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On the Notion of Bimodel for Functorial Semantics

FRANCIS BORCEUX* and ENRICO M. VITALE
Institut de Mathématique Pure et Appliquée, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

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Abstract. We discuss the notion of bimodel in order to obtain a classification of the equivalences between categories of models in the sense of functorial semantics.

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1. Introduction

There are various doctrines wherein theories are small categories and the models of a theory constitute a full subcategory of the category of functors from the theory to sets. We define the notion of bimodel at the nondoctrinaire level where those specified subcategories are arbitrary; as a first application we look for a classification of the equivalences between categories of models. We obtain the expected formulation under two distinct hypotheses on these subcategories of models: the case where the subcategory is reflective (the importance of which was pointed out by Pultr in [14]) is treated in Section 5 and the second case, where the subcategory contains all representable functors and has certain colimits, is treated in Section 6. In Section 7 we give relevant examples which in particular illustrate that neither of these two cases includes the other. In the last section we study in detail the case of algebraic theories.

2. Some Classical Facts

In order to support intuition in the following sections, we recall the classic Morita theorem: let A and B be two rings; the categories of left-modules A-mod and B-mod are equivalent if and only if there exist two bimodules $M \in B$-mod-A and $N \in A$-mod-B and two isomorphisms of bimodules $M \otimes_A N \simeq B$ and $N \otimes_B M \simeq A$.

The proof of this theorem (cf. [2]) is based on the fact that A is a regular generator for A-mod and the functor $M \otimes_A - : A\text{-mod} \to B\text{-mod}$ preserves colimits.

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A way to present the categories of modules is to observe that A-mod is equivalent to the category of functors from $\mathbb{A}^{op}$ to $\mathcal{SET}$ preserving finite-products, where $\mathbb{A}$ is the full subcategory of finitely generated free A-modules; from this point of view, a bimodule $M \in B$-mod-A is exactly a functor $\mathbb{A}^{op} \xrightarrow{\tilde{M}} B$-mod preserving finite-coproducts and the functor $M \otimes_A -$ is the Kan-extension of $\tilde{M}$ along the contravariant Yoneda-embedding $\mathbb{A}^{op} \xrightarrow{Y} A$-mod.

3. Notations and Preliminary Facts

In the following $\mathbb{T}, \mathbb{S}, \ldots$ are always small categories; $\mathbb{T}^{op} \xrightarrow{Y_{\mathbb{T}}} \hat{\mathbb{T}}$ is the contravariant Yoneda embedding of $\mathbb{T}$ in the category $\hat{\mathbb{T}}$ of all covariant functors from $\mathbb{T}$ to $\mathcal{SET}$ and natural transformations; $\vec{\mathbb{T}} \overset{i_{\mathbb{T}}}{\rightarrowtail} \hat{\mathbb{T}}$ is the inclusion of a full subcategory $\vec{\mathbb{T}}$ in $\hat{\mathbb{T}}$.

The definitions and basic properties of Kan extensions and dense functors can be found in section 10 of [12] and in section 17 of [15]; nevertheless we recall that:

3.1. If $\mathcal{D}$ is a cocomplete category, for each pair of functors $\mathbb{T} \xrightarrow[\mathcal{C}]{G}$ and $\mathbb{T} \xrightarrow[\mathcal{D}]{F}$, the Kan extension of F along G exists and can be computed pointwise.

3.2. $Y_{\mathbb{T}}$ is a dense functor.

3.3. Consider two functors $\mathcal{E} \xrightarrow[\mathcal{C}]{D} \xrightarrow[\mathcal{D}]{F}$ such that D is dense and F has a full faithful right adjoint; then $D \cdot F$ is dense.

3.4. Consider three functors $\mathbb{T} \xrightarrow[\mathcal{C}]{D} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ such that D is dense, F and G preserve colimits and $\mathcal{D}$ is cocomplete; if $D \cdot F \simeq D \cdot G$, then $F \simeq G$.

3.5. Consider a functor $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$; the Kan extension $\hat{\varphi}$ of φ along $Y_{\mathbb{T}}$ has as a right adjoint the functor $\mathrm{Hom}(\varphi, -)$ defined, $\forall\, K \in \hat{\mathbb{S}}$ and $\forall\, T \in \mathbb{T}$, by $\mathrm{Hom}(\varphi, K)(T) = \mathrm{Nat}(\varphi(T), K)$.

$$
\begin{tikzcd}
\mathbb{T}^{op} \arrow[r, "Y_{\mathbb{T}}"] \arrow[dr, "\varphi"'] & \hat{\mathbb{T}} \arrow[d, shift right, "\hat{\varphi}"'] \\
& \hat{\mathbb{S}} \arrow[u, shift right, "\mathrm{Hom}(\varphi, -)"']
\end{tikzcd}
$$

For some basic facts on the Hom-Tensor calculus, see also [7].

4. Bimodels

DEFINITION 4.1. Consider a functor $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \mathbb{S}$ and the composition $\varphi = \tilde{\varphi} \cdot i_{\mathbb{S}}$; we say that $\tilde{\varphi}$ is a $\mathbb{S} - \mathbb{T}$-bimodel (or, in short, a bimodel) if the functor $\mathrm{Hom}(\varphi, -)$ defined in 3.5 can be restricted to the categories of models

$$\begin{tikzcd}
\hat{\mathbb{T}} & \vec{\mathbb{T}} \arrow[l, tail, "i_{\mathbb{T}}"'] \\
\hat{\mathbb{S}} \arrow[u, "\mathrm{Hom}(\varphi,-)"] & \vec{\mathbb{S}} \arrow[l, tail, "i_{\mathbb{S}}"] \arrow[u]
\end{tikzcd}$$

We write the restriction $\vec{\mathbb{S}} \to \vec{\mathbb{T}}$ as $\vec{\mathrm{Hom}}(\varphi, -)$.

