\vec{c} = m_2 \cdot \pi_2$,
- $\vec{e}$ is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$\begin{array}{ccccc} B_1 & \xrightarrow{<d,1>} & B_0 \times B_1 & \xrightarrow{e\times 1} & B_1 \times_{c,d} B_1 \\ {\scriptstyle <1,c>}\downarrow & & & & \downarrow{\scriptstyle m} \\ B_1 \times B_0 & \xrightarrow[1\times e]{} & B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

- we leave to the reader the task of describing $\vec{m}$ and $\vec{i}$.

3.5 The internal groupoid $\vec{\mathbb{B}}$ is equipped with two internal functors $\delta, \gamma\colon \vec{\mathbb{B}} \to \mathbb{B}$ specified by

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\delta_1=m_2\cdot\pi_1} & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\delta_0=d]{} & B_0 \end{array} \qquad \begin{array}{ccc} \vec{B}_1 & \xrightarrow{\gamma_1=m_1\cdot\pi_2} & B_1 \\ {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ B_1 & \xrightarrow[\gamma_0=c]{} & B_0 \end{array}$$

and it tourns out that to give an internal natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{A} \to \mathbb{B}$ is the same as giving an internal functor $\alpha\colon \mathbb{A} \to \vec{\mathbb{B}}$ such that $\alpha \cdot \delta = F$ and $\alpha \cdot \gamma = G$. Indeed, the internal functor α is specified by

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & \vec{B}_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle \vec{d}}\downarrow\downarrow{\scriptstyle \vec{c}} \\ A_0 & \xrightarrow[\alpha]{} & B_1 \end{array}$$

where α_1 is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{<1,c>} & A_1 \times A_0 & \xrightarrow{F_1\times\alpha} & B_1 \times_{c,d} B_1 \\ {\scriptstyle <d,1>}\downarrow & & & & \downarrow{\scriptstyle m} \\ A_0 \times A_1 & \xrightarrow[\alpha\times G_1]{} & B_1 \times_{c,d} B_1 & \xrightarrow[m]{} & B_1 \end{array}$$

3.6 We are ready to prove Proposition 3.1. We use the notations of 2.6.

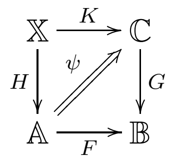
Proof. Given $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in $\mathit{Grpd}(\mathcal{C})$, a bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

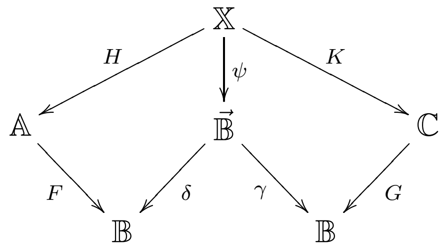
is given by the following limit in $\mathit{Grpd}(\mathcal{C})$ (recall that $\mathit{Grpd}(\mathcal{C})$ has limits computed componentwise in $\mathcal{C}$)

$$\begin{array}{ccccccc} & & & \mathbb{P} & & & \\ & & {\scriptstyle G'}\swarrow & \downarrow{\scriptstyle \varphi} & \searrow{\scriptstyle F'} & & \\ & \mathbb{A} & & \vec{\mathbb{B}} & & \mathbb{C} & \\ & {\scriptstyle F}\searrow & \swarrow{\scriptstyle \delta} & & {\scriptstyle \gamma}\searrow & \swarrow{\scriptstyle G} & \\ & & \mathbb{B} & & \mathbb{B} & & \end{array}$$

Indeed, any diagram

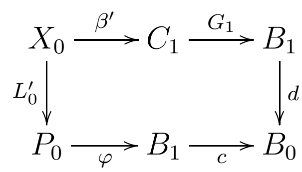


produces a commutative diagram

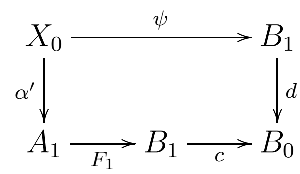


so that following the universal property of $\mathbb{P}$ as a limit there exists a unique $L\colon \mathbb{X} \to \mathbb{P}$ such that $L \cdot G' = H$, $L \cdot F' = K$ and $L \cdot \varphi = \psi$. (In other words, $(\mathbb{P}, G', F', \varphi)$ is the standard homotopy pullback of F and G.)
Clearly, $(L, \alpha = id, \beta = id)$ is a fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. Let (L', α', β') be another fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. We have to show that there exists a unique $\lambda\colon L' \Rightarrow L$ such that $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. Define:

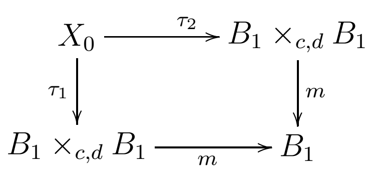
- τ_1 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram



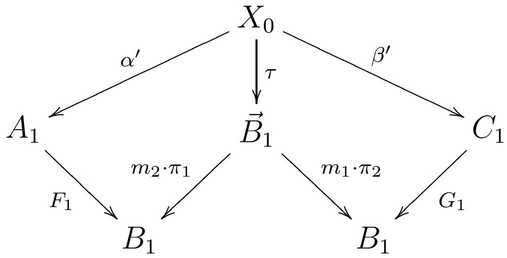
- τ_2 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram



- τ to be the unique factorization through $\vec{B}_1$ of the following diagram



Finally, λ is the unique factorization through P_1 of the following diagram



Clearly, $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. To check that $\lambda \cdot d = L'_0$ and $\lambda \cdot c = L_0$, the naturality of λ, and its uniqueness is a diagram chasing using that $\{G'_1, \varphi_1, F'_1\}$, $\{m_1, m_2\}$ and $\{\pi_1, \pi_2\}$ are jointly monomorphic. ∎

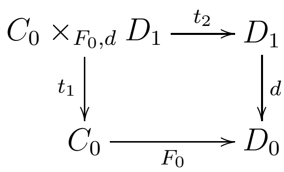
4. Weak equivalences in *Grpd*($\mathcal{C}$)

Definition 4.1 *(Bunge-Paré)* Let $F \colon \mathbb{C} \to \mathbb{B}$ be in *Grpd*($\mathcal{C}$).

1. F is essentially surjective on objects if

$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

   is a regular epimorphism, where t_2 is given by the following pullback



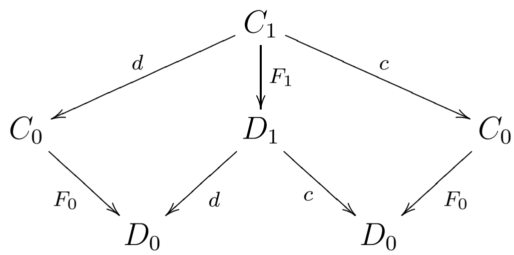
2. F is a weak equivalence if it is full and faithful (see 2.7) and essentially surjective on objects.

The previous definition is due to M. Bunge and R. Paré (see [10]). In [13] a more general notion of weak equivalence involving a Grothendieck topology on $\mathcal{C}$ has been considered. Since in Sections 6 and 7 the base category $\mathcal{C}$ is regular, I adopt for the moment the definition of Bunge and Paré. More on this point is contained in 5.10.

Next lemma is well-known and we only sketch the proof.

Lemma 4.2 *Let $F \colon \mathbb{C} \to \mathbb{D}$ be in Grpd($\mathcal{C}$).*

1. F is full and faithful if and only if the following is a limit diagram

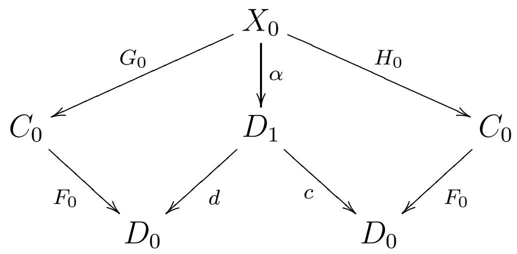


2. F is an equivalence if and only if it is full and faithful and

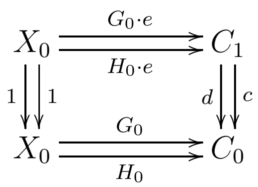
$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

is a split epimorphism.

Proof. 1. If the diagram is a limit diagram and $\alpha\colon G \cdot F \Rightarrow H \cdot F \colon \mathbb{X} \to \mathbb{D}$ is an internal natural transformation, then $\alpha \cdot d = G_0 \cdot F_0$ and $\alpha \cdot c = H_0 \cdot F_0$. By the universal property of C_1 we get a unique $\beta\colon X_0 \to C_1$ such that $\beta \cdot d = G_0$, $\beta \cdot d = H_0$ and $\beta \cdot F_1 = \alpha$. So we have $\beta\colon G \Rightarrow H$ such that $\beta \cdot F = \alpha$. (The naturality of β follows from that of α.)
Conversely, any commutative diagram



gives rise to internal functors $G, H\colon \mathbb{X} \to \mathbb{C}$ with discrete domain



and to an internal natural transformation $\alpha\colon G \cdot F \Rightarrow H \cdot F$. To give an internal natural transformation $\beta\colon G \Rightarrow H$ such that $\beta \cdot F = \alpha$ means precisely to give a factorization $\beta\colon X_0 \to C_1$ of (G_0, α, H_0) through (d, F_1, c).
2. Let F be an equivalence and consider an internal natural transformation $\beta\colon G \cdot F \Rightarrow Id_{\mathcal{D}}$. Since $\beta \cdot d = G_0 \cdot F_0$, there exists a unique $j\colon D_0 \to C_0 \times_{F_0,d} D_1$ such that $j \cdot t_1 = G_0$ and $j \cdot t_2 = \beta$. Therefore $j \cdot t_2 \cdot c = \beta \cdot c = id$.
Conversely, if $j\colon D_0 \to C_0 \times_{F_0,d} D_1$ such that $j \cdot t_2 \cdot c = id$, we can construct a quasi-inverse internal functor $G\colon \mathbb{D} \to \mathbb{C}$ as follows: first define G_0 by

$$G_0 = j \cdot t_1 \colon D_0 \to C_0 \times_{F_0,d} D_1 \to C_0$$

Then, define $j_1\colon D_1 \to D_1$ by

$$j_1 = < d \cdot j \cdot t_2, 1, c \cdot j \cdot t_2 \cdot i > \cdot (m \times 1) \cdot m \colon D_1 \to D_1 \times_{c,d} D_1 \times_{c,d} D_1 \to D_1$$

