A DUALITY RELATIVE TO A LIMIT DOCTRINE

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Abstract. We give a unified proof of Gabriel-Ulmer duality for locally finitely presentable categories, Adámek-Lawvere-Rosický duality for varieties and Morita duality for presheaf categories. As an application, we compare presheaf categories and varieties.

1. Introduction

The celebrated Gabriel-Ulmer duality states that there is a contravariant biequivalence

$$\mathbf{Lex}^{op} \longrightarrow \mathbf{LFP}$$

where **Lex** is the 2-category of small finitely complete categories, finite limit preserving functors and natural transformations, and **LFP** is the 2-categories of locally finitely presentable categories, finitary right adjoint functors and natural transformations (see [7]). More recently, Adámek, Lawvere and Rosický established a similar biequivalence

$$\mathbf{Th}^{op} \longrightarrow \mathbf{VAR}$$

where **Th** is the 2-category of small Cauchy complete categories with finite products, finite product preserving functors and natural transformations, and **VAR** is the 2-category of multisorted finitary varieties, algebraically exact functors and natural transformations (see [2]).

The first one of these dualities is based on the interplay between finite limits and filtered colimits, which are those colimits commuting in **Set** with finite limits and which are used to define locally finitely presentable categories. Similarly, the second duality is based on finite products and sifted colimits, which can be defined as those colimits commuting in **Set** with finite products and which can be used to give an abstract characterization of finitary multisorted varieties.

It seems natural to look for a common generalization of these dualities, starting from a good class $\mathbb{D}$ of limits and using $\mathbb{D}$-filtered colimits, i.e. those colimits commuting in **Set** with $\mathbb{D}$-limits. The notion of locally finitely presentable category has been recently generalized by Adámek, Borceux, Lack and Rosický in [1], where the basic theory of locally $\mathbb{D}$-presentable categories is developed. In particular, Adámek, Borceux, Lack and Rosický have the expected representation theorem of locally $\mathbb{D}$-presentable categories as categories of $\mathbb{D}$-continuous set-valued functors. The aim of this note is just to put this theorem in its natural context, which is a biequivalence

$$\mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

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where $\mathbb{D}$-**Th** is the 2-category of essentially small Cauchy complete and $\mathbb{D}$-complete categories, $\mathbb{D}$-continuous functors and natural transformations, and $\mathbb{D}$-**LP** is the 2-category of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations.

In Section 2, we recall from [1] the basic definitions and results we need. In section 3 we state and prove the duality between $\mathbb{D}$-**Th** and $\mathbb{D}$-**LP**. The proof is achieved thanks to several simple lemmas. These lemmas are inserted when needed in the proof, but their own proofs are postponed to Section 4, so to make the reading of the main result easier. We follow the lines of the remarkable exposition of Gabriel-Ulmer duality given in [3]. In the last section, we give a simple application of the duality theorem comparing presheaf categories and varieties.

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2. Preliminaries

All definitions and results contained in this section can be found in [1].

Definition 1 A collection $\mathbb{D}$ of small categories is called a *limit doctrine*, or just *doctrine*, if $\mathbb{D}$, regarded as full subcategory of **Cat**, is essentially small. A $\mathbb{D}$-limit is the limit of a functor with domain in $\mathbb{D}$. A category which has all $\mathbb{D}$-limits is said $\mathbb{D}$-*complete*, and a functor between $\mathbb{D}$-complete categories is called $\mathbb{D}$-*continuous* if it preserves all $\mathbb{D}$-limits. Dually, there are the notions of $\mathbb{D}$-cocompleteness and $\mathbb{D}$-cocontinuity. $\mathbb{D}^{op}$ stands for the doctrine consisting of all categories $\mathcal{D}^{op}$ for $\mathcal{D} \in \mathbb{D}$.

As general consideration, let us point out that a category $\mathcal{A}$ is $\mathbb{D}$-complete if and only if its dual category $\mathcal{A}^{op}$ is $\mathbb{D}^{op}$-cocomplete.

Definition 2 We say that a small category $\mathcal{C}$ is $\mathbb{D}$-*filtered* if $\mathcal{C}$-colimits commute in **Set** with $\mathbb{D}$-limits, i.e. for any $\mathcal{D} \in \mathbb{D}$ and any functor $F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathbf{Set}$ the canonical map

$$\operatorname*{colim}_{C\in\mathcal{C}} \lim_{D\in\mathcal{D}} F(C,D) \longrightarrow \lim_{D\in\mathcal{D}} \operatorname*{colim}_{C\in\mathcal{C}} F(C,D)$$

is an isomorphism.

