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Cahiers de topologie et géométrie différentielle catégoriques, tome 33, nº 4 (1992), p. 331-343.

<http://www.numdam.org/item?id=CTGDC_1992__33_4_331_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VOL. XXXIII-4 (1992)

MONOIDAL CATEGORIES FOR MORITA THEORY

by Enrico M. VITALE

Résumé. Dans cet article on vise à introduire axiomatiquement un environnement dans lequel développer les points fondamentaux de la théorie de Morita telle qu'elle est présentée dans les cas classiques ou d'autres semblables, c'est-à-dire l'étude des équivalences entre catégories de modules. Dans ce but on discute en détail des conditions d'exactitude et de linéarité permettant de classer ces équivalences au moyen des bimodules. En particulier, on montre que la notion de module fidèlement projective est liée aux identités triangulaires d'une équivalence adjointe.

Introduction

The aim of this work is to introduce axiomatically an environment in which to develop the basic points of Morita theory as it appears in the classic [1] or other similar cases [2], i.e. the study of equivalences between categories of modules.

In the following, we will not compare systematically the notions to be introduced with the existing literature on the subject: we will only notice here that the basic reference for closed categories is [3] and that two generalizations of Morita's problem are investigated in [4] and [6]; for a detailed introduction to monoidal categories, we refer to [5].

1 Tensor product

1.1 Let $\mathbb{C}$ be a monoidal category, i.e. $\mathbb{C} = (\mathbb{C}, \otimes, I,$ coherent natural isomorphisms of associativity and right and left identities) with $\mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}$ bifunctor and $I \in \mathbb{C}$ neutral object for $\otimes$; we can define in $\mathbb{C}$ what a monoid and a morphism of monoids are: for example, a monoid is a triple $(A, A \otimes A \xrightarrow{m_A} A, I \xrightarrow{\mathrm{id}_A} A)$ of objects and arrows of $\mathbb{C}$ making commutative the usual diagrams of identities and associativity; again with arrows and commutative diagrams in $\mathbb{C}$ we can define, if A and B are monoids, the categories of left-A-modules ($A - \mathrm{mod}$); right-B-modules ($\mathrm{mod} - B$) and left-A-right-B-bimodules ($A - \mathrm{mod} - B$).

The identity and associativity isomorphisms give I a structure of monoid for which we have $\mathbb{C} \simeq I - \mathrm{mod} - I \simeq I - \mathrm{mod} \simeq \mathrm{mod} - I$.

1.2 Axiom: *now we require the existence of some coequalizers in* $\mathbb{C}$; *in fact, if* $M \in \mathrm{mod} - B$ *and* $N \in B - \mathrm{mod}$, *we can define a new object* $M \otimes_B N$ *of* $\mathbb{C}$ *as the codomain of the universal bimorpism, i.e. the coequalizer*

$$M \otimes B \otimes N \overset{\mu_B \otimes 1_N}{\underset{1_M \otimes \eta_B}{\rightrightarrows}} M \otimes N \xrightarrow{q} M \otimes_B N$$

where $\mu_B : M \otimes B \longrightarrow M$ and $\eta_B : B \otimes N \longrightarrow N$ are the actions (as usual, we omit the associativity isomorphism for $\otimes$).

Such a construction, for the universal property of the coequalizer, gives rise to a functor $\mathrm{mod} - B \times B - \mathrm{mod} \xrightarrow{\otimes_B} \mathbb{C}$ for which B acts as neutral element.

We want that, if $N \in B - \mathrm{mod} - C$, then $M \otimes_B N$ inherits a $\mathrm{mod} - C$ structure from that of N : to this end we need a stability of coequalizers with respect to tensor product expressed with the following

1.3 Axiom: *if $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Q$ is a coequalizer in $\mathbb{C}$, then for each $Z \in \mathbb{C}$, $X \otimes Z \overset{f \otimes 1_Z}{\underset{g \otimes 1_Z}{\rightrightarrows}} Y \otimes Z \xrightarrow{q \otimes 1_Z} Q \otimes Z$ is a coequalizer.*