Finally, since F is full and faithful, by the first part of the lemma we get a unique arrow $G_1\colon D_1 \to C_1$ such that $G_1 \cdot d = d \cdot G_0$, $G_1 \cdot F_1 = j_1$ and $G_1 \cdot c = c \cdot G_0$. ■

Corollary 4.3 *Every equivalence in Grpd($\mathcal{C}$) is a weak equivalence. The converse is true provided that in $\mathcal{C}$ the axiom of choice holds (i.e., regular epimorphisms split).*

4.4 Regular categories have been introduced by M. Barr in [2] (see also Ch. 2 in [6]). In a regular category regular epimorphisms behave well: they are closed under composition and finite products, stable under pullbacks, and if a composite arrow $f \cdot g$ is a regular epimorphism, then g is a regular epimorphism. It follows that if $F\colon \mathbb{C} \to \mathbb{D}$ is in *Grpd*($\mathcal{C}$) with $\mathcal{C}$ regular and if F_0 is a regular epimorphism, then F is essentially surjective on objects.

Proposition 4.5 *Let $\mathcal{C}$ be a regular category and let Σ be the class of weak equivalences in Grpd($\mathcal{C}$). Then Σ has a right calculus of fractions.*

Proof. Since by Proposition 3.1 *Grpd*($\mathcal{C}$) has bipullbacks, to prove that Σ has a right calculus of fractions we check conditions BP1–BP5 in Proposition 2.8.
BP1 is given by Corollary 4.3, BP4 follows from 2.7 and BP5 is an exercise for the reader.
BP2: full and faithful internal functors are closed under composition because so they are in *Grpd*(*Set*). Assume now that $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ are essentially surjective. Consider the following pullbacks

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_0 \times_{G_0,d} C_1 & \xrightarrow{t_2} & C_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ B_0 & \xrightarrow[G_0]{} & C_0 \end{array} \qquad \begin{array}{ccc} A_0 \times_{F_0G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0\cdot G_0]{} & C_0 \end{array}$$

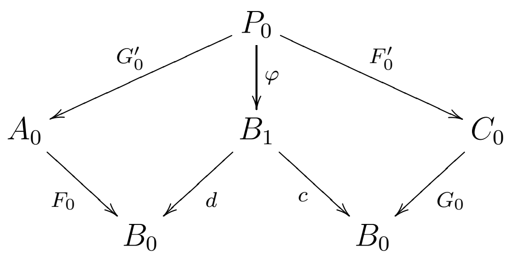
The essential surjectivity of $F \cdot G$ comes from the commutativity of the following diagram

$$\begin{array}{ccccc} A_0 \times_{F_0,d} B_1 \times_{G_1c,d} C_1 & \xrightarrow{t_2\times 1} & B_1 \times_{G_1c,d} C_1 & \xrightarrow{c\times 1} & B_0 \times_{G_0,d} C_1 \\ {\scriptstyle 1\times G_1\times 1}\downarrow & & & & \downarrow{\scriptstyle t_2} \\ A_0 \times_{F_0G_0,d} C_1 \times_{c,d} C_1 & & & & C_1 \\ {\scriptstyle 1\times m}\downarrow & & & & \downarrow{\scriptstyle c} \\ A_0 \times_{F_0G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

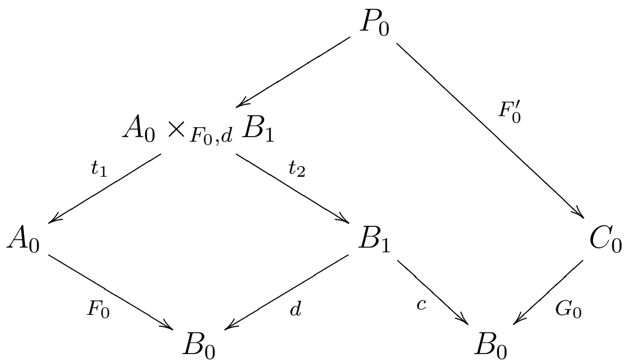
BP3: full and faithful internal functors are stable under bipullbacks because so they are in *Grpd*(*Set*) (use 3.3) and *Grpd*($\mathcal{C}$)($\mathbb{X}, -$): *Grpd*($\mathcal{C}$) $\to$ *Grpd*(*Set*) preserves bipullbacks. Consider now a bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

and assume that F is essentially surjective. Following the description of $\mathbb{P}$ given at the beginning of 3.6, we have a limit diagram in $\mathcal{C}$



But such a limit can be obtained performing two pullbacks as follows



Since by assumption $t_2 \cdot c\colon A_0 \times_{F_0,d} B_1 \to B_1 \to B_0$ is a regular epimorphism, F'_0 also is a regular epimorphism and then F' is essentially surjective (see 4.4). ■

5. Bipullback congruences

Next definition is the direct bicategorical generalization of the notion of pullback congruence introduced by J. Bénabou in [4].