Definition 3 Let $\mathcal{K}$ and $\mathcal{K}'$ be categories with $\mathbb{D}$-filtered colimits. A functor $F\colon\mathcal{K} \longrightarrow \mathcal{K}'$ is $\mathbb{D}$-*accessible* if it preserves $\mathbb{D}$-filtered colimits. An object K of $\mathcal{K}$ is $\mathbb{D}$-*presentable* if the representable functor $\mathcal{K}(K,-) : \mathcal{K} \longrightarrow \mathbf{Set}$ is $\mathbb{D}$-accessible.

Lemma 4 *A $\mathbb{D}^{op}$-colimit of $\mathbb{D}$-presentable objects is $\mathbb{D}$-presentable itself.*

Definition 5 A doctrine $\mathbb{D}$ is *sound* if for a functor $F : \mathcal{A} \longrightarrow \mathbf{Set}$, with $\mathcal{A}$ a small category, the left Kan extension $\mathrm{Lan}_{\mathrm{Y}}F : [\mathcal{A}^{op}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ of F along the Yoneda embedding preserves $\mathbb{D}$-limits if and only if it preserves $\mathbb{D}$-limits of representables.

From now on, we assume the doctrine $\mathbb{D}$ to be sound.

Lemma 6 *Let $\mathcal{A}$ be a small and $\mathbb{D}$-complete category. A functor $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$ is $\mathbb{D}$-continuous if and only if it is a $\mathbb{D}$-filtered colimit of representable functors.*

Definition 7 To say that a locally small category $\mathcal{K}$ is $\mathbb{D}$-*accessible* is to say that it has $\mathbb{D}$-filtered colimits and admits a small set $\mathcal{A}$ of $\mathbb{D}$-presentable objects such that any object of $\mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$.

Lemma 8 *Let $\mathcal{K}$ be a $\mathbb{D}$-accessible category. The full subcategory $\mathcal{K}_{\mathbb{D}}$ of $\mathbb{D}$-presentable objects is essentially small.*

Definition 9 A locally small category $\mathcal{K}$ is *locally* $\mathbb{D}$-*presentable* if it is $\mathbb{D}$-accessible and cocomplete (equivalently, if it is $\mathbb{D}$-accessible and complete).

Observe that, by Lemma 4, if $\mathcal{K}$ is locally $\mathbb{D}$-presentable, then $\mathcal{K}_{\mathbb{D}}$ is $\mathbb{D}^{op}$-cocomplete.

If $\mathcal{A}$ is a small and $\mathbb{D}$-complete category, we write $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ for the full subcategory of the functor category $[\mathcal{A}, \mathbf{Set}]$ given by $\mathbb{D}$-continuous functors. By general reasons (see for example [8]), the full inclusion

$$i_{\mathcal{A}}\colon \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}] \longrightarrow [\mathcal{A}, \mathbf{Set}]$$

has a left adjoint, which we denote by $r_{\mathcal{A}}$.

Theorem 10

1. *Let $\mathcal{A}$ be a small and $\mathbb{D}$-complete category. Then the category $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ is locally $\mathbb{D}$-presentable.*

2. *Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category. Then $\mathcal{K}$ is equivalent to $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{K}_{\mathbb{D}}^{op}, \mathbf{Set}]$.*

3. The duality theorem

Let $\mathbb{D}$ be a sound doctrine, we denote by $\mathbb{D}$-**LP** the 2-category of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations. Whereas, write $\mathbb{D}$-**Th** for the 2-category of essentially small $\mathbb{D}$-complete and Cauchy complete categories, $\mathbb{D}$-continuous functors and natural transformations. Recall that a category $\mathcal{C}$ is Cauchy complete if all idempotents of $\mathcal{C}$ split (see [4]).

Theorem 11 *There is a biequivalence of 2-categories*

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

Proof. Let $\mathcal{A}$ be in $\mathbb{D}$-**Th**. We put $\varepsilon(\mathcal{A}) = \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$, which is locally $\mathbb{D}$-presentable by part 1 of Theorem 10. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism in $\mathbb{D}$-**Th**. To define $\varepsilon(F)$, consider the following diagram

$$\begin{tikzcd}
\mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}] \arrow[r, shift right, "i_B"'] \arrow[d, dashed, "\varepsilon(F)"'] & {[\mathcal{B}, \mathbf{Set}]} \arrow[l, shift right, "r_B"'] \arrow[d, shift left, "- \cdot F"] \\
\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}] \arrow[r, "i_A"'] & {[\mathcal{A}, \mathbf{Set}]} \arrow[u, shift left, "T"]
\end{tikzcd} \tag{1}$$