With such a condition we obtain the desired structure; in fact, we can consider the coequalizer

$$\begin{tikzcd} M \otimes B \otimes N \otimes C \arrow[r, shift left, "\mu_B \otimes 1_N \otimes 1_C"] \arrow[r, shift right, "1_M \otimes \eta_B \otimes 1_C"'] & M \otimes N \otimes C \arrow[r, "q'"] \arrow[dr, "q \otimes 1_C"'] & M \otimes_B (N \otimes C) \arrow[d, "\lambda"] \\ & & (M \otimes_B N) \otimes C \end{tikzcd}$$

but now λ is an isomorphism and we can define the action

$$(M \otimes_B N) \otimes C \xrightarrow{\cong} M \otimes_B (N \otimes C) \xrightarrow{1_M \otimes_B \eta_C} M \otimes_B N$$

With an analogous stability condition on the left, we have a bifunctor $A - \mathrm{mod} - B \times B - \mathrm{mod} - C \xrightarrow{\otimes_B} A - \mathrm{mod} - C$ with isomorphisms $(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P)$, $M \simeq M \otimes_B B, \ldots$ in the suitable categories of bimodules.

Note that the stability condition on the left certainly holds if the functor $Z \otimes -$ has a right adjoint.

2 Closure

2.1 Now we look at the closure of the categories of modules, i.e., if $M \in A - \mathrm{mod} - B$, we look at the existence of a right adjoint $M \supset_A -$ to the functor $M \otimes_B - :$ $B - \mathrm{mod} - C \longrightarrow A - \mathrm{mod} - C$. We will find that, as the definition of $\otimes_B$ is linked to the existence of some coequalizers, its closure is linked to the existence of some equalizers.

Let us suppose that for every $X \in \mathbb{C}$ the functor $X \otimes - : \mathbb{C} \longrightarrow \mathbb{C}$ has a right adjoint $X \supset -$; let us consider $X, Y \in A-$mod and a pair of arrows $X \supset Y \overset{a_1}{\underset{a_2}{\rightrightarrows}} (A \otimes X) \supset Y$ corresponding, in the adjunction $X \otimes - \dashv X \supset -$, to

$$A \otimes X \otimes (X \supset Y) \xrightarrow{1_A \otimes \epsilon} A \otimes Y \xrightarrow{\quad \eta_A \quad} Y \qquad \text{and}$$

$$A \otimes X \otimes (X \supset Y) \xrightarrow{\mu_A \otimes 1_{X \supset Y}} X \otimes (X \supset Y) \xrightarrow{\quad \epsilon \quad} Y$$

(with ϵ we always denote the counity of some adjunction); the following properties hold

2.2 Proposition: *if the functor $X \otimes - : \mathbb{C} \longrightarrow A-$ mod has a right adjoint $X \supset_A -$, then $X \supset_A Y$ is the equalizer in $\mathbb{C}$ of $X \supset Y \overset{a_1}{\underset{a_2}{\rightrightarrows}} (A \otimes X) \supset Y$.*

Proof: it suffices to define $X \supset_A Y \xrightarrow{e} X \supset Y$ as the correspondent, in $X \otimes - \dashv X \supset -$, of the counity of $X \otimes - \dashv X \supset_A -$. ■

The abuse of notation by which we denote with $X \otimes -$ two different functors is justified by the existence of the obvious forgetful functor $A-$mod$\longrightarrow \mathbb{C}$.

2.3 Proposition: *if, for every $Y \in A-$ mod, the equalizer of $X \supset Y \overset{a_1}{\underset{a_2}{\rightrightarrows}} (A \otimes X) \supset Y$ exists and, moreover, X is an $A-B-$ bimodule, then the functor $X \otimes_B - : B-mod-C \longrightarrow A-mod-C$ has a right adjoint $X \supset_A -$; moreover, this functor gives rise to a bifunctor $- \supset_A - : (A-mod-B)^{op} \times (A-mod-C) \longrightarrow B-mod-C$ and the bijection*

$$\frac{X \otimes_B Z \longrightarrow Y}{Z \longrightarrow X \supset_A Y}$$

is natural also in X.