Definition 5.1 Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. The class Σ is a bipullback congruence if the following conditions hold:

BC1. Σ contains all equivalences;

BC2. Σ satisfies the "$2 \Rightarrow 3$" property: let $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$ be 1-cells in $\mathcal{B}$; if two of F, G and $F \cdot G$ are in Σ, then the third one is in Σ;

BC3. Σ is stable under bipullbacks;

BC4. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

Proposition 5.2 *Let $\mathcal{B}$ be a bicategory with bipullbacks. Any bipullback congruence has a right calculus of fractions.*

Proof. It is enough to prove that a bipullback congruence Σ satisfies condition BP3 in Proposition 2.8. Let $W\colon \mathbb{B} \to \mathbb{A}$ be in Σ and let $(D_W\colon \mathbb{B} \to \mathbb{K}, \delta_1\colon D_W \cdot W_1 \Rightarrow id, \delta_2\colon D_W \cdot W_2 \Rightarrow id)$ be the diagonal fill-in as in 2.7. By BC1, $id \in \Sigma$, and then by BC4 $D_W \cdot W_1 \in \Sigma$. Since by BC3 $W_1 \in \Sigma$, we conclude by BC2 that $D_W \in \Sigma$. ■

5.3 Protomodular categories have been introduced by D. Bourn in [8] (see also [7]). Since we are concerned only with regular categories, we can consider the next lemma, proved in [9], as a definition of protomodular category. This lemma makes also evident the analogy between bipullback congruences and regular protomodular categories: in a regular protomodular category pullbacks satisfies the "2 ⇒ 3" property. This analogy will be made precise in Proposition 5.5.

Lemma 5.4 *(Bourn-Gran) Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

1. *$\mathcal{C}$ is protomodular;*
2. *In any commutative diagram*

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow{\scriptstyle b} & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

where b is a regular epimorphism, if the left hand square and the outer rectangle are pullbacks, then the right hand square is a pullback.

Proposition 5.5 *Let $\mathcal{C}$ be a regual protomodular category. The class of weak equivalences in Grpd($\mathcal{C}$) is a bipullback congruence.*

Let $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ be in *Grpd*($\mathcal{C}$). In order to prove Proposition 5.5 we need two lemmas on the shape of certains limits. The proof is routine.

Lemma 5.6 *Consider the pullbacks*

$$\begin{array}{ccc} A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\ A_0 & \xrightarrow[F_0]{} & B_0 \end{array} \qquad \begin{array}{ccc} B_1 \times_{c,F_0} A_0 & \xrightarrow{s_2} & A_0 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ B_1 & \xrightarrow[c]{} & B_0 \end{array}$$

and the commutative diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow{c} & A_0 \\ {\scriptstyle <d,F_1>}\downarrow & (1) & \downarrow{\scriptstyle F_0} \\ A_0 \times_{F_0,d} B_1 \xrightarrow[t_2]{} B_1 & \xrightarrow[c]{} & B_0 \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{d} & A_0 \\ {\scriptstyle <F_1,c>}\downarrow & (2) & \downarrow{\scriptstyle F_0} \\ B_1 \times_{c,F_0} A_0 \xrightarrow[s_1]{} B_1 & \xrightarrow[d]{} & B_0 \end{array}$$

Then $F\colon \mathbb{A} \to \mathbb{B}$ is full and faithful iff (1) is a pullback iff (2) is a pullback.

Lemma 5.7 *Consider the pullbacks*

$$\begin{array}{ccc} A_0\times_{F_0,d}B_1 & \xrightarrow{t_2} & B_1\\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d}\\ A_0 & \xrightarrow[F_0]{} & B_0\end{array}\qquad \begin{array}{ccc} C_1\times_{c,G_0}B_0 & \xrightarrow{s_2} & B_0\\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle G_0}\\ C_1 & \xrightarrow[c]{} & C_0\end{array}\qquad \begin{array}{ccc} A_0\times_{F_0G_0,d}C_1 & \xrightarrow{\tau_2} & C_1\\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle d}\\ A_0 & \xrightarrow[F_0\cdot G_0]{} & C_0\end{array}$$

and the commutative diagrams

$$\begin{array}{ccccc} A_0\times_{F_0,d}B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{\langle G_1,c\rangle} & C_1\times_{c,G_0}B_0\\ & & & & \downarrow{\scriptstyle s_1}\\ {\scriptstyle t_1}\downarrow & & (3) & & C_1\\ & & & & \downarrow{\scriptstyle d}\\ A_0 & \xrightarrow[F_0]{} & B_0 & \xrightarrow[G_0]{} & C_0\end{array}\qquad \begin{array}{ccccc} A_0\times_{F_0,d}B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0\\ {\scriptstyle 1\times G_1}\downarrow & & (4) & & \downarrow{\scriptstyle G_0}\\ A_0\times_{F_0G_0,d}C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0\end{array}$$