where T is the left Kan extension of $Y_{\mathcal{B}} \cdot F^{op}\colon \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op} \longrightarrow [\mathcal{B}, \mathbf{Set}]$ along the Yoneda embedding $Y_{\mathcal{A}}\colon \mathcal{A}^{op} \longrightarrow [\mathcal{A}, \mathbf{Set}]$. Since F is $\mathbb{D}$-continuous, composing with F on the right actually restricts to a functor $\varepsilon(F)\colon \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$. We wish to show that $\varepsilon(F)$ is in fact a morphism of $\mathbb{D}$-**LP**. For this, we need two lemmas. The second one is part of Proposition 3.3 in [1], which states that $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ is the free completion of $\mathcal{A}^{op}$ under $\mathbb{D}$-filtered colimits.

Lemma 12 *Consider the following diagram of categories and functors:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift right, "i"'] \arrow[d, "h"'] & \mathcal{B} \arrow[l, shift right, "r"'] \arrow[d, shift left, "g"] \\
\mathcal{C} \arrow[r, "j"'] & \mathcal{D} \arrow[u, shift left, "f"]
\end{tikzcd}$$

Suppose: j full and faithful, $g \cdot i \simeq j \cdot h$, $r \dashv i$ and $f \dashv g$. Then, the composite $r \cdot f \cdot j$ satisfies $r \cdot f \cdot j \dashv h$.

Lemma 13 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. The full embedding $i_{\mathcal{A}} : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}] \longrightarrow [\mathcal{A}, \mathbf{Set}]$ preserves and reflects $\mathbb{D}$-filtered colimits.*

Since T is left adjoint to $- \cdot F\colon [\mathcal{B}, \mathbf{Set}] \longrightarrow [\mathcal{A}, \mathbf{Set}]$, we can apply Lemma 12 to diagram (1) and we get a left adjoint to $\varepsilon(F)\colon \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$. Now, observe that $- \cdot F\colon [\mathcal{B}, \mathbf{Set}] \longrightarrow [\mathcal{A}, \mathbf{Set}]$ preserves all colimits (because thay are computed pointwise in **Set**) and then, by Lemma 13, $i_{\mathcal{A}} \cdot \varepsilon(F) \simeq (- \cdot F) \cdot i_{\mathcal{B}}$ preserves $\mathbb{D}$-filtered colimits. Finally, since $i_{\mathcal{A}}$ reflects $\mathbb{D}$-filtered colimits (Lemma 13 again), $\varepsilon(F)$ preserves them.
This complete the definition of

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

on objects and morphisms. It is defined in the obvious (covariant) way on 2-cells and it is clearly a 2-functor.
Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category and consider its full subcategory $\mathcal{K}_{\mathbb{D}}$ of $\mathbb{D}$-presentable objects. To prove that ε is surjective on objects up to equivalence, we apply part 2 of Theorem 10 and we have only to check that $\mathcal{K}_{\mathbb{D}}^{op}$ is Cauchy complete. For later use, we state a slightly more general fact.

Lemma 14 *Consider two functors $F, G\colon \mathcal{K} \longrightarrow \mathbf{Set}$, with $\mathcal{K}$ a locally small category with $\mathbb{D}$-filtered colimits. If F is $\mathbb{D}$-accessible and G is a retract of F, then G is $\mathbb{D}$-accessible.*

This lemma implies that $\mathcal{K}_{\mathbb{D}}$ is closed in $\mathcal{K}$ under retracts, so that it is Cauchy complete (and then $\mathcal{K}_{\mathbb{D}}^{op}$ is Cauchy complete too).
It remains to prove that

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

is locally an equivalence, that is, for $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{D}$-$\mathbf{Th}$, the functor

$$\varepsilon_{\mathcal{A},\mathcal{B}}\colon \mathbb{D}\text{-}\mathbf{Th}(\mathcal{A},\mathcal{B}) \longrightarrow \mathbb{D}\text{-}\mathbf{LP}(\varepsilon(\mathcal{B}),\varepsilon(\mathcal{A}))$$

is an equivalence of categories.
Proving that $\varepsilon_{\mathcal{A},\mathcal{B}}$ is full and faithful is routine calculation based on Yoneda Lemma, which can be used because the Yoneda embedding $\mathcal{A}^{op} \longrightarrow [\mathcal{A}, \mathbf{Set}]$ clearly factors through $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$.
We prove now that $\varepsilon_{\mathcal{A},\mathcal{B}}$ is essentially surjective. For this, consider a morphism $R\colon \varepsilon(\mathcal{B}) \longrightarrow \varepsilon(\mathcal{A})$ in $\mathbb{D}$-$\mathbf{LP}$, together with its left adjoint L and the corestricted Yoneda embeddings as in the following diagram