5. The Reflective Case

Now let us suppose that $\vec{\mathbb{T}}$ is a reflective subcategory of $\hat{\mathbb{T}}$, i.e. that the inclusion $\vec{\mathbb{T}} \overset{i_{\mathbb{T}}}{\rightarrowtail} \hat{\mathbb{T}}$ has a left adjoint $\hat{\mathbb{T}} \xrightarrow{r_{\mathbb{T}}} \vec{\mathbb{T}}$. The key property of a bimodel is the following

PROPOSITION 5.1. *Let* $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \mathbb{S}$ *be a bimodel and* $\vec{\varphi}$ *its Kan extension along* $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$; *then* $\vec{\varphi}$ *is the left adjoint of* $\vec{Hom}(\varphi, -)$

Proof. we need two lemmas for which we refer to the following diagram

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[r, "Y_{\mathbb{T}}"] \arrow[dr, "\varphi"'] & \hat{\mathbb{T}} \arrow[r, shift right, "r_{\mathbb{T}}"'] \arrow[d, shift right, "\hat{\varphi}"'] \arrow[dr, "\underline{\varphi}"] & \vec{\mathbb{T}} \arrow[l, shift right, "i_{\mathbb{T}}"'] \arrow[d, shift right, "\vec{\varphi}"'] \\
& \hat{\mathbb{S}} \arrow[u, shift right, "\mathrm{Hom}(\varphi,-)"'] \arrow[r, shift right, "r_{\mathbb{S}}"'] & \vec{\mathbb{S}} \arrow[l, shift right, "i_{\mathbb{S}}"'] \arrow[u, shift right, "\vec{\mathrm{Hom}}(\varphi,-)"']
\end{tikzcd}$$

LEMMA. *Let* $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$ *be a functor and* $\underline{\varphi}$ *the Kan extension of* $\varphi \cdot r_{\mathbb{S}}$ *along* $Y_{\mathbb{T}}$; $\hat{\varphi} \cdot r_{\mathbb{S}} = \underline{\varphi}$ *and then* $\underline{\varphi}$ *is the left adjoint of* $i_{\mathbb{S}} \cdot Hom(\varphi, -)$.

Proof. From the pointwise formula of the Kan extension and using the fact that $r_{\mathbb{S}}$ preserves colimits.

LEMMA. *Let* $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$ *be a functor and* $\vec{\varphi}$ *the Kan extension of* $\varphi \cdot r_{\mathbb{S}}$ *along* $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$; *then* $i_{\mathbb{T}} \cdot \underline{\varphi} = \vec{\varphi}$.

Proof. From the pointwise formula of Kan extension taking into account that $\forall\, F \in \vec{\mathbb{T}}$, the comma categories $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}/F$ and $Y_{\mathbb{T}}/i_{\mathbb{T}}(F)$ are isomorphic and such an isomorphism commutes with the forgetful functors towards $\mathbb{T}$ involved in that formula.

Proof of Proposition 5.1. Let $F \in \vec{\mathbb{T}}$ and $H \in \vec{\mathbb{S}}$; then

$$\begin{aligned}
\mathrm{Nat}(F, \vec{\mathrm{Hom}}(\varphi, H)) &= \mathrm{Nat}(i_{\mathbb{T}}(F), i_{\mathbb{T}}(\vec{\mathrm{Hom}}(\varphi, H))) && \vec{\mathbb{T}} \text{ is full in } \hat{\mathbb{T}}\\
&= \mathrm{Nat}(i_{\mathbb{T}}(F), \mathrm{Hom}(\varphi, i_{\mathbb{S}}(H))) && \tilde{\varphi} \text{ is a bimodel}\\
&= \mathrm{Nat}(\underline{\varphi}(i_{\mathbb{T}}(F)), H) && \text{first lemma}\\
&= \mathrm{Nat}(\vec{\varphi}(F), H) && \text{second lemma}
\end{aligned}$$
□

COROLLARY 5.2. *Let $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ be a bimodel and $\vec{\varphi}$ its Kan extension along $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$; then (1) $\vec{\varphi}$ preserves colimits; (2) the following diagram is commutative*

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[r, "Y_{\mathbb{T}}"] \arrow[rrd, "\tilde{\varphi}"'] & \hat{\mathbb{T}} \arrow[r, "r_{\mathbb{T}}"] & \vec{\mathbb{T}} \arrow[d, "\vec{\varphi}"] \\
& & \vec{\mathbb{S}}
\end{tikzcd}$$

and (3) $\vec{\varphi}$ is the only functor (up to isomorphisms) satisfying the two conditions above.

Proof. (1) holds because $\vec{\varphi}$ is a left adjoint.

(2) $\vec{\mathrm{Hom}}(\varphi, -) \cdot i_{\mathbb{T}} = i_{\mathbb{S}} \cdot \mathrm{Hom}(\varphi, -)$ because $\tilde{\varphi}$ is a bimodel; $r_{\mathbb{T}} \cdot \vec{\varphi} = \hat{\varphi} \cdot r_{\mathbb{S}}$ and then $Y_{\mathbb{T}} \cdot r_{\mathbb{T}} \cdot \vec{\varphi} = Y_{\mathbb{T}} \cdot \hat{\varphi} \cdot r_{\mathbb{S}} = \varphi \cdot r_{\mathbb{S}} = \tilde{\varphi} \cdot i_{\mathbb{S}} \cdot r_{\mathbb{S}} = \tilde{\varphi}$.