Then (3) *is a pullback iff* (4) *is a pullback.*

5.8 We are ready to prove Proposition 5.5.

Proof. Let Σ be the class of weak equivalences in *Grpd*($\mathcal{C}$). We have to show that condition BC2 holds, since the other conditions have been checked in the proof of Proposition 4.5. More precisely, given $F\colon \mathbb{A}\to\mathbb{B}$ and $G\colon \mathbb{B}\to\mathbb{C}$ in *Grpd*($\mathcal{C}$) such that $F\cdot G\in\Sigma$, we have to prove that $F\in\Sigma$ iff $G\in\Sigma$. There are two not obvious steps. (The protomodularity of $\mathcal{C}$ is needed only for the first step.)

1. If $F\cdot G$ is full and faithful and F is a weak equivalence, then G is full and faithful. Consider the following commutative diagram

$$\begin{array}{ccccc} A_1 & & \xrightarrow{c} & & A_0\\ {\scriptstyle \langle d,F_1\rangle}\downarrow & & & & \downarrow{\scriptstyle F_0}\\ A_0\times_{F_0,d}B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0\\ {\scriptstyle 1\times G_1}\downarrow & & & & \downarrow{\scriptstyle G_0}\\ A_0\times_{F_0G_0,d}C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0\end{array}$$

Since F is full and faithful, by Lemma 5.6 the top square is a pullback. Since $F\cdot G$ is full and faithful, by Lemma 5.6 the outer rectangle is a pullback. Since F is essentially surjective, the second row is a regular epimorphism. Following Lemma 5.4 the bottom square is a pullback. Therefore, by Lemma 5.7, the outer rectangle of the following commutative diagram is a pullback

$$\begin{array}{ccccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{<G_1,c>} & C_1 \times_{c,G_0} B_0 \\
 & & & & \downarrow{\scriptstyle s_1} \\
{\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} & & C_1 \\
 & & & & \downarrow{\scriptstyle d} \\
A_0 & \xrightarrow[F_0]{} & B_0 & \xrightarrow[G_0]{} & C_0
\end{array}$$

Since the left hand square is a pullback by definition and the second column is a split epimorphism, by Lemma 5.4 the right hand square is a pullback. By Lemma 5.6 again we conclude that G is full and faithful.
2. If $F \cdot G$ is essentially surjective and G is full and faithful, then F is essentially surjective. Consider the following pullback (notations as in Lemma 5.7)

$$\begin{array}{ccccc}
Q & & \xrightarrow{\lambda_2} & & B_0 \\
{\scriptstyle \lambda_1}\downarrow & & & & \downarrow{\scriptstyle G_0} \\
A_0 \times_{F_0G_0,d} C_1 & \xrightarrow[\tau_2]{} & C_1 & \xrightarrow[c]{} & C_0
\end{array}$$

By assumption $\tau_2 \cdot c$ is a regular epimorphism, so that λ_2 also is a regular epimorphism. Since G is full and faithful, there exists $\lambda\colon Q \to B_1$ such that $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$, $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$ and $\lambda \cdot c = \lambda_2$. From the first equation on λ, we deduce the existence of $\mu\colon Q \to A_0 \times_{F_0,d} B_1$ such that $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$ and $\mu \cdot t_2 = \lambda$. Finally, $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$, so that $t_2 \cdot c$ is a regular epimorphism. (Note that we need only the existence of λ, not its uniqueness. In other words we only use the "fullness" of G, and not its "faithfulness".) ■

5.9 Observe that, contrarily to Lemma 5.4, Proposition 5.5 is not a characterization of regular protomodular categories. Indeed, if $\mathcal{C}$ is *Set* (more generally, if in $\mathcal{C}$ the axiom of choice holds) then weak equivalences in *Grpd*($\mathcal{C}$) are the same that equivalences (see Corollary 4.3), and the class of equivalences obviously is a bipullback congruence.

5.10 G. Janelidze pointed out to me that condition 2 in Lemma 5.4 holds in any protomodular (not necessarily regular) category $\mathcal{C}$ provided that the arrow b is a pullback stable strong epimorphism. This fact has an interesting consequence. Indeed, Proposition 4.5 holds when $\mathcal{C}$ is any finitely complete category and Σ is the class of "weak $\mathcal{E}$-equivalences", where:

- $\mathcal{E}$ is any class of arrows that behaves well (in the sense explained in 4.4) and contains the split epimorphisms,
- an internal functor F is a weak $\mathcal{E}$-equivalence if it is full and faithful and essentially $\mathcal{E}$-surjective (that is, the arrow $t_2 \cdot c\colon C_0 \times_{F_0,d} D_1 \to D_1 \to D_0$ of Definition 4.1 is in $\mathcal{E}$).