$$\begin{tikzcd}
\mathcal{B}^{op} \arrow[r, tail, "Y_{\mathcal{B}}"] & \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}] \arrow[d, shift left=2, "R"] \arrow[d, phantom, "\dashv"] \\
\mathcal{A}^{op} \arrow[u, dashed, "L_{|}"] \arrow[r, tail, "Y_{\mathcal{A}}"'] & \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}] \arrow[u, shift left=2, "L"]
\end{tikzcd}$$

We claim that L restricts to a functor $\mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$. For this, we use the following lemmas. The first one can be deduced from Lemma 3.9 in [1]. We state it explicitly for sake of clearness.

Lemma 15 *Let $\mathcal{A}$ be a locally small and $\mathbb{D}$-complete category.*

1. *Any representable functor is a $\mathbb{D}$-presentable object in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$.*
2. *Any $\mathbb{D}$-presentable object of $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ is a retract of a representable functor.*

Lemma 16 *Let $\mathcal{K}$ and $\mathcal{K}'$ be locally small categories with $\mathbb{D}$-filtered colimits, and consider two functors $R\colon \mathcal{K} \longrightarrow \mathcal{K}'$, $L\colon \mathcal{K}' \longrightarrow \mathcal{K}$ with $L \dashv R$. If R is $\mathbb{D}$-accessible, then L preserves $\mathbb{D}$-presentable objects.*

Consider now an object A in $\mathcal{A}$. By the first part of Lemma 15 and by Lemma 16, $L(\mathcal{A}(A, -))$ is $\mathbb{D}$-presentable. Then, by the second part of Lemma 15, $L(\mathcal{A}(A, -))$ is a retract of a representable functor. Since $\mathcal{B}$ is a Cauchy complete category, $L(\mathcal{A}(A, -))$ is itself representable. It follows then that for every $A \in \mathcal{A}$ we can choose an object $B := FA$ in $\mathcal{B}$ such that $\mathcal{B}(B, -) \simeq L(\mathcal{A}(A, -))$. Let us fix a natural isomorphism

$l_A : L(\mathcal{A}(A,-)) \longrightarrow \mathcal{B}(FA,-)$ for all $A \in \mathcal{A}$. For a morphism $f : A \longrightarrow A'$ in $\mathcal{A}$, the Yoneda Lemma guarantees that there exists a unique map $Ff : FA \longrightarrow FA'$ for which the natural transformation

$$l_A \cdot L(\mathcal{A}(f,-)) \cdot l_{A'}^{-1} : \mathcal{B}(FA',-) \longrightarrow \mathcal{B}(FA,-)$$

is given by composition with Ff. In this way, we get a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, which will show to be $\mathbb{D}$-continuous by making use of the following lemma.

Lemma 17 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. The restricted Yoneda embedding*

$$Y_{\mathcal{A}}\colon \mathcal{A}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$$

preserves and reflects $\mathbb{D}^{op}$-colimits.

Since L is a left adjoint and, by Lemma 17, $Y_{\mathcal{A}}$ preserves $\mathbb{D}^{op}$-colimits, the composite $LY_{\mathcal{A}}$ preserves $\mathbb{D}^{op}$-colimits. Thus $Y_{\mathcal{B}}F^{op}$ preserves $\mathbb{D}^{op}$-colimits and, by Lemma 17 again, $Y_{\mathcal{B}}$ reflects them. So F^{op} also preserves them: that is, F preserves $\mathbb{D}$-limits.
Finally, it only remains to prove that the functor $\varepsilon(F)\colon \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ is isomorphic to R. Since $L \dashv R$, it is sufficient to prove that $\varepsilon(F)$ is right adjoint to L, i.e., that there is an isomorphism $\mathbf{Nat}(L(M), G) \simeq \mathbf{Nat}(M, G \cdot F)$ natural in the variables $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ and $G \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{Set}]$. First, remark that for a representable functor $M = \mathcal{A}(A,-)$, where $A \in \mathcal{A}$, this is just the Yoneda Lemma. For an arbitrary $\mathbb{D}$-continuous functor, one just use this fact together Lemma 6 (or the fact that $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ is the free completion of $\mathcal{A}^{op}$ under $\mathbb{D}$-filtered colimits). The proof of Theorem 11 is now complete. ∎

Example 18

1. If we take as $\mathbb{D}$ the doctrine of finite limits, then Theorem 11 is exactly Gabriel-Ulmer duality.

2. If we take as $\mathbb{D}$ the doctrine of finite products, then Theorem 11 is exactly Adamek-Lawvere-Rosicky duality between algebraic theories and multisorted finitary varieties.