Proof: the conditions of adjunction are verified if the equalizer $X \supset_A Y$ is enriched with the $B-$mod$-C$ structure coming from that of mod$-B$ of X and mod$-C$ of Y; as far as the functor $- \supset Y : \mathbb{C}^{op} \longrightarrow \mathbb{C}$ is concerned, we observe only that its definition on the arrows is forced by the request for the bijection $\frac{X \otimes Z \longrightarrow Y}{Z \longrightarrow X \supset Y}$ to be natural also in X. ■

3 Internal composition

3.1 Using the tensor product and their adjoints, we can build up an internal composition being associative and with identities: more precisely, if A, B, C, D are monoids in $\mathbb{C}$, and $M \in A - \text{mod} - B$, $N \in A - \text{mod} - C$, $P \in A - \text{mod} - D$, we can define a morphism in $B - \text{mod} - D$

$$(M \supset_A N) \otimes_C (N \supset_A P) \xrightarrow{c} M \supset_A P$$

To construct c and verify the claimed properties, we can proceed as follows: let us define $(M \supset N) \otimes (N \supset P) \xrightarrow{\bar{c}} M \supset P$ as the correspondent, in $M \otimes - \dashv M \supset -$, of

$$M \otimes (M \supset N) \otimes (N \supset P) \xrightarrow{\epsilon \otimes 1_{N \supset P}} N \otimes (N \supset P) \xrightarrow{\epsilon} P$$

It is easy to prove that $\bar{c}$ satisfies the claimed properties and then they pass to c which is the factorization of $\bar{c}$ through suitable equalizers and coequalizers.

In particular, $M \supset_A M$ is a monoid and $A \supset_A A$ and A are isomorphic as monoids.

4 Normal functors

4.1 Let us observe more carefully the functor $t = N \otimes_A - : A-\text{mod} \longrightarrow B-\text{mod}$ with $N \in B - \text{mod} - A$: it induces, for each pair $X, Y \in A - \text{mod}$, an arrow $\tau_{X,Y} : X \supset_A Y \longrightarrow tX \supset_B tY$ in $\mathbb{C}$ defined by the adjunction $tX \otimes - \dashv tX \supset_B -$; such a family $\tau = \{\tau_{X,Y}\}_{X,Y \in A-\text{mod}}$ satisfies the three following properties:

- it is natural in X and Y
- it is functorial, i.e. it commutes with the internal composition and the identity of the monoids like $X \supset_A X$
- it is normal, i.e. if $X \xrightarrow{f} Y \in A-\text{mod}$, then the following diagram commutes

$$\begin{tikzcd} I \arrow[rr, "\widetilde{tf}"] \arrow[dr, "\tilde{f}"'] & & tX \supset_B tY \\ & X \supset_A Y \arrow[ur, "\tau_{X,Y}"'] & \end{tikzcd}$$

(where $\tilde{f}$ is the factorization through $X \supset_A Y \xrightarrow{e} X \supset Y$ of the arrow corresponding to $X \otimes I \simeq X \xrightarrow{f} Y$ in $X \otimes - \dashv X \supset -$).

4.2 Definition: *a normal functor is a pair $\langle t, \tau \rangle$: $A - mod \longrightarrow B - mod$ where t is an application from the objects of $A - mod$ to those of $B - mod$ and τ is a family $\{\tau_{X,Y} : X \supset_A Y \longrightarrow tX \supset_B tY\}_{X,Y \in A-mod}$ of arrows in $\mathbb{C}$ satisfying the three conditions above; a natural transformation between two normal functors*

$\langle t,\tau\rangle \xrightarrow{\alpha} \langle r,\rho\rangle : A-mod \longrightarrow B-mod$ *is a family of arrows in* $B-mod$ $\alpha=\{\alpha_X : tX \longrightarrow rX\}_{X\in A-mod}$ *such that for every* $X,Y \in A-mod$ *the following square commutes*

$$\begin{tikzcd}
X\supset_A Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "\rho_{X,Y}"'] & tX\supset_B tY \arrow[d, "tX\supset_B\alpha_Y"] \\
rX\supset_B rY \arrow[r, "\alpha_X\supset_B rY"'] & tX\supset_B rY
\end{tikzcd}$$

