A CHARACTERIZATION OF LOCALLY $\mathbb{D}$-PRESENTABLE CATEGORIES

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Abstract. Locally finitely presentable categories have been generalized in [1], under the name of locally $\mathbb{D}$-presentable categories, replacing filtered colimits by colimits commuting in *Set* with limits indexed by an arbitrary doctrine $\mathbb{D}$. In this note, we characterize locally $\mathbb{D}$-presentable categories as cocomplete categories with a strong generator consisting of $\mathbb{D}$-presentable objects. This extends known results on locally finitely presentable categories, varieties and presheaf categories.

1. Introduction

There are many features that varieties, presheaf categories and locally finitely presentable categories of Gabriel and Ulmer have in common, and they are at the base of the general framework introduced in [1] under the name of locally $\mathbb{D}$-presentable categories. Our aim is to bring one additional feature to the list presented in [1].

Locally finitely presentable categories can be defined as cocomplete categories with a small (up to isomorphism) set of finitely presentable objects such that each object is a filtered colimit of finitely presentable ones. In [1], for a limit doctrine $\mathbb{D}$, locally $\mathbb{D}$-presentable categories are defined as cocomplete categories with a small set of $\mathbb{D}$-presentable objects such that each object is a $\mathbb{D}$-filtered colimit of $\mathbb{D}$-presentable ones (a small category $\mathcal{C}$ is $\mathbb{D}$-filtered if $\mathcal{C}$-colimits commute in *Set* with $\mathbb{D}$-limits; an object G is $\mathbb{D}$-presentable if the functor $hom(G,-)$ preserves $\mathbb{D}$-filtered colimits). A basic theorem, due to Gabriel and Ulmer, asserts that a cocomplete category is locally finitely presentable if and only if it has a strong generator consisting of finitely presentable objects, see [9, 2, 4]. There is a similar result for (finitary, multi-sorted) varieties: a category is equivalent to a variety if and only if it is cocomplete and has a strong generator consisting of strongly finitely presentable objects (that is, objects G such that $hom(G,-)$ preserves those colimits commuting in *Set* with finite products, see [7, 11]). Since locally finitely presentable categories and varieties are special cases of locally $\mathbb{D}$-presentable categories (choose as $\mathbb{D}$, respectively, the doctrine of finite limits and the doctrine of finite products), it is natural to look for a generalization of the previous results to locally $\mathbb{D}$-presentable categories. This is what we do in this note.

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2. The characterization theorem

We recall from [1] the following definitions.

Definition 1

1. A collection $\mathbb{D}$ of small categories is called a *doctrine* if it is small up to isomorphism.

2. Let $\mathbb{D}$ be a doctrine. A $\mathbb{D}$-limit is a limit of a functor with domain in $\mathbb{D}$. A category which has all $\mathbb{D}$-limits is said $\mathbb{D}$-*complete*, and a functor between $\mathbb{D}$-complete categories is called $\mathbb{D}$-*continuous* if it preserves all $\mathbb{D}$-limits. Dually, there are the notions of $\mathbb{D}$-cocompleteness and $\mathbb{D}$-cocontinuity. $\mathbb{D}^{op}$ stands for the doctrine consisting of all categories $\mathcal{D}^{op}$, for $\mathcal{D} \in \mathbb{D}$.

3. We say that a small category $\mathcal{C}$ is $\mathbb{D}$-*filtered* if $\mathcal{C}$-colimits commute in Set with $\mathbb{D}$-limits.

4. Let $\mathcal{K}$ and $\mathcal{K}'$ be categories with $\mathbb{D}$-filtered colimits. A functor $F\colon \mathcal{K} \to \mathcal{K}'$ is $\mathbb{D}$-*accessible* if it preserves $\mathbb{D}$-filtered colimits. An object K of $\mathcal{K}$ is $\mathbb{D}$-*presentable* if the representable functor $\mathcal{K}(K,-)\colon \mathcal{K} \to Set$ is $\mathbb{D}$-accessible.

5. A doctrine $\mathbb{D}$ is said to be *sound* if a small category $\mathcal{C}$ is $\mathbb{D}$-filtered whenever the category of cocones of any functor $\mathcal{D}^{op} \to \mathcal{C}$, with $\mathcal{D} \in \mathbb{D}$, is connected.

6. A locally small category $\mathcal{K}$ is *locally* $\mathbb{D}$-*presentable* if it is cocomplete and admits a small set $\mathcal{M}$ of $\mathbb{D}$-presentable objects such that any object of $\mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{M}$.