(3) $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$ is dense (cf. 3.2 and 3.3) and then we apply 3.4. □

In order to define a composition of bimodels, let us look at the stability of this notion.

PROPOSITION 5.3. *$Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$ is a bimodel and its Kan extension along itself is the identity functor.*

Proof. Let $\tilde{\varphi} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$, then $Y_{\mathbb{T}} \cdot \hat{\varphi} \cdot r_{\mathbb{T}} = \varphi \cdot r_{\mathbb{T}} = \tilde{\varphi} \cdot i_{\mathbb{T}} \cdot r_{\mathbb{T}} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}} \cdot i_{\mathbb{T}} \cdot r_{\mathbb{T}} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$; now applying 3.4 to the dense functor $Y_{\mathbb{T}}$, we have $\hat{\varphi} \cdot r_{\mathbb{T}} = r_{\mathbb{T}}$ and considering the right adjoint we have the required restriction. □

PROPOSITION 5.4. *Let $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ be a bimodel and $\vec{\mathbb{S}} \xrightarrow{\epsilon} \vec{\mathbb{R}}$ a functor with right adjoint $\bar{\epsilon}$; then $\tilde{\varphi} \cdot \epsilon$ is a bimodel.*

Proof. Let $\tilde{\psi} = \tilde{\varphi} \cdot \epsilon$; for the hypotheses on $\tilde{\varphi}$ and ϵ, we can verify that $\bar{\epsilon} \cdot \vec{\mathrm{Hom}}(\varphi, -)$ is the required restriction of $\mathrm{Hom}(\psi, -)$. □

For corollary 5.2 and the above stability properties, we can define an associative composition of bimodels.

DEFINITION 5.5. Let $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ and $\mathbb{S}^{op} \xrightarrow{\tilde{\psi}} \vec{\mathbb{R}}$ be two bimodels; their composition as bimodels is $\tilde{\varphi} \cdot \vec{\psi}$ (where $\vec{\psi}$ is the Kan extension of $\tilde{\psi}$ along $Y_{\mathbb{S}} \cdot r_{\mathbb{S}}$) and we write it as $\tilde{\varphi} * \tilde{\psi}$; we write again the bimodel $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$ as $1_{\mathbb{T}}$; it acts as neutral element with regard to such a composition.

THEOREM 5.6. *$\vec{\mathbb{T}}$ and $\vec{\mathbb{S}}$ are equivalent categories if and only if there exist two bimodels $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ and $\mathbb{S}^{op} \xrightarrow{\tilde{\psi}} \vec{\mathbb{T}}$ such that $\tilde{\varphi} * \tilde{\psi} = 1_{\mathbb{T}}$ and $\tilde{\psi} * \tilde{\varphi} = 1_{\mathbb{S}}$.*

Proof. It follows immediately from 5.2, 5.3 and 5.4. □

6. The "Representable" Case

We sketch now the same arguments as in the section above starting from a full subcategory $\vec{\mathbb{T}}$ of $\hat{\mathbb{T}}$ not necessarily reflective, but now we want that the Yoneda embedding factorizes in $\vec{\mathbb{T}}$

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[rr, "Y_{\mathbb{T}}"] \arrow[dr, "\tilde{Y}_{\mathbb{T}}"'] & & \hat{\mathbb{T}} \\
& \vec{\mathbb{T}} \arrow[ur, "i_{\mathbb{T}}"'] &
\end{tikzcd}$$

and that $\vec{\mathbb{T}}$ has enough colimits for the pointwise formula of all the Kan extensions involved. The definition of a bimodel $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ is again the same (it does not depend on the reflectivity) and we have the basic property

PROPOSITION 6.1. *Let $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ be a bimodel and $\vec{\varphi}$ its Kan extension along $\tilde{Y}_{\mathbb{T}}$; then $\vec{\varphi}$ is the left adjoint of $\overrightarrow{Hom}(\varphi, -)$.*

The proof of this proposition is based on two lemmas analogous to those used for 5.1, but now they have to be proved in a different way (with a straightforward verification of the natural isomorphism of the adjunction using the pointwise formulas for the first one; directly from the definition of Kan extension for the second one); for this property we can characterize the Kan extension of bimodels as in 5.2 and, with analogous stability conditions, we arrive at the

DEFINITION 6.2. Let $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ and $\mathbb{S}^{op} \xrightarrow{\tilde{\psi}} \vec{\mathbb{R}}$ be two bimodels; their composition $\tilde{\varphi} * \tilde{\psi}$ is $\tilde{\varphi} \cdot \vec{\psi}$ (where $\vec{\psi}$ is the Kan extension of $\tilde{\psi}$ along $\tilde{Y}_{\mathbb{S}}$) and the neutral element $1_{\mathbb{T}}$ is now $\tilde{Y}_{\mathbb{T}}$.