In particular, the normality condition allows us to build up a functor, in the usual sense, from a normal one. Moreover, if $\langle t,\tau\rangle : A-\text{mod}\longrightarrow B-\text{mod}$ is a normal functor, tA is not only in $B-\text{mod}$, but it is in $B-\text{mod}-A$ with the action $tA\otimes A\longrightarrow tA$ induced by $A\simeq A\supset_A A \xrightarrow{\tau_{A,A}} tA\supset_B tA \xrightarrow{e} tA\supset tA$.

4.3 Proposition: *a sufficient condition for a normal colimit preserving functor* $\langle t,\tau\rangle : A-mod\longrightarrow B-mod$ *to be* $tA\otimes_A- : A-mod\longrightarrow B-mod$ *(and then to be completely determined by* tA *and* $\tau_{A,A}$*) is that* A *is a regular generator for* $A-mod$*, i.e. every* $X\in A-mod$ *can be seen as a suitable coequalizer* $\coprod_I A \rightrightarrows \coprod_J A\longrightarrow X$.

Proof: taking into account that the functor $tA\otimes_A-$, as a left adjoint, preserves colimits, we have that the canonical isomorphism $tA\otimes_A A\xrightarrow{\cong} tA$ gives rise to a natural isomorphism $tA\otimes_A - \xrightarrow{\cong} t$. ■

In the following, with functor we always mean normal functor.

5 Natural transformations

5.1 To classify equivalences between categories of modules through suitable bimodules, we have again a problem: let us consider $M,N\in A-\text{mod}-B$ and the functors $B-\text{mod} \underset{N\otimes_B-}{\overset{M\otimes_B-}{\rightrightarrows}} A-\text{mod}$; we want that if $M\otimes_B-$ and $N\otimes_B-$ are isomorphic as functors, then M and N are isormorphic as bimodules.

5.2 For this, let $\rho : M\otimes_B - \longrightarrow N\otimes_B-$ be a natural transformation; in particular, $\rho_B : M\otimes_B B\longrightarrow N\otimes_B B\in A-\text{mod}$; so we need to prove that ρ_B is B-linear on the right and we wonder if the naturality of ρ is enough to this end.

The diagram we want to be commutative is the diagram hereunder on the left;the naturality of ρ, on the contrary, makes commutative the diagram on the right

$$
\begin{tikzcd}
M\otimes_B B\otimes B \arrow[r, "\rho_B\otimes 1_B"'] \arrow[d, "M\otimes_B m_B"'] & N\otimes_B B\otimes B \arrow[d, "N\otimes_B m_B"] \\
M\otimes_B B \arrow[r, "\rho_B"'] & N\otimes_B B
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes_B B\otimes B \arrow[r, "\rho_{B\otimes B}"'] \arrow[d, "M\otimes_B m_B"'] & N\otimes_B B\otimes B \arrow[d, "N\otimes_B m_B"] \\
M\otimes_B B \arrow[r, "\rho_B"'] & N\otimes_B B
\end{tikzcd}
$$

and so it is not sufficient because, in general, it is not true that $\rho_B \otimes 1_B = \rho_{B\otimes B}$.

Nevertheless, we can observe that the naturality of ρ implies, for every $I \xrightarrow{b} B$, the commutativity of

$$
\begin{tikzcd}
M\otimes_B B\otimes I \simeq M\otimes_B B \arrow[r, "\rho_B"] \arrow[d, "M\otimes_B 1_B\otimes b"'] & N\otimes_B B \simeq N\otimes_B B\otimes I \arrow[d, "N\otimes_B 1_B\otimes b"] \\
M\otimes_B B\otimes B \arrow[d, "M\otimes_B m_B"'] & N\otimes_B B\otimes B \arrow[d, "N\otimes_B m_B"] \\
M\otimes_B B \arrow[r, "\rho_B"'] & N\otimes_B B
\end{tikzcd}
$$

and so we obtain the B-linearity on the right of ρ_B if we assume a condition like