3. If we take as $\mathbb{D}$ the empty doctrine, we have a duality between the 2-category of small Cauchy complete categories, functors and natural transformations, and the 2-category of categories of the form $[\mathcal{A}, \mathbf{Set}]$ for $\mathcal{A}$ small, cocontinuous right adjoint functors and natural transformations. This duality refines the well-known fact that two small categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, that is $[\mathcal{A}, \mathbf{Set}] \simeq [\mathcal{B}, \mathbf{Set}]$, if and only if they are Cauchy equivalent, that is they have equivalent Cauchy completions. (A Cauchy completion of a small category $\mathcal{A}$ is given by the full subcategory of $[\mathcal{A}, \mathbf{Set}]$ of retracts of representable functors, see [4].)

4. Proof of some lemmas

Proof of Lemma 12: It follows from the following bijections natural in $C \in \mathcal{C}$ and $A \in \mathcal{A}$.

$$\mathcal{A}(r \cdot f \cdot j(C), A) \simeq \mathcal{B}(f \cdot j(C), i(A)) \simeq \mathcal{D}(j(C), g \cdot i(A)) \simeq \mathcal{D}(j(C), j \cdot h(A)) \simeq \mathcal{C}(C, h(A))$$

■

Proof of Lemma 14: Consider

$$\begin{tikzcd} G \arrow[r, shift right, "\alpha"'] & F \arrow[l, two heads, shift right, "\beta"'] \end{tikzcd}$$

where $\beta \cdot \alpha = 1_G$. We wish to show that $G : \mathcal{K} \longrightarrow \mathbf{Set}$ preserves $\mathbb{D}$-filtered colimits. For it, take $\mathcal{J}$ a $\mathbb{D}$-filtered category and a functor $H : \mathcal{J} \longrightarrow \mathcal{K}$. For any $J \in \mathcal{J}$ applying G to the colimit injection $\sigma_J : H(J) \longrightarrow \mathrm{colim}_{J\in\mathcal{J}} H(J)$ gives an arrow $G(\sigma_J) : G(H(J)) \longrightarrow G(\mathrm{colim}_{J\in\mathcal{J}} H(J))$. In this way, we get a unique comparison $\varphi : \mathrm{colim}_{J\in\mathcal{J}} G(H(J)) \longrightarrow G(\mathrm{colim}_{J\in\mathcal{J}} H(J))$ such that $\varphi \cdot \delta_J = G(\sigma_J)$, where δ_J are the colimit injections corresponding to the diagram $G \cdot H$. We will show that φ is an isomorphism. The natural transformations α and β not only induce arrows, for any $J \in \mathcal{J}$, as in the following diagram

$$\begin{tikzcd} \mathrm{colim}_{J\in\mathcal{J}} G(H(J)) \arrow[r, shift right, "\hat{\alpha}"'] & \mathrm{colim}_{J\in\mathcal{J}} F(H(J)) \arrow[l, shift right, "\hat{\beta}"'] \\ G(H(J)) \arrow[u, "\delta_J"] \arrow[r, shift right, "\alpha_{H(J)}"'] & F(H(J)) \arrow[u, "\gamma_J"'] \arrow[l, two heads, shift right, "\beta_{H(J)}"'] \end{tikzcd}$$

which gives, by the universal property of the colimits involved, two morphisms $\hat{\alpha}$ and $\hat{\beta}$ making the correspondent squares commute, but also, for any $X \in \mathcal{K}$ they provide $G(X)$ as retract of $F(X)$, since $\beta_X \cdot \alpha_X = (\beta \cdot \alpha)_X = (1_G)_X = 1_{G(X)}$. Consider then the diagram

$$\begin{tikzcd} G(\mathrm{colim}_{J\in\mathcal{J}} H(J)) & \mathrm{colim}_{J\in\mathcal{J}} G(H(J)) \arrow[l, "\varphi"'] \\ F(\mathrm{colim}_{J\in\mathcal{J}} H(J)) \arrow[u, "\beta_{\mathrm{colim}_{J\in\mathcal{J}} H(J)}"] & \mathrm{colim}_{J\in\mathcal{J}} F(H(J)) \arrow[l, "\simeq"] \arrow[u, "\hat{\beta}"'] \\ G(\mathrm{colim}_{J\in\mathcal{J}} H(J)) \arrow[u, "\alpha_{\mathrm{colim}_{J\in\mathcal{J}} H(J)}"] & \mathrm{colim}_{J\in\mathcal{J}} G(H(J)) \arrow[l, "\varphi"] \arrow[u, "\hat{\alpha}"'] \end{tikzcd}$$