THEOREM 6.3. *$\vec{\mathbb{T}}$ and $\vec{\mathbb{S}}$ are equivalent categories if and only if there exist two bimodels $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ and $\mathbb{S}^{op} \xrightarrow{\tilde{\psi}} \vec{\mathbb{T}}$ such that $\tilde{\varphi} * \tilde{\psi} = 1_{\mathbb{T}}$ and $\tilde{\psi} * \tilde{\varphi} = 1_{\mathbb{S}}$.*

7. Examples

I – Let $\mathcal{P}(\mathbb{T})$ be a set of cones in $\mathbb{T}$ on functors $\mathcal{X} \xrightarrow[\mathbb{T}]{L}$ defined on small categories $\mathcal{X}$; the models are the functors carrying the cones of $\mathcal{P}(\mathbb{T})$ into limits in $\mathcal{SET}$; this is an example of the reflective case (cf.[1]) but not, in general, of the representable case.

PROPOSITION. *A functor $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ is a bimodel if and only if it carries the (co) cones of $\mathcal{P}(\mathbb{T})$ into colimits in $\vec{\mathbb{S}}$.*

Proof. $(\Leftarrow)$: Let $(l \xrightarrow{p_x} Lx)$ be a cone on L in $\mathcal{P}(\mathbb{T})$ and let $H \in \vec{\mathbb{S}}$: $\mathrm{Hom}(\varphi, i_{\mathbb{S}}(H))(l \xrightarrow{p_x} Lx) = \mathrm{Nat}(\varphi(l \xrightarrow{p_x} Lx), i_{\mathbb{S}}(H)) = \mathrm{Nat}(\tilde{\varphi}(l \xrightarrow{p_x} Lx), H) = \mathrm{Nat}(\mathrm{colim}(L \cdot \tilde{\varphi}), H) = \lim(L \cdot \tilde{\varphi} \cdot \mathrm{Nat}(-, H))$ and so $\mathrm{Hom}(\varphi, i_{\mathbb{S}}(H)) \in \vec{\mathbb{T}}$.

$(\Rightarrow)$: $\tilde{\varphi}$ bimodel means that $\forall\, H \in \vec{\mathbb{S}}$, $\mathrm{Nat}(\tilde{\varphi}-, H) \in \vec{\mathbb{T}}$; so, with the previous notations, $\mathrm{Nat}(\tilde{\varphi}(l \xrightarrow{p_x} Lx), H) = \lim(L \cdot \mathrm{Nat}(\tilde{\varphi}-, H)) = \lim(L \cdot \tilde{\varphi} \cdot \mathrm{Nat}(-, H)) = \mathrm{Nat}(\mathrm{colim}(L \cdot \tilde{\varphi}), H)$ and so $\tilde{\varphi}(l \xrightarrow{p_x} Lx) = \mathrm{colim}(L \cdot \tilde{\varphi})$. □

COROLLARY. *Let $t : \vec{\mathbb{T}} \to \vec{\mathbb{S}}$ be a functor and let us write $\tilde{\varphi} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}} \cdot t$; then*

1. *if t preserves the colimits of $r_{\mathbb{T}}(Y_{\mathbb{T}}(\mathcal{P}(\mathbb{T})))$, then $\tilde{\varphi}$ is a bimodel;*
2. *if t preserves colimits, then $t \simeq \vec{\varphi}$ and then it has a right adjoint.*

II – Like some particular cases of example I, we can write

a – $\mathcal{P}(\mathbb{T})$ is a set of limits in $\mathbb{T}$ (cf. [8]);

b – $\mathbb{T}$ has all finite limits and $\mathcal{P}(\mathbb{T})$ is the set of all such limits;

c – $\mathbb{T}$is an algebraic theory, i.e. a category with an enumerable set of objects $\langle T^0, T^1, T^2, \ldots\rangle$ such that $\forall\, n \in \mathbb{N}$, $T^n = (T^1)^n$; the models are functors preserving finite-products; a bimodel is exactly a functor $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$ preserving finite coproducts (cf. [9]);

d – $\mathcal{P}(\mathbb{T})$ is empty; then the category of models is $\hat{\mathbb{T}}$ and a bimodel is exactly a distributor, i.e. any functor $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$. In this case Theorem 5.6 is the translation of Cauchy-equivalences $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$ in terms of distributors. (for an explanation of the term "Cauchy-equivalence", see [10]) We observe here that also the translation of Cauchy-equivalences $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$ via equivalences $\mathcal{R}(\mathbb{T}) \simeq \mathcal{R}(\mathbb{S})$ (where $\mathcal{R}(\mathbb{T})$ is the full subcategory of $\hat{\mathbb{T}}$ of all retracts of representable functors, cf. [6]) can be deduced from this theorem: in fact, in the non-restrictive hypothesis that the equivalence $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$ is an adjoint equivalence, starting from $\varphi * \psi = 1_{\mathbb{T}}$, i.e. from an isomorphism $\varphi \cdot \hat{\psi} \simeq Y_{\mathbb{T}}$, and evaluating on an object $T \in \mathbb{T}$, we have an isomorphism $\hat{\psi}(\varphi(T)) \xrightarrow{\cong} Y_{\mathbb{T}}(T)$; now, taking into account that the colimits involved in the Kan extension $\hat{\psi}$ are in $\hat{\mathbb{T}}$ and therefore they are pointwise colimits and also considering that we have an isomorphism $\hat{\psi} \simeq \mathrm{Hom}(\varphi, -)$, evaluating again on the same $T \in \mathbb{T}$, we obtain an isomorphism from a quotient of

$$\coprod_{S \in \mathbb{S}} [\mathrm{Nat}(\varphi(T), Y_{\mathbb{S}}(S)) \times \mathrm{Nat}(Y_{\mathbb{S}}(S), \varphi(T))]$$

(i.e. the explicit formula for the composition of distributors, cf. [3]) to $\mathrm{Nat}(\varphi(T),\ \varphi(T))$; it is possible to show that this isomorphism is, on the equivalence classes, exactly the composition; so, considering the counter-image of the identity of $\varphi(T)$, we have that $\varphi(T) \in \mathcal{R}(\mathbb{S})$.