Therefore, Proposition 5.5 holds for weak $\mathcal{E}$-equivalences in any protomodular category $\mathcal{C}$ provided that $\mathcal{E}$ behaves well, contains the split epimorphisms and is contained in the class of pullback stable strong epimorphisms. Examples are:

i. the class of pullback stable regular epimorphisms,

ii. the class of pullback stable regular epimorphisms that are effective descent morphisms.

6. Monoidal functors

All along this section we fix $\mathcal{C} = Grp$, the category of groups, which is a regular and protomodular category. I use additive notation for groups.

6.1 The aim of this section is to prove that the 2-category *MON* described hereunder is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences.

1. Objects of *MON* are internal groupoids in *Grp*. Note that since the forgetful functor $Grp \to Set$ preserves finite limits, any object of *MON* is also a groupoid in the usual sense.

2. 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ in *MON* are monoidal functors, that is, pairs (F, F_2) where F is a (not necessarily internal) functor and

$$F_2 = \{F_2^{a,b}\colon Fa + Fb \to F(a+b)\}_{a,b \in A_0}$$

is a natural family of arrows in $\mathbb{B}$ satisfying the cocycle condition

$$\begin{array}{ccc} Fa + Fb + Fc & \xrightarrow{1+F_2^{b,c}} & Fa + F(b+c) \\ {\scriptstyle F_2^{a,b}+1}\downarrow & & \downarrow{\scriptstyle F_2^{a,b+c}} \\ F(a+b) + Fc & \xrightarrow[F_2^{a+b,c}]{} & F(a+b+c) \end{array}$$

(and suitable $F_0\colon 0 \to F0$ is uniquely determined by F and F_2).

3. 2-cells $\lambda\colon F \Rightarrow G$ in *MON* are monoidal natural transformations, that is, natural transformations such that the following diagram commutes

$$\begin{array}{ccc} Fa + Fb & \xrightarrow{F_2^{a,b}} & F(a+b) \\ {\scriptstyle \lambda_a+\lambda_b}\downarrow & & \downarrow{\scriptstyle \lambda_{a+b}} \\ Ga + Gb & \xrightarrow[G_2^{a,b}]{} & G(a+b) \end{array}$$

Remark 6.2 1. The 2-category *Grpd*($\mathcal{C}$) embeds into the 2-category *MON*: internal functors $F\colon \mathbb{A} \to \mathbb{B}$ are precisely those monoidal functors for which all the $F_2^{a,b}$ are identites. Indeed, in this case the naturality of F_2 corresponds to the fact that $F_1\colon A_1 \to B_1$ is a group homomorphism, and the cocycle condition is verified because $e\colon B_0 \to B_1$ is a group homomorphism.

2. The embedding $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* is full and faithful on 2-cells. Indeed, if $F_2^{a,b} = id = G_2^{a,b}$, then the fact that λ is monoidal corresponds to the fact that $\lambda\colon A_0 \to B_1$ is a group homomorphism.

3. The embedding $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* preserves weak equivalences. In fact, the forgetful functor *Grp* $\to$ *Set* preserves and reflects finite limits and regular epimorphisms (this is because *Grp* is an algebraic category, see Ch. 3 in [6]), so that weak equivalences in *Grpd*($\mathcal{C}$) and in *MON* are 1-cells which are full, faithful and essentially surjective in the usual sense.

4. In *MON* weak equivalences coincide with equivalences. Indeed, if $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence, any quasi-inverse $G\colon \mathbb{B} \to \mathbb{A}$ can be equipped with a monoidal structure as follows: choose, for each $x \in B_0$, an arrow $\beta_x\colon F(Gx) \to x$ so to have a natural transformation $\beta\colon G \cdot F \Rightarrow Id$. Then define

$$G_2^{x,y}\colon Gx + Gy \to G(x+y)$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc} F(Gx+Gy) & \xrightarrow{F(G_2^{x,y})} & F(G(x+y)) \\ {\scriptstyle F_2^{Gx,Gy}}\uparrow & & \downarrow{\scriptstyle \beta_{x+y}} \\ F(Gx)+F(Gy) & \xrightarrow[\beta_x+\beta_y]{} & x+y \end{array}$$

It is straightforward to check naturality and cocycle condition for G_2 and that β is monoidal. Moreover, we get a monoidal natural transformation $\alpha\colon F \cdot G \Rightarrow Id$ via the equation $F(\alpha_a) = \beta_{Fa}$.

5. The above construction of G_2 makes clear that even if F is a weak equivalence in *Grpd*($\mathcal{C}$) in general G is in *MON* but not in *Grpd*($\mathcal{C}$).