5.3 Axiom: *let B be a monoid and $M, N \in mod-B$ with action $M\otimes B \xrightarrow{\mu_B} M$ and $N\otimes B \xrightarrow{\eta_B} N$; let $M \xrightarrow{\alpha} N$ be an arrow in $\mathbb{C}$; if for each $I \xrightarrow{b} B$ the following diagram is commutative*

$$
\begin{tikzcd}
M\otimes I \arrow[r, "\alpha\otimes 1_I"] \arrow[d, "1_M\otimes b"'] & N\otimes I \arrow[d, "1_N\otimes b"] \\
M\otimes B \arrow[d, "\mu_B"'] & N\otimes B \arrow[d, "\eta_B"] \\
M \arrow[r, "\alpha"'] & N
\end{tikzcd}
$$

then $M \xrightarrow{\alpha} N \in mod-B$.

5.4 For this condition, $\overline{\rho} : M \simeq M\otimes_B B \xrightarrow{\rho_B} N\otimes_B B \simeq N$ is a bimodule morphism and, if B is a regular generator for mod $-$ B, the natural transformation induced by $M \xrightarrow{\overline{\rho}} N$ coincides with $M\otimes_B - \xrightarrow{\rho} N\otimes_B -$; in particular we have the result requested in 5.1.

6 Morita categories

We can summarize the above discussion in the following

6.1 Definition: *a Morita category is a monoidal closed category with enough coequalizers (axiom 1.2) which are stable under the tensor product (axiom 1.3) and enough equalizers (as in section 2) in which each monoid A is a regular generator for the category $A-\mathrm{mod}$ and in which the linearity condition stated in 5.3 holds.*

7 Equivalences

Finally we can consider an equivalence

$$t : A - \mathrm{mod} \xrightarrow{\cong} B - \mathrm{mod}.$$

7.1 Theorem: *in a Morita category, such an equivalence condition can be translated into the existence of two bimodule isomorphisms*

$$P \otimes_B Q \simeq A \textit{ and } Q \otimes_A P \simeq B.$$

Proof: t and t^{-1} are colimit preserving normal functors, so for proposition 4.3, we can write $t = Q\otimes_A -$ and $t^{-1} = P\otimes_B -$ where $Q = tA \in B - \mathrm{mod} - A$ and $P = t^{-1}B \in A - \mathrm{mod} - B$; therefore the equivalence conditions $t \cdot t^{-1} \simeq \mathrm{id}_{A-\mathrm{mod}}$ and $t^{-1} \cdot t \simeq \mathrm{id}_{B-\mathrm{mod}}$ become $P\otimes_B Q \otimes_A - \simeq A\otimes_A -$ and $Q\otimes_A P\otimes_B - \simeq B\otimes_B -$ and for 5.4 we have the announced isomorphisms. ■

7.2 As $P\otimes_B -$ and $Q\otimes_A -$ are equivalences, we also have the following isomorphisms of monoids: $A \simeq Q \supset_B Q$ and $B \simeq P \supset_A P$; moreover, taking into account that the right adjoints of $Q\otimes_A -$ and $P\otimes_B -$ are $Q\supset_B -$ and $P\supset_A -$ and in the non-restrictive hypothesis that the equivalence t is an adjoint equivalence, we have the following isomorphisms of bimodules: $P \simeq Q\supset_B B$ and $Q \simeq P\supset_A A$.

Let us look more carefully at the data $P \in A - \mathrm{mod} - B$, $Q \in B - \mathrm{mod} - A$, $f : P\otimes_B Q \xrightarrow{\cong} A$, $g : Q\otimes_A P \xrightarrow{\cong} B$ that induce the equivalence $A - \mathrm{mod} \simeq B - \mathrm{mod}$.