Analogously, starting from $\psi * \varphi = 1_{\mathbb{S}}$, we obtein that, for every $S \in \mathbb{S}$, there exists $T \in \mathbb{T}$ such that $Y_{\mathbb{S}}(S)$ is a retract of $\varphi(T)$. Therefore, we have theorem 3.6 of [6], i.e. Cauchy-equivalent small categories have equivalent idempotent completions.

III – Another example of the reflective case is given (up to replacing the contravariant Yoneda embedding with the covariant one) by the Grothendieck topoi, which are exactly localizations of presheaf categories (cf. [1]).

IV – To give an example of the representable case but not, in general, of the reflective case, let $\vec{\mathbb{T}}$ be the category of flat functors from $\mathbb{T}$ to $\mathcal{SET}$ (cf. [13]). This is a very particular case; in fact, for every functor $\mathbb{T}^{op} \xrightarrow{\tilde{\varphi}} \vec{\mathbb{S}}$, the Kan extension $\vec{\varphi}$ of $\tilde{\varphi}$ along $\tilde{Y}_{\mathbb{T}}$ is the restriction of the Kan extension $\hat{\varphi}$ of $\varphi = \tilde{\varphi} \cdot i_{\mathbb{S}}$ along $Y_{\mathbb{T}} = \tilde{Y}_{\mathbb{T}} \cdot i_{\mathbb{T}}$

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[r, "\tilde{Y}_{\mathbb{T}}"] \arrow[dr, "\tilde{\varphi}"'] & \vec{\mathbb{T}} \arrow[r, tail, "i_{\mathbb{T}}"] \arrow[d, "\vec{\varphi}"] & \hat{\mathbb{T}} \arrow[d, "\hat{\varphi}"] \\
 & \vec{\mathbb{S}} \arrow[r, tail, "i_{\mathbb{S}}"'] & \hat{\mathbb{S}}
\end{tikzcd}$$

This holds because the colimits involved in the pointwise formula of the Kan extensions are, in this case, filtered colimits and $\vec{\mathbb{S}}$ is closed in $\hat{\mathbb{S}}$ with respect to such colimits.

This commutativity has two immediate consequences:

1. each Morita-equivalence $\vec{\mathbb{T}} \simeq \vec{\mathbb{S}}$ can be extended to a Cauchy-equivalence $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$;

2. if a Cauchy-equivalence is induced by two distributors factorizing into the categories of models, then it can be restricted to a Morita-equivalence (and the factorizations of the distributors are automatically two bimodels).

Of course, example II-b can be obtained also as a particular case of this last example.

8. A Comparison between Cauchy-Equivalences and Morita-Equivalences

In example IV we have compared, in a very particular case, Cauchy-equivalences and Morita-equivalences; now let us try to compare them in the reflective case defined in Section 5.

DEFINITION 8.1. A good distributor is a functor $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$ such that $\varphi \cdot r_{\mathbb{S}}$ is a bimodel.

LEMMA 8.2. *If $\mathbb{T}^{op} \xrightarrow{\varphi} \hat{\mathbb{S}}$ is a good distributor, then the following diagram of Kan extensions is commutative in every part*

$$\begin{tikzcd} \mathbb{T}^{op} \arrow[r, "Y_{\mathbb{T}}"] \arrow[dr, "\varphi"'] & \hat{\mathbb{T}} \arrow[r, "r_{\mathbb{T}}"] \arrow[d, "\hat{\varphi}"] & \vec{\mathbb{T}} \arrow[d, "\vec{\varphi}"] \\ & \hat{\mathbb{S}} \arrow[r, "r_{\mathbb{S}}"'] & \vec{\mathbb{S}} \end{tikzcd}$$

Proof. For corollary 5.2 and for 3.4 applied to the dense functor $Y_{\mathbb{T}}$. □

PROPOSITION 8.3. *The following conditions are equivalent*

(1) a Cauchy-equivalence $F : \hat{\mathbb{T}} \xrightarrow{\cong} \hat{\mathbb{S}}$ can be restricted to a Morita-equivalence $F_| : \vec{\mathbb{T}} \xrightarrow{\cong} \vec{\mathbb{S}}$;

(2) $Y_{\mathbb{T}} \cdot F$ and $Y_{\mathbb{S}} \cdot F^{-1}$ are good distributors.

Proof. (1) ⇒ (2): we have that $i_{\mathbb{S}} \cdot F^{-1} = F_|^{-1} \cdot i_{\mathbb{T}}$; then, considering the left adjoints, $F \cdot r_{\mathbb{S}} = r_{\mathbb{T}} \cdot F_|$ and so $Y_{\mathbb{T}} \cdot F \cdot r_{\mathbb{S}} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}} \cdot F_|$ is a bimodel for 5.3 and 5.4.