where we know that $F(\mathrm{colim}_{J\in\mathcal{J}} H(J)) \simeq \mathrm{colim}_{J\in\mathcal{J}} F(H(J))$, since F is $\mathbb{D}$-accessible. To exhibit $\hat{\beta} \cdot \alpha_{\mathrm{colim}_{J\in\mathcal{J}} H(J)}$ as the inverse of φ, it is sufficient to show that both squares commute. Commutativity of the bottom square is evident by naturality of α once we compose with the colimit injections δ_J, as well as commutativity of the upper square derives from naturality of β by composing with the colimit injections γ_J. ■

Proof of Lemma 16: Let $M \in \mathcal{K}'$ be $\mathbb{D}$-presentable. To prove that $\mathcal{K}(L(M), -) : \mathcal{K} \longrightarrow \mathbf{Set}$ preserves $\mathbb{D}$-filtered colimits, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a functor $H : \mathcal{J} \longrightarrow \mathcal{K}$. We get:

$$\begin{aligned}
\mathcal{K}(L(M), \operatorname*{colim}_{J\in\mathcal{J}} H(J)) &\simeq \mathcal{K}'(M, R(\operatorname*{colim}_{J\in\mathcal{J}} H(J))) \\
&\simeq \mathcal{K}'(M, \operatorname*{colim}_{J\in\mathcal{J}} R\cdot H(J)) \\
&\simeq \operatorname*{colim}_{J\in\mathcal{J}} \mathcal{K}'(M, R\cdot H(J)) \\
&\simeq \operatorname*{colim}_{J\in\mathcal{J}} \mathcal{K}(L(M), H(J))
\end{aligned}$$

(where the second isomorphism follows from R being $\mathbb{D}$-accessible and the third one from M being $\mathbb{D}$-presentable). ■

Proof of Lemma 17: (Note that, when $\mathcal{A}$ is Cauchy complete, this is Lemma 4.) For any $\mathcal{D} \in \mathbb{D}$ and any functor $S : \mathcal{D} \longrightarrow \mathcal{A}$, consider its $\mathbb{D}$-limit cone $(\lim_{D\in\mathcal{D}} S(D); \lambda_D : \lim_{D\in\mathcal{D}} S(D) \longrightarrow S(D))$ in $\mathcal{A}$. We wish to show that applying Yoneda to this diagram produces a $\mathbb{D}^{op}$-colimit cone in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$, namely that

$$(\mathcal{A}(\lim_{D\in\mathcal{D}} S(D), -); \mathcal{A}(\lambda_D, -) : \mathcal{A}(S(D), -) \longrightarrow \mathcal{A}(\lim_{D\in\mathcal{D}} S(D), -))$$

is the $\mathbb{D}^{op}$-colimit of the diagram $Y \cdot S$. This is obviously a cocone in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$. To show that it actually coincides with $\operatorname{colim}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D), -)$ (which exists, $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ being cocomplete) we take any object $E \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$ and just use the Yoneda Lemma, once we have remarked that the functor E is in fact $\mathbb{D}$-continuous:

$$\begin{aligned}
\mathbf{Nat}(\mathcal{A}(\lim_{D\in\mathcal{D}} S(D), -), E) &\simeq E(\lim_{D\in\mathcal{D}} S(D)) \\
&\simeq \lim_{D\in\mathcal{D}} E(S(D)) \\
&\simeq \lim_{D\in\mathcal{D}} \mathbf{Nat}(\mathcal{A}(S(D), -), E) \\
&\simeq \mathbf{Nat}(\operatorname*{colim}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D), -), E)
\end{aligned}$$

Since this is valid for any $E \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{Set}]$, again from Yoneda Lemma we can derive that $\mathcal{A}(\lim_{D\in\mathcal{D}} S(D), -) \simeq \operatorname{colim}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D), -)$, as desired. ■

Remark 19 To end this section, let us point out a simple consequence of Lemmas 13 and 16. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category and consider a reflective subcategory $i : \mathcal{K}' \longrightarrow \mathcal{K}$. If the functor i is $\mathbb{D}$-accessible, then $\mathcal{K}'$ is locally $\mathbb{D}$-presentable. When $\mathbb{D}$ is such that $[\mathcal{A}, \mathbf{Set}]$ is locally $\mathbb{D}$-presentable for any $\mathcal{A}$ in $\mathbb{D}$-$\mathbf{Th}$, then locally $\mathbb{D}$-presentable categories are precisely the $\mathbb{D}$-accessible reflections of presheaf categories. This is the case when $\mathbb{D}$ is as in Example 18. (B. Mesablishvili pointed out to us that, when $\mathbb{D}$ is the empty doctrine, the previous remark can be generalized to the enriched context, replacing **Set** by any complete and cocomplete symmetric monoidal closed category. One has just to observe that $\mathcal{K}'$ is monadic over $\mathcal{K}$ and then use Theorem 3.12 in [6].)