7.3 Proposition: *by the isomorphisms listed in 7.2, we can build up the diagrams*

$$\begin{tikzcd} P\otimes_B Q \arrow[r, "f"] \arrow[d, "\cong"] & A \arrow[d, "\cong"] \\ (A\supset_A P)\otimes_B(P\supset_A A) \arrow[r, "c"'] & A\supset_A A \end{tikzcd} \qquad \begin{tikzcd} Q\otimes_A P \arrow[r, "g"] \arrow[d, "\cong"] & B \arrow[d, "\cong"] \\ (P\supset_A A)\otimes_A(A\supset_A P) \arrow[r, "c"'] & P\supset_A P \end{tikzcd}$$

whose commutativity gives us the link between f and g and the internal compositions (the same can be done with Q).

Proof: for the first square, it suffices to notice that f and c are, up to isomorphisms, the counity, at the level of A, of the adjunction $P\otimes_B - \dashv Q\otimes_A -$ that is $P\otimes_B - \dashv P\supset_A -$; the second square requires more attention because to build it we use g (explicitely) but also f (in the isomorphism $Q \simeq P\supset_A A$) and so it gives us a link between f and g; in fact the proof of the second commutativity is based on the two triangular identities of the adjoint equivalence t which can be expressed in the following commutative squares

$$\begin{tikzcd} P\otimes_B Q\otimes_A P \arrow[r, "P\otimes_B g"] \arrow[d, "f\otimes_A P"'] & P\otimes_B B \arrow[d, "\cong"] \\ A\otimes_A P \arrow[r, "\cong"] & P \end{tikzcd} \qquad \begin{tikzcd} Q\otimes_A P\otimes_B Q \arrow[r, "Q\otimes_A f"] \arrow[d, "g\otimes_B Q"'] & Q\otimes_A A \arrow[d, "\cong"] \\ B\otimes_B Q \arrow[r, "\cong"] & Q \end{tikzcd}$$

■

7.4 Definition: *a module $P \in A-mod$ is faithfully projective if the compositions*

$$(A\supset_A P)\otimes_{P\supset_A P}(P\supset_A A) \xrightarrow{c} A\supset_A A \ , \ (P\supset_A A)\otimes_A(A\supset_A P) \xrightarrow{c} P\supset_A P$$

are isomorphisms.

8 Faithfully projective modules

8.1 We wonder if the notion of faithfully projective module is stable under equivalences; we begin with the construction of an equivalence by a faithfully projective module.

Let $P \in A-\mathrm{mod}$; such a module has always a structure of right module on $P\supset_A P$ given by the counity $P \otimes (P\supset_A P) \longrightarrow P$, so we can consider the functor $P\supset_A - : A-\mathrm{mod} \longrightarrow (P\supset_A P)-\mathrm{mod}$. The results of the above section allow us to say that this functor is an equivalence if and only if P is faithfully projective and that all the equivalences between categories of modules are of this kind.

8.2 With this characterization of faithfully projective modules, the stability of the notion follows from the commutativity of the following diagram

$$\begin{tikzcd}
A - \mathrm{mod} \arrow[rr, "\cong t"] \arrow[d, "P\supset_A -"', "\cong"] & & B - \mathrm{mod} \arrow[d, "tP\supset_B -"] \\
(P\supset_A P) - \mathrm{mod} \arrow[rr, "\cong"] & & (tP\supset_B tP) - \mathrm{mod}
\end{tikzcd}$$

8.3 Corollary: *an equivalence $A - mod \simeq B - mod$ can be restricted to an equivalence between the full subcategories of the faithfully projective modules $A - mod_{f.p.} \simeq B - mod_{f.p.}$.*

8.4 In order to establish the opposite of 8.3 using the same arguments as in theorem 7.1, i.e. to extend an equivalence $A - \mathrm{mod}_{f.p.} \simeq B - \mathrm{mod}_{f.p.}$ to an equivalence between the whole categories, we need a stronger generating condition: as $A \in A - \mathrm{mod}_{f.p.}$, such a condition can consist in finding, for each $X \in A - \mathrm{mod}_{f.p.}$, a presenting base $\coprod_I A \rightrightarrows \coprod_J A$ entirely contained in $A - \mathrm{mod}_{f.p.}$.

8.5 As far as the notion of faithfully projective is concerned, more generally we can consider $M, N \in A - \mathrm{mod}$ and then $M\supset_A N$ and $N\supset_A M$; now we can enrich them respectively with a structure of $(M\supset_A M) - \mathrm{mod} - (N\supset_A N)$ and $(N\supset_A N) - \mathrm{mod} - (M\supset_A M)$ using the internal composition: we say that M and N are projectively equivalent if the compositions

$$(M\supset_A N)\otimes_{N\supset_A N}(N\supset_A M) \longrightarrow M\supset_A M$$

$$(N\supset_A M)\otimes_{M\supset_A M}(M\supset_A N) \longrightarrow N\supset_A N$$

are isomorphisms; this is equivalent to say that the functors

$$(N\supset_A N) - \mathrm{mod} \underset{(M\supset_A N)\otimes_{N\supset_A N}-}{\overset{(N\supset_A M)\otimes_{M\supset_A M}-}{\rightleftarrows}} (M\supset_A M) - \mathrm{mod}$$

constitute an equivalence.

9 The commutative case

9.1 To examine the commutative case, we begin with some general considerations. If $\mathbf{C}$ is a Morita category which is symmetric as a monoidal category, i.e. if there exist natural coherent isomorphisms $\gamma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$, the right adjoints $X \supset -$ and $- \subset X$ of the functors $X\otimes-$ and $-\otimes X$ are connected by a natural isomorphism $i_{X,Y} : X \supset Y \longrightarrow Y \subset X$ depending on γ because $X \supset Y \xrightarrow{i_{X,Y}} Y \subset X$ is the correspondent of

$$(X \supset Y) \otimes X \xrightarrow{\gamma_{X \supset Y, X}} X \otimes (X \supset Y) \xrightarrow{\epsilon} Y$$

in $- \otimes X \dashv - \subset X$.

9.2 Now, if $X, Y \in A - \mathrm{mod} - B$, we can build up two commutative squares

$$\begin{tikzcd} (A \otimes X) \supset Y \arrow[r, "b_i"] & (A \otimes X \otimes B) \supset Y \\ X \supset Y \arrow[u, "a_i"] \arrow[r, "\alpha_i"'] & (X \otimes B) \supset Y \arrow[u, "\beta_i"'] \end{tikzcd} \qquad i = 1, 2$$

where the a_i's are defined in 2.1, the α_i's correspond, in $(X \otimes B) \otimes - \dashv (X \otimes B) \supset -$, to

$$X \otimes B \otimes (X \supset Y) \simeq X \otimes (X \supset Y) \otimes B \xrightarrow{\epsilon \otimes 1_B} Y \otimes B \xrightarrow{\eta_B} Y$$

and $\qquad X \otimes B \otimes (X \supset Y) \xrightarrow{\mu_B \otimes 1_{X \supset Y}} X \otimes (X \supset Y) \xrightarrow{\epsilon} Y;$

the b_i's and the β_i's are defined analogously.

We define $X \supset_{A,B} Y$ as the equalizer

$$\begin{tikzcd} & (A \otimes X) \supset Y \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & (A \otimes X \otimes B) \supset Y \\ & X \supset Y \arrow[u, shift left, "a_2"'] \arrow[u, shift right, "a_1"] \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & (X \otimes B) \supset Y \arrow[u, shift left, "\beta_2"'] \arrow[u, shift right, "\beta_1"] \\ X \supset_{A,B} Y \arrow[r] & X \supset_A Y \arrow[u] \arrow[r, shift left, "\gamma_1"] \arrow[r, shift right, "\gamma_2"'] & (X \otimes B) \supset_A Y \arrow[u] \end{tikzcd}$$

9.3 Proposition: *if $M \in C - mod - A$, consider $M \otimes_A - : A - mod \longrightarrow C - mod$ and $\tau_{X,Y} : X \supset_A Y \longrightarrow (M \otimes_A X) \supset_C (M \otimes_A Y)$ as in 4.1; then this arrow can be restricted to*

$$\begin{array}{ccc} X\supset_A Y & \xrightarrow{\tau_{X,Y}} & (M\otimes_A X)\supset_C(M\otimes_A Y) \\ \uparrow & & \uparrow \\ X\supset_{A,B} Y & \longrightarrow & (M\otimes_A X)\supset_{C,B}(M\otimes_A Y) \end{array}$$

Proof: we have to verify the double commutativity ot τ at the level of the bases defining $X\supset_{A,B}Y$ and $(M\otimes_A X)\supset_{C,B}(M\otimes_A Y)$; for this it is useful to explicit γ_1 and γ_2 as the correspondent, in $(X \otimes B)\otimes_B - \dashv (X \otimes B)\supset_B -$, of

$$X \otimes B\otimes_B(X\supset_A Y) \simeq B \otimes X\otimes_B(X\supset_A Y) \xrightarrow{1_B \otimes \epsilon} B \otimes Y \simeq Y \otimes B \xrightarrow{\eta_B} Y$$

and $$X \otimes B\otimes_B(X\supset_A Y) \xrightarrow{\mu_B\otimes_B 1_{X\supset_A Y}} X\otimes_B(X\supset_A Y) \xrightarrow{\epsilon} Y.$$

■

Of course, we can use $Y \subset X$ to build up two squares analogous to those built up in 9.2 and then define $Y\subset_{A,B}X$.

9.4 Lemma: *$X\supset_{A,B}Y$ and $Y\subset_{A,B}X$ are isomorphic.*

Proof: it suffices to observe that the isomorphisms $i_{X,Y}$ commute with the basic squares and then use the interchange rule for equalizers. ■

From the isomorphism $X\supset_{A,B}Y \simeq Y\subset_{A,B}X$ and its construction, we have that the following diagram is a pullback

$$\begin{array}{ccc} X\supset_{A,B} Y & \longrightarrow & X\supset_A Y \\ \downarrow\cong & & \downarrow \\ Y\subset_{A,B} X & & X \supset Y \\ \downarrow & & \downarrow i_{X,Y} \\ Y\subset_B X & \longrightarrow & Y \subset X \end{array}$$

9.5 As the equalizer $A\supset_A A \longrightarrow A \supset A$ is exactly the arrow $A \xrightarrow{\chi} A \supset A$ corresponding, in $A \otimes - \dashv A \supset -$, to the multiplication $A \otimes A \xrightarrow{m_A} A$, we have that the left one of the following two diagrams is a pullback; so we have a definition of the centre of the monoid A; in fact the diagram on the right is a pullback if and only if A is commutative

$$
\begin{tikzcd}
A\supset_{A,A}A \arrow[r] \arrow[d, "\cong"] & A \arrow[d, "\chi"] \\
A\subset_{A,A}A \arrow[d] & A\supset A \arrow[d, "i_{A,A}"] \\
A \arrow[r, "\chi'"'] & A\subset A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[dd, "1_A"'] & A \arrow[d, "\chi"] \\
 & A\supset A \arrow[d, "i_{A,A}"] \\
A \arrow[r, "\chi'"'] & A\subset A
\end{tikzcd}
$$

9.6 Let us now consider an equivalence

$$A-\mathrm{mod}\underset{Q\otimes_A -}{\overset{P\otimes_B -}{\rightleftarrows}}B-\mathrm{mod};$$

as $P\otimes_B Q\simeq A$ and $Q\otimes_A P\simeq B$ (cf. theorem 7.1), we have that

$$\mathrm{mod}-A\underset{-\otimes_A P}{\overset{-\otimes_B Q}{\rightleftarrows}}\mathrm{mod}-B$$

is also an equivalence; so we have a pair of isomorphisms of monoids $A\supset_A A\simeq Q\supset_B Q$ and $B\subset_B B\simeq Q\subset_A Q$ that, for proposition 9.3, can be restricted to two isomorphisms $A\supset_{A,A}A\simeq Q\supset_{A,B}Q$ and $B\subset_{B,B}B\simeq Q\subset_{A,B}Q$ and then, for lemma 9.4, we have

$$A\supset_{A,A}A\simeq B\subset_{B,B}B.$$

This last monoid isomorphism, taking account of 9.5, allows us to conclude that, in a symmetric Morita category, Morita-equivalent monoids have isomorphic centres.

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