(2) ⇒ (1): $\varphi = Y_{\mathbb{T}} \cdot F$ and $\psi = Y_{\mathbb{S}} \cdot F^{-1}$ are good distributors; then, applying Lemma 8.2,

$$Y_{\mathbb{T}} \cdot r_{\mathbb{T}} \cdot \vec{\varphi} \cdot \vec{\psi} = Y_{\mathbb{T}} \cdot \hat{\varphi} \cdot \hat{\psi} \cdot r_{\mathbb{T}} = Y_{\mathbb{T}} \cdot F \cdot F^{-1} \cdot r_{\mathbb{T}} = Y_{\mathbb{T}} \cdot r_{\mathbb{T}},$$

but $\vec{\varphi}$ preserves colimits (as Kan extension of the bimodel $Y_{\mathbb{T}} \cdot F \cdot r_{\mathbb{S}}$ along $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$) and the same holds for $\vec{\psi}$. Moreover, $Y_{\mathbb{T}} \cdot r_{\mathbb{T}}$ is a dense functor and so $\vec{\varphi} \cdot \vec{\psi} \simeq \mathrm{id}_{\vec{\mathbb{T}}}$

and in the same way $\vec{\psi} \cdot \vec{\varphi} \simeq \mathrm{id}_{\vec{\mathbb{S}}}$; so $\vec{\mathbb{T}} \xrightarrow{\vec{\varphi}} \vec{\mathbb{S}} \xrightarrow{\vec{\psi}} \vec{\mathbb{T}}$ is an equivalence and it is the restriction of $F : \hat{\mathbb{T}} \xrightarrow{\cong} \hat{\mathbb{S}}$ (because $r_{\mathbb{S}} \cdot \vec{\psi} = F^{-1} \cdot r_{\mathbb{T}}$ and then $\vec{\varphi} \cdot i_{\mathbb{S}} = i_{\mathbb{T}} \cdot F$). □

Taking into account that $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$ if and only if $\mathcal{R}(\mathbb{T}) \simeq \mathcal{R}(\mathbb{S})$, condition (2) of 8.3 can be written as

(2′) $\mathbb{T}^{op} \underset{\mathcal{R}}{\xrightarrow{Y_{\mathbb{T}}}} (\mathbb{T}) \underset{\mathcal{R}}{\overset{f}{\rightarrowtail}} (\mathbb{S})\hat{\mathbb{S}}$ and $\mathbb{S}^{op} \underset{\mathcal{R}}{\xrightarrow{Y_{\mathbb{S}}}} (\mathbb{S}) \underset{\mathcal{R}}{\overset{f^{-1}}{\rightarrowtail}} (\mathbb{T})\hat{\mathbb{T}}$ are good distributors (where f is the restriction of F).

Moreover, if the representable functors are models, then condition 2′) can be written as

(2″) $\mathbb{T}^{op} \underset{\mathcal{R}}{\xrightarrow{\tilde{Y}_{\mathbb{T}}}} (\mathbb{T}) \underset{\mathcal{R}}{\overset{f}{\rightarrowtail}} (\mathbb{S})\vec{\mathbb{S}}$ and $\mathbb{S}^{op} \underset{\mathcal{R}}{\xrightarrow{\tilde{Y}_{\mathbb{S}}}} (\mathbb{S}) \underset{\mathcal{R}}{\overset{f^{-1}}{\rightarrowtail}} (\mathbb{T})\vec{\mathbb{T}}$ are bimodels.

The different formulations of Proposition 8.3 can be used to study sufficient conditions to obtain examples of Morita-equivalences in the examples of Section 7; in particular we have

PROPOSITION 8.4. *Let $\mathbb{T}$ and $\mathbb{S}$ be two algebraic theories; if they are Cauchy-equivalent, i.e. $\hat{\mathbb{T}} \simeq \hat{\mathbb{S}}$, then they are also Morita-equivalent, i.e. $\vec{\mathbb{T}} \simeq \vec{\mathbb{S}}$.*

Proof. As in this case a bimodel is a functor preserving finite coproducts, then, for condition (2″) of 8.3, it suffices to notice that $\mathcal{R}(\mathbb{T})$ is closed in $\vec{\mathbb{T}}$ with respect to finite coproducts. □

This last proposition will be inverted in the following section, where we examine more carefully the case of algebraic theories.

9. Faithfully Projective Models of Algebraic Theories

In this section we show that all the equivalences between algebraic categories can be built up through the notion of faithfully projective model. In the following $\mathbb{T}$ is always an algebraic theory and $Y : \mathbb{T}^{op} \to \vec{\mathbb{T}}$ is the corresponding Yoneda embedding in the category of models. Let us recall two well-known properties (cf. vol.II, chapter 3 of [4]).

DEFINITION 9.1. A model $M \in \vec{\mathbb{T}}$ is regular projective if, for each regular epimorphism $M_1 \xrightarrow{p} M_2$ and for each morphism $M \xrightarrow{f} M_2$, there exists a morphism $M \xrightarrow{p'} M_1$ such that $p' \cdot p = f$.

PROPOSITION 9.2. *For each set X, the free $\mathbb{T}$-model $L(X)$ is regular projective.*

DEFINITION 9.3. A model $P \in \vec{\mathbb{T}}$ is finitely presentable if it can be obtained via a coequalizer diagram

$$Y(T^m) \rightrightarrows Y(T^n) \to P.$$

PROPOSITION 9.4. *A model $P \in \vec{\mathbb{T}}$ is finitely presentable if and only if the representable functor $Nat(P,-) : \vec{\mathbb{T}} \to \mathcal{SET}$ preserves filtered colimits.*

DEFINITION 9.5.a. A $\mathbb{T}$-model P is finitely generated and projective if it is a retract of a representable functor, i.e. if $P \underset{a}{\overset{b}{\rightleftarrows}} Y(T^n)$ with $a \cdot b = \mathrm{id}_P$.