Theorem 2 *Let $\mathbb{D}$ be a sound doctrine and $\mathcal{K}$ a locally small category. The following conditions are equivalent:*

1. *$\mathcal{K}$ is locally $\mathbb{D}$-presentable;*

2. *$\mathcal{K}$ is cocomplete and has a strong generator $\mathcal{G}$ consisting of $\mathbb{D}$-presentable objects.*

Proof. $1 \Rightarrow 2$: Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category. By Theorem 5.5 in [1], $\mathcal{K}$ is equivalent to a reflection of the functor category $[\mathcal{A}, Set]$, for $\mathcal{A}$ a small and $\mathbb{D}$-complete category. Since the set of representable functors is a strong generator in $[\mathcal{A}, Set]$, the set of their reflections is a strong generator in $\mathcal{K}$. Moreover, a representable functor is an absolutely presentable object (that is, the corresponding representable functor preserves all colimits) and the inclusion of $\mathcal{K}$ in $[\mathcal{A}, Set]$ is $\mathbb{D}$-accessible (Lemma 3.3 in [6]). By Lemma 3.6 in [6], we can conclude that the reflection of a representable functor is a $\mathbb{D}$-presentable object in $\mathcal{K}$.
$2 \Rightarrow 1$: Let $\mathcal{K}$ be a cocomplete category with a strong generator $\mathcal{G}$ consisting of $\mathbb{D}$-presentable objects. We write $\tilde{\mathcal{G}}$ for its $\mathbb{D}^{op}$-colimit completion in $\mathcal{K}$, that is $\tilde{\mathcal{G}}$ is the smallest full subcategory of $\mathcal{K}$ which contains $\mathcal{G}$ and is closed in $\mathcal{K}$ under $\mathbb{D}^{op}$-colimits. Stated otherwise, $\tilde{\mathcal{G}}$ is the iterative closure of $\mathcal{G}$ under $\mathbb{D}^{op}$-colimits, that is, it can be inductively constructed in the following way:

- $\mathcal{G}_0$ is $\mathcal{G}$, regarded as full subcategory of $\mathcal{K}$;

- $\mathcal{G}_{\lambda+1}$ is obtained by adding to $\mathcal{G}_\lambda$ a colimit of any functor $F\colon \mathcal{D}^{op} \to \mathcal{K}$ which factors through $\mathcal{G}_\lambda$, for any $\mathcal{D}$ in $\mathbb{D}$;

- If λ is a limit ordinal, then $\mathcal{G}_\lambda = \bigcup_{\xi<\lambda} \mathcal{G}_\xi$.

It is a result due to Ehresmann that $\tilde{\mathcal{G}}$ is small, see [8]. Since each object of $\mathcal{G}$ is $\mathbb{D}$-presentable, and a $\mathbb{D}^{op}$-colimit of $\mathbb{D}$-presentable objects is $\mathbb{D}$-presentable (Lemma 1.6 in [1]), we have that any object of $\tilde{\mathcal{G}}$ is $\mathbb{D}$-presentable in $\mathcal{K}$. Moreover, for any object $K \in \mathcal{K}$, the comma category $\tilde{\mathcal{G}}/K$ is $\mathbb{D}$-filtered. In fact, it is $\mathbb{D}^{op}$-cocomplete (because $\tilde{\mathcal{G}}$ is so) and then $\mathbb{D}$-filtered by Proposition 2.5 in [1].

We prove now that $\tilde{\mathcal{G}}$ is a dense generator in $\mathcal{K}$. This means to prove that, for any object $K \in \mathcal{K}$, the colimit of the forgetful functor

$$\Gamma_K\colon \tilde{\mathcal{G}}/K \to \tilde{\mathcal{G}} \to \mathcal{K} \qquad \Gamma_K(A, a\colon A \to K) = A$$

is precisely $\langle K, (a\colon A \to K)_{(A,a)\in\tilde{\mathcal{G}}/K}\rangle$. Let $\langle L, (s_{(A,a)})_{(A,a)\in\tilde{\mathcal{G}}/K}\rangle$ be a colimit of Γ_K in $\mathcal{K}$. Since the morphisms $a\colon \Gamma_K(A,a) \to K$ constitute a cocone on Γ_K, we get a canonical factorization $\lambda\colon L \to K$, and we have to prove that λ is an isomorphism. Consider the diagram

$$\begin{tikzcd}
\coprod_{(G,g)} G \arrow[r, "v"] \arrow[dr, "u"'] & L \arrow[d, "\lambda"] \\
& K
\end{tikzcd}$$