5. A simple application

To end this note, we sketch a simple application of the duality theorem. In [9, 10], Pedicchio and Wood establish the following precise comparison between the 2-category **VAR** of varieties and the 2-category **LFP** of locally finitely presentable categories.

1. The (not full) inclusion of **VAR** into **LFP** has a left biadjoint;
2. A small category with finite limits $\mathcal{C}$ is such that $\mathbf{Lex}[\mathcal{C}, \mathbf{Set}]$ is equivalent to a variety if and only if every object of $\mathcal{C}$ is a regular subobject of an *effective injective* (an object $E \in \mathcal{C}$ is effective injective if $\mathcal{C}(-, E) \colon \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ preserves coequalizers of reflexive graphs).

Here is the general setting: consider two sound doctrines $\mathbb{D}_1 \hookrightarrow \mathbb{D}_2$, so that there are a not full inclusion $\mathbb{D}_2\text{-}\mathbf{Th} \longrightarrow \mathbb{D}_1\text{-}\mathbf{Th}$ and a corresponding 2-functor $j \colon \mathbb{D}_2\text{-}\mathbf{LP} \longrightarrow \mathbb{D}_1\text{-}\mathbf{LP}$ defined by $j(\mathbb{D}_2\text{-}\mathbf{cont}[\mathcal{C}, \mathbf{Set}]) = \mathbb{D}_1\text{-}\mathbf{cont}[\mathcal{C}, \mathbf{Set}]$. If every $\mathcal{A} \in \mathbb{D}_1\text{-}\mathbf{Th}$ has a $\mathbb{D}_1$-conservative $\mathbb{D}_2$-completion, that is if there is a $\mathbb{D}_1\text{-}\mathbf{Th}$ morphism $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$ with $\overline{\mathcal{A}} \in \mathbb{D}_2\text{-}\mathbf{Th}$ universal w.r.t. $\mathbb{D}_2$-complete categories (see [11]), then $\mathbb{D}_2\text{-}\mathbf{Th} \longrightarrow \mathbb{D}_1\text{-}\mathbf{Th}$ has a left biadjoint. Via the duality, we have:

1. $j \colon \mathbb{D}_2\text{-}\mathbf{LP} \longrightarrow \mathbb{D}_1\text{-}\mathbf{LP}$ has a right biadjoint j^* which is an “inclusion” in the sense that $j^*(\mathcal{K}) \simeq \mathcal{K}$ for any $\mathcal{K} \in \mathbb{D}_1\text{-}\mathbf{LP}$;
2. $\mathcal{C} \in \mathbb{D}_2\text{-}\mathbf{Th}$ is such that $\mathbb{D}_2\text{-}\mathbf{cont}[\mathcal{C}, \mathbf{Set}] \in \mathbb{D}_1\text{-}\mathbf{LP}$ if and only if $\mathcal{C}$ is of the form $\overline{\mathcal{A}}$ for some $\mathcal{A} \in \mathbb{D}_1\text{-}\mathbf{Th}$.

The case studied in [9, 10] corresponds to choosing as $\mathbb{D}_1$ the doctrine of finite products and as $\mathbb{D}_2$ the doctrine of finite limits. The completion is then the completion under equalizers of coreflexive graphs and the interesting result of [9] is the characterization of those finitely complete categories which are completion of categories with finite products.

The other relevant (even if much more easy) example is when $\mathbb{D}_1$ is the empty doctrine and $\mathbb{D}_2$ is the doctrine of finite products, so that $\mathbb{D}_1\text{-}\mathbf{LP}$ is the 2-category **PS** of presheaf categories and $\mathbb{D}_2\text{-}\mathbf{LP}$ is **VAR**. The next lemma provides the context for this example. We state it in terms of the more popular dual completion: we write $Fam_f\mathcal{A}$ for the category of finite families of objects of $\mathcal{A}$, which is the completion of $\mathcal{A}$ under finite coproducts. Part 2 of the lemma is simply the finitary version of Proposition 6.1.5 in [5]. We call an object *finitely connected* if the associated representable functor preserves finite coproducts.