Lemma 6.3 *The 2-category MON has bipullbacks. Moreover, given 1-cells* $F\colon \mathbb{A} \to \mathbb{B}$ *and* $G\colon \mathbb{C} \to \mathbb{B}$*, it is possible to choose a bipullback of* F *and* G

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in such a way that F' *and* G' *are internal functors in Grp.*

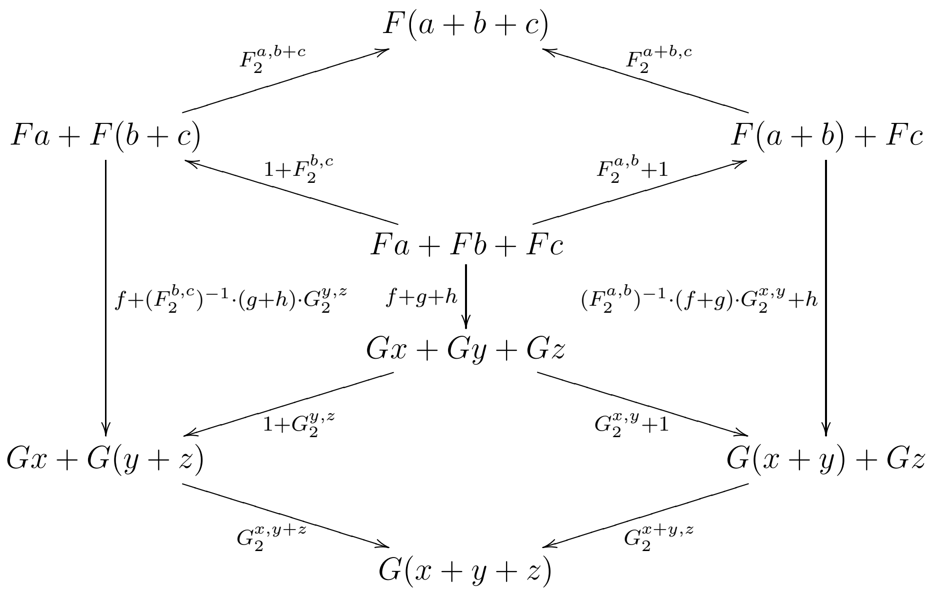
Proof. The construction of the pullback $\mathbb{P}$ is as in 3.3. The interesting point is that, even if F and G are monoidal (not necessarily internal) functors, $\mathbb{P}$ is an internal groupoid in *Grp* and not just a monoidal category. Indeed, if

$$(a, f\colon Fa \to Gx, x) \text{ and } (b, g\colon Fb \to Gy, y)$$

are objects in $\mathbb{P}$, their tensor product $(a, f\colon Fa \to Gx, x) + (b, g\colon Fb \to Gy, y)$ is given by

$$(a+b,\ F(a+b) \xrightarrow{(F_2^{a,b})^{-1}} Fa + Fb \xrightarrow{f+g} Gx + Gy \xrightarrow{G_2^{x,y}} G(x+y)\ , x+y)$$

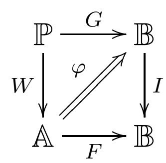
If $(c, h\colon Fc \to Gz, z)$ is a third object in $\mathbb{P}$, to check that the above tensor product is strictly associative easily reduces to the commutativity of the following diagram



that is, to the cocycle condition on F_2 and G_2.
The fact that F' and G' are internal functors is obvious. ∎

Proposition 6.4 *The embedding $\mathcal{F}\colon$ Grpd($\mathcal{C}$) $\to$ MON is the bicategory of fractions of Grpd($\mathcal{C}$) with respect to the class of weak equivalences.*

Proof. Let Σ be the class of weak equivalences in *Grpd*($\mathcal{C}$). From Proposition 4.5 we know that Σ has a right calculus of fractions. Moreover, by 6.2.3 and 6.2.4, $\mathcal{F}(W)$ is an equivalence for every $W \in \Sigma$. It remains to check conditions EF1–EF3 in Proposition 2.5: EF1 is obvious and EF2 is precisely 6.2.2. As far as EF3 is concerned, consider a 1-cell $F\colon \mathbb{A} \to \mathbb{B}$ in *MON* and perform the bipullback of F along the identity 1-cell I as in Lemma 6.3



so that both W and G are internal functors. Since equivalences are stable under bipullbacks, W is an equivalence in *MON* and therefore it is a weak equivalence in *Grpd*($\mathcal{C}$). Finally, $\varphi\colon \mathcal{F}(W)\cdot F \Rightarrow \mathcal{F}(G)$ is the 2-cell needed in EF3. Following Proposition 2.5, $\mathcal{F}\colon$ *Grpd*($\mathcal{C}$) $\to$ *MON* is the bicategory of fractions with respect to Σ. ■

Remark 6.5 Observe that we cannot expect to describe a class larger than the class of monoidal functors as fractions of internal functors with respect to weak equivalences. Indeed, the existence of a 2-cell $\mathcal{F}(W)\cdot F \Rightarrow \mathcal{F}(G)$ as in condition EF3 implies that F is monoidal.

7. Homomorphisms of strict Lie 2-algebras

In this section the base category $\mathcal{C}$ is the category *Lie* of Lie algebras over a fixed field K, which is a regular and protomodular category. The situation is completely analogous to the situation described in Section 6 for groups. The reason is that the forgetful functors *Lie* $\to$ *Vect* (where *Vect* is the category of vector spaces over K) and *Vect* $\to$ *Set* preserve and reflect finite limits and regular epimorphisms (because *Lie* and *Vect* are algebraic categories) and moreover in *Vect* the axiom of choice holds (because every vector space is free and therefore regular projective).

7.1 The aim of this section is to prove that the 2-category *LIE* described hereunder is the bicategory of fractions of *Grpd*($\mathcal{C}$) with respect to weak equivalences.

1. Objects of *LIE* are internal groupoids in *Lie*, also called strict Lie 2-algebras in [1].
2. 1-cells $F\colon \mathbb{A} \to \mathbb{B}$ in *LIE* are internal functors in *Vect* equipped with a family of arrows in $\mathbb{B}$

   $$F_2 = \{F_2^{a,b}\colon [Fa,Fb] \to F[a,b]\}_{a,b\in A_0}$$

   which is natural, bilinear, antisymmetric, and satisfies the following Jacobi condition

   $$\begin{array}{ccc} [Fa,[Fb,Fc]] & = & [[Fa,Fb],Fc]+[Fb,[Fa,Fc]] \\ \Big\downarrow{\scriptstyle [1,F_2^{b,c}]} & & \Big\downarrow{\scriptstyle [F_2^{a,b},1]+[1,F_2^{a,c}]} \\ [Fa,F[b,c]] & & [F[a,b],Fc]+[Fb,F[a,c]] \\ \Big\downarrow{\scriptstyle F_2^{a,[b,c]}} & & \Big\downarrow{\scriptstyle F_2^{[a,b],c}+F_2^{b,[a,c]}} \\ F[a,[b,c]] & = & F[[a,b],c]+F[b,[a,c]] \end{array}$$

   These 1-cells are simply called homomorphisms in [1], where in fact they are defined for more general semi-strict Lie 2-algebras.

3. 2-cells $\lambda \colon F \Rightarrow G$ in *LIE* are internal natural transformations in *Vect* such that the following diagram commutes

$$\begin{array}{ccc} [Fa, Fb] & \xrightarrow{F_2^{a,b}} & F[a,b] \\ {\scriptstyle [\lambda_a, \lambda_b]}\downarrow & & \downarrow{\scriptstyle \lambda_{[a,b]}} \\ [Ga, Gb] & \xrightarrow[G_2^{a,b}]{} & G[a,b] \end{array}$$

Remark 7.2 1. The 2-category *Grpd*($\mathcal{C}$) embeds into the 2-category *LIE*: internal functors $F \colon \mathbb{A} \to \mathbb{B}$ are precisely those homomorphisms for which all the $F_2^{a,b}$ are identites. The embedding $\mathcal{F} \colon$ *Grpd*($\mathcal{C}$) $\to$ *LIE* is full and faithful on 2-cells, and preserves weak equivalences.

2. In *LIE* weak equivalences coincide with equivalences. Indeed, let $F \colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in *LIE*. Then F is also a weak equivalence in the 2-category of internal groupoids and internal functors in *Vect*. Since in *Vect* the axiom of choice holds, F has a quasi-inverse $G \colon \mathbb{B} \to \mathbb{A}$ which is an internal functor in *Vect* (see Corollary 4.3). Now G can be equipped with a structure of homomorphism as follows: consider the internal (in *Vect*) natural transformation $\beta \colon G \cdot F \Rightarrow Id$ and define

$$G_2^{x,y} \colon [Gx, Gy] \to G[x,y]$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc} F[Gx, Gy] & \xrightarrow{F(G_2^{x,y})} & F(G[x,y]) \\ {\scriptstyle F_2^{Gx,Gy}}\uparrow & & \downarrow{\scriptstyle \beta_{[x,y]}} \\ [F(Gx), F(Gy)] & \xrightarrow[{[\beta_x, \beta_y]}]{} & [x,y] \end{array}$$

Lemma 7.3 *The 2-category LIE has bipullbacks. Moreover, given 1-cells* $F \colon \mathbb{A} \to \mathbb{B}$ *and* $G \colon \mathbb{C} \to \mathbb{B}$, *it is possible to choose a bipullback of* F *and* G

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ {\scriptstyle G'}\downarrow & \varphi \;\Nearrow & \downarrow{\scriptstyle G} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

in such a way that F' *and* G' *are internal functors in Lie.*

Proof. Once again the point is that, even if F and G are homomorphisms, the bipullback $\mathbb{P}$ constructed as in 3.3 is an internal groupoid in *Lie* and not just a semi-strict Lie 2-algebra. Indeed, the Lie operation in $\mathbb{P}$ is defined by

$$([a,b],\ F[a,b] \xrightarrow{(F_2^{a,b})^{-1}} [Fa, Fb] \xrightarrow{[f,g]} [Gx, Gy] \xrightarrow{G_2^{x,y}} G[x,y]\,, [x,y])$$

and the Jacobi identity is strictly verified thanks to the Jacobi condition on F_2 and G_2. ■

Proposition 7.4 *The embedding $\mathcal{F}\colon$ Grpd($\mathcal{C}$) $\rightarrow$ LIE is the bicategory of fractions of Grpd($\mathcal{C}$) with respect to the class of weak equivalences.*

Proof. The proof is analogous to that of Proposition 6.4 and we omit details. ■

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