DEFINITION 9.5.b. A $\mathbb{T}$-model P is a generator if $Y(T^1)$ is a retract of a finite sum mP of P, i.e. $Y(T^1) \underset{c}{\overset{d}{\rightleftarrows}} mP$ with $c \cdot d = \mathrm{id}_{Y(T^1)}$.

DEFINITION 9.5.c. A $\mathbb{T}$-model P is faithfully projective if it is at the same time a generator and a finitely generated and projective model.

We write $\mathcal{R}(\mathbb{T})$, $\mathcal{G}(\mathbb{T})$ and $\mathcal{F}(\mathbb{T}) = \mathcal{R}(\mathbb{T}) \cap \mathcal{G}(\mathbb{T})$ for the full subcategories of $\vec{\mathbb{T}}$ respectively defined in 9.5.a, 9.5.b and 9.5.c; we use these names to emphasize that, if $\vec{\mathbb{T}}$ is a category of modules on a ring, we have the usual notions for modules (cf.[2]). We write $\mathcal{F}_*(\mathbb{T})$ for the full subcategory of $\vec{\mathbb{T}}$ whose objects are the faithfully projective models and the initial object $Y(T^0)$.

PROPOSITION 9.6. *A $\mathbb{T}$-model $P \in \mathcal{R}(\mathbb{T})$ if and only if P is finitely presentable and, moreover, if P is regular projective.*

Proof. Because the free $\mathbb{T}$-model $L(n)$ is the representable functor $Y(T^n)$. □

PROPOSITION 9.7. *A $\mathbb{T}$-model $P \in \mathcal{G}(\mathbb{T})$ if and only if, for each $M \in \vec{\mathbb{T}}$, there exists a filtered diagram $\mathcal{D}_M$ in $\vec{\mathbb{T}}$ whose objects are finite sums of P and there exists a regular epimorphism $colim_{\mathcal{D}_M}(mP) \xrightarrow{q_M} M$ from the colimit of $\mathcal{D}_M$ to M.*

Proof. Let $U : \vec{\mathbb{T}} \to \mathcal{SET}$ be the usual forgetful functor and $L : \mathcal{SET} \to \vec{\mathbb{T}}$ its left adjoint; the counit $L(U(M)) \xrightarrow{\epsilon_M} M$ of this adjunction is a regular epimorphism (cf. [4]). As $U(M) = \mathrm{colim}_n(n)$ in $\mathcal{SET}$, where $n \subseteq U(M)$ is finite, and such a colimit is filtered, we have $\mathrm{colim}_n(Y(T^n)) \xrightarrow{\epsilon_M} M$. Now, if $P \in \mathcal{G}(\mathbb{T})$, then $Y(T^1)$ is a retract of mP and then $\mathrm{colim}_n(Y(T^n))$ is a retract of $\mathrm{colim}_n(n \cdot mP)$, so we have a regular epimorphism $\mathrm{colim}_n(nmP) \xrightarrow[n]{q} (Y(T^n))$ and, composing with ϵ_M, we have the first implication. Conversely, we can consider the regular epimorphism

$$\mathrm{colim}_{\mathcal{D}_{Y(T^1)}(mP)} \xrightarrow{q_{Y(T^1)}} Y(T^1),$$

but $Y(T^1)$ is regular projective and so we can invert such morphism

$$Y(T^1) \xrightarrow[\mathcal{D}_{Y(T^1)}]{s} (mP) \quad s \cdot q_{Y(T^1)} = \mathrm{id}_{Y(T^1)}.$$

As $Y(T^1)$ is finitely presentable, the functor Nat$(Y(T^1), -) : \vec{\mathbb{T}} \to \mathcal{SET}$ preserves filtered colimits and so s can be factorized in a canonical inclusion

$$\begin{tikzcd}
Y(T^1) \arrow[rr, "s"] \arrow[dr, "s'"'] & & \mathrm{colim}_{\mathcal{D}_{Y(T^1)}(mP)} \\
& mP \arrow[ur, "j"'] &
\end{tikzcd}$$

and we have the retraction $Y(T^1) \underset{s'}{\overset{j\cdot q_{Y(T^1)}}{\leftrightarrows}} mP$; in fact, $s' \cdot j \cdot q_{Y(T^1)} = s \cdot q_{Y(T^1)} = \mathrm{id}_{Y(T^1)}$. □

THEOREM 9.8. *Let* $\mathbb{T}$ *and* $\mathbb{S}$ *be two algebraic theories;* $\vec{\mathbb{T}}$ *and* $\vec{\mathbb{S}}$ *are equivalent categories if and only if* $\mathcal{F}_*(\mathbb{T})$ *and* $\mathcal{F}_*(\mathbb{S})$ *are equivalent.*