Lemma 20

1. *If $\mathcal{A}$ is small and Cauchy complete, then $Fam_f\mathcal{A}$ is small and Cauchy complete. Conversely, if $\mathcal{C}$ has finite coproducts and is Cauchy complete, then the full subcategory of finitely connected objects is Cauchy complete.*

2. *A category $\mathcal{C}$ with finite coproducts is of the form $Fam_f\mathcal{A}$ if and only if each object is a finite coproduct of finitely connected objects; when this is the case, one can choose as $\mathcal{A}$ the full subcategory of finitely connected objects.*

Proof of 1: One implication can be proved using the same arguments as in the proof of Lemma 14. Conversely, assume that $\mathcal{A}$ is small and Cauchy complete. An object in $Fam_f\mathcal{A}$ is a pair (I,f) with I a finite set and $f\colon I \longrightarrow \mathcal{A}$ a functor; an arrow $(a,\alpha)\colon (I,f) \longrightarrow (J,g)$ is given by a functor $a\colon I \longrightarrow J$ and a natural transformation $\alpha\colon f \Rightarrow g \cdot a$. Clearly, $Fam_f\mathcal{A}$ is small because its objects are in bijection with $\coprod_I (Ob\mathcal{A})^I$ for I varying in the category of finite sets, which is (essentially) small. Consider now an idempotent $(a,\alpha)\colon (I,f) \longrightarrow (I,f)$. The condition $(a,\alpha)\cdot(a,\alpha) = (a,\alpha)$ means

$$\begin{cases} a \cdot a = a & \text{that is } a \text{ is an idempotent in } \mathbf{Set} \\ \alpha_{a(i)} \cdot \alpha_i = \alpha_i & \forall i \in I \end{cases} \tag{2}$$

We can consider the splitting of the idempotent a in **Set**, which is given by the following equalizer

$$\begin{tikzcd} J \arrow[rr, "c"] & & I \arrow[r, shift left, "a"] \arrow[r, shift right, "1_I"'] & I \\ & I \arrow[ul, "b"] \arrow[ur, "a"'] & & \end{tikzcd}$$

so that $J = \{j \in I \mid a(j) = j\}, c \cdot b = a, b \cdot c = 1_J$. Now, for each $j \in J$, the second equation in (2) says that $\alpha_j\colon f(j) \longrightarrow f(j)$ is an idempotent in $\mathcal{A}$, and then it splits. Let us consider its splitting

$$\begin{tikzcd} X_j \arrow[rr, "s_j"] & & f(j) \arrow[r, shift left, "\alpha_j"] \arrow[r, shift right, "1_{f(j)}"'] & f(j) \\ & f(j) \arrow[ul, "t_j"] \arrow[ur, "\alpha_j"'] & & \end{tikzcd}$$

that is $s_j \cdot t_j = \alpha_j$ and $t_j \cdot s_j = 1_{X_j}$. In this way we get an object $(J, g\colon J \longrightarrow \mathcal{A}\colon j \mapsto X_j)$ in $Fam_f\mathcal{A}$ and two arrows $(b,\beta)\colon (I,f) \longrightarrow (J,g)$ and $(c,\gamma)\colon (J,g) \longrightarrow (I,f)$ defined by

- $b\colon I \longrightarrow J \quad b(i) = a(i)\ ,\quad \beta_i = t_{a(i)} \cdot \alpha_i$

- $c\colon J \longrightarrow I \quad c(j) = j\ ,\quad \gamma_j = s_j$

and one can check that $(c,\gamma)\cdot(b,\beta) = (a,\alpha), (b,\beta)\cdot(c,\gamma) = 1_{(J,g)}$. ■

Accordingly with the general setting, we have:

1. the inclusion $\mathbf{PS} \longrightarrow \mathbf{VAR}$ is right biadjoint to $j: \mathbf{VAR} \longrightarrow \mathbf{PS}$;

2. An algebraic theory $\mathcal{C}$ is such that the corresponding variety is equivalent to a presheaf category if and only if each object of $\mathcal{C}$ is a finite product of "finitely coconnected" objects (that is, objects X such that $\mathcal{C}(-,X): \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ preserves finite coproducts).

Putting together the **PS** - **VAR** comparison and the **VAR** - **LFP** comparison, we have:

Corollary 21 *Let $\mathcal{C}$ be a finitely complete small category. The following conditions are equivalent:*

1. $\mathbf{Lex}[\mathcal{C}, \mathbf{Set}]$ *is equivalent to