where the coproduct is indexed by all pairs (G,g) with $G \in \mathcal{G}$ and $g\colon G \to K$. Denote by $\rho_{(G,g)}\colon G \to \coprod G$ the coproduct injections, u is the unique morphism such that $u\cdot\rho_{(G,g)} = g$ and v is the unique morphism such that $v \cdot \rho_{(G,g)} = s_{(G,g)}$. Since the diagram commutes (compose with $\rho_{(G,g)}$) and u is a strong epimorphism, λ also is a strong epimorphism, so that it only remains to prove that λ is a monomorphism.
Consider two morphisms $x, y\colon X \to L$ in $\mathcal{K}$ such that $\lambda \cdot x = \lambda \cdot y$. To prove that $x = y$ is to prove that $x \cdot z = y \cdot z$ for any $G \in \mathcal{G}$ and any $z\colon G \to X$, because $\mathcal{G}$ is a generator. Since $\tilde{\mathcal{G}}/K$ is $\mathbb{D}$-filtered and any $G \in \mathcal{G}$ is $\mathbb{D}$-presentable, we have

$$\mathcal{K}(G, \mathrm{colim}_{(A,a)}\Gamma_K(A,a)) \simeq \mathrm{colim}_{(A,a)}\mathcal{K}(G,A)\,.$$

Therefore, both of $x \cdot z$ and $y \cdot z$ factor through some term of the colimit, i.e. there exist $(A,a) \in \tilde{\mathcal{G}}/K$ and $x'\colon G \to A$ such that $s_{(A,a)} \cdot x' = x \cdot z$ and analogously, there exist $(B,b) \in \tilde{\mathcal{G}}/K$ and $y'\colon G \to B$ such that $s_{(B,b)} \cdot y' = y \cdot z$. This gives rise to an object $(G, \lambda\cdot x\cdot z) = (G, \lambda\cdot y\cdot z)$ and two arrows $x'\colon (G, \lambda\cdot x\cdot z) \to (A,a)$ and $y'\colon (G,\lambda\cdot y\cdot z) \to (B,b)$ in $\tilde{\mathcal{G}}/K$. Finally, we have $x \cdot z = s_{(A,a)} \cdot x' = s_{(G,\lambda\cdot x\cdot z)} = s_{(G,\lambda\cdot y\cdot z)} = s_{(B,b)} \cdot y' = y \cdot z$, as desired. ∎

Remark 3 Let $\mathbb{D}$ be a sound doctrine. The following facts can be deduced directly from the definition of locally $\mathbb{D}$-presentable category or, more easily, by using Theorem 2.

1. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category; for any object $A \in \mathcal{K}$, the slice category $\mathcal{K}/A$ is locally $\mathbb{D}$-presentable.

2. Locally $\mathbb{D}$-presentable categories are precisely the $\mathbb{D}$-accessible reflections of presheaf categories (i.e., reflective subcategories closed under $\mathbb{D}$-filtered colimits).

3. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category; for any small category $\mathcal{C}$, the functor category $[\mathcal{C}, \mathcal{K}]$ is locally $\mathbb{D}$-presentable.

Example 4 We consider here the main examples of sound doctrines.

1. Let $\mathbb{D}$ be the doctrine of finite categories. Then locally $\mathbb{D}$-presentable precisely means locally finitely presentable, and Theorem 2 is the classical result on locally finitely presentable categories quoted in the Introduction.

2. Let $\mathbb{D}$ be the doctrine of finite discrete categories. Then locally $\mathbb{D}$-presentable categories are multisorted finitary varieties (see [3]). Theorem 2 asserts that a category is equivalent to a variety if and only if it is cocomplete and has a strong generator consisting of strongly finitely presentable objects, that is objects G such that the representable functor $hom(G, -)$ preserves sifted colimits.

3. Let $\mathbb{D}$ be the empty doctrine. Then locally $\mathbb{D}$-presentable categories are precisely presheaf categories. Theorem 2 characterizes them as those cocomplete categories having a strong generator consisting of absolutely presentable objects, that is objects G such that the representable functor $hom(G, -)$ preserves all small colimits.

Remark 5 Up to minor modifications, the previous characterization of presheaf categories is due to Bunge [5], and the characterization of varieties is due to Diers [7]. A different proof for the characterization of varieties has also been proposed by Pedicchio and Wood [11]. Our general proof is quite close to the one in [7], which is based on general properties of dense functors. The proofs in [5, 11] are substantially different: the first one is based on the special adjoint functor theorem and on Beck monadicity theorem, the second one consists in proving that equivalence relations are effective and makes use of the original characterization of varieties due to Lawvere [10].

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