Proof. the translation for the notion of faithfully projective given in 9.6 and 9.7 is stable under the equivalence $\vec{\mathbb{T}} \xrightarrow{\cong} \vec{\mathbb{S}}$ (it is evident for the notion of regular projective and for the notion introduced in 9.7; it depends on 9.4 for the notion of finitely presentable), so such an equivalence can be restricted to an equivalence $\mathcal{F}(\mathbb{T}) \simeq \mathcal{F}(\mathbb{S})$ and also to an equivalence $\mathcal{F}_*(\mathbb{T}) \simeq \mathcal{F}_*(\mathbb{S})$; conversely, if $\mathcal{F}_*(\mathbb{T}) \xrightarrow[*]{\cong_f} (\mathbb{S})$ is an equivalence, then $\mathbb{T}^{op} \xrightarrow[*]{Y_{\mathbb{T}}} (\mathbb{T}) \overset{f}{\underset{*}{\rightarrowtail}} (\mathbb{S})\vec{\mathbb{S}}$ and $\mathbb{S}^{op} \xrightarrow[*]{Y_{\mathbb{S}}} (\mathbb{S}) \overset{[}{\rightarrowtail} heads = vee]f^{-1}\mathcal{F}_*(\mathbb{T})\vec{\mathbb{T}}$ are bimodels (respectively because $\mathcal{F}^*(\mathbb{S})$ and $\mathcal{F}^*(\mathbb{T})$ are closed in $\vec{\mathbb{S}}$ and in $\vec{\mathbb{T}}$ under finite coproducts) and give rise to the desired equivalence $\vec{\mathbb{T}} \xrightarrow{\cong} \vec{\mathbb{S}}$. □

REMARK. Considering separately Proposition 9.6, we have that, if $\vec{\mathbb{T}} \simeq \vec{\mathbb{S}}$, then $\mathcal{R}(\mathbb{T}) \simeq \mathcal{R}(\mathbb{S})$. Via the example II.d, this gives us the converse of Proposition 8.4, so that two algebraic theories are Morita-equivalent if and only if they are Cauchy-equivalent. This result is obtained also in [5], where, starting from the notion of algebraic theory introduced in [11], a syntactical characterization of algebraic theories with equivalent categories of models is given.

We want to discuss now two consequences of the previous theorem.

DEFINITION 9.9. A bimodel $\mathbb{T}^{op} \xrightarrow{\varphi} \vec{\mathbb{S}}$ is faithfully projective if its restriction to $\mathbb{T} - T^0$ factorizes through the inclusion $\mathcal{F}(\mathbb{S}) \overset{i}{\rightarrowtail} \vec{\mathbb{S}}$ or, equivalently, if $\varphi(T^1) \in \mathcal{F}(\mathbb{S})$.

PROPOSITION 9.10. *If* $\mathbb{T}^{op} \xrightarrow{\varphi} \vec{\mathbb{S}}$ *and* $\mathbb{S}^{op} \xrightarrow{\psi} \vec{\mathbb{T}}$ *are two bimodels inducing* *an equivalence $\vec{\mathbb{T}} \simeq \vec{\mathbb{S}}$, then (1) φ and ψ are faithfully projective and (2) $\mathbb{T}^{op}$ is equivalent to the full subcategory of $\vec{\mathbb{S}}$ spanned by $\varphi(T^1)$ via finite coproducts, and the same holds for $\mathbb{S}^{op}$.*

Proof. (1) because the equivalence $\vec{\mathbb{T}} \simeq \vec{\mathbb{S}}$ can be restricted to $\mathcal{F}(\mathbb{T}) \simeq \mathcal{F}(\mathbb{S})$
(2) because $\varphi = Y \cdot \vec{\varphi}$ is a full and faithful functor. □

PROPOSITION 9.11. *Let $\mathbb{T}$ be an algebraic theory and $P \in \vec{\mathbb{T}}$; we can consider a new algebraic theory $\mathbb{T}_P$ defined as the dual of the full subcategory of $\vec{\mathbb{T}}$ spanned by P via finite coproducts and we can consider the bimodel*

$$\mathbb{T}_P^{op} \overset{i}{\rightarrowtail} \vec{\mathbb{T}} \quad \textit{(full inclusion);}$$

such a bimodel induces an equivalence $\vec{\mathbb{T}}_P \simeq \vec{\mathbb{T}}$ if and only if $P \in \mathcal{F}(\mathbb{T})$.

Proof. if i induces an equivalence, then it is a faithfully projective bimodel and then $i(P) \in \mathcal{F}(\mathbb{T})$. Conversely, if $P \in \mathcal{F}(\mathbb{T})$, then $P \in \mathcal{G}(\mathbb{T})$, i.e. $Y(T^1) \underset{c}{\overset{d}{\rightleftarrows}} mP$ with $c \cdot d = \mathrm{id}_{Y(T^1)}$, and $Y(T^1) \xrightarrow{c} mP \underset{\mathrm{id}_{mP}}{\overset{d \cdot c}{\rightrightarrows}} mP$ is an absolute equalizer. So to each functor $\mathbb{T}_P \xrightarrow[\mathcal{SET}]{H}$ we can associate a functor $\mathbb{T} \xrightarrow[\mathcal{SET}]{\overline{H}}$ and the same holds for natural transformations. Moreover, if H preserves finite products, so does $\overline{H}$ for the interchange property of limits. On the other hand, $P \in \mathcal{R}(\mathbb{T})$, i.e. $P \underset{ca}{\overset{b}{\rightleftarrows}} Y(T^n)$ with $a \cdot b = \mathrm{id}_P$, then $P \xrightarrow{a} Y(T^n) \underset{\mathrm{id}_{Y(T^n)}}{\overset{b \cdot a}{\rightrightarrows}} Y(T^n)$ is an absolute equalizer. So to each functor $\mathbb{T} \xrightarrow[\mathcal{SET}]{F}$ we can associate a functor $\mathbb{T}_P \xrightarrow[\mathcal{SET}]{\tilde{F}}$ and again if F preserves finite products, so does $\tilde{F}$. It is straightforward to prove that these two constructions give rise to an equivalence $\vec{\mathbb{T}}_P \simeq \vec{\mathbb{T}}$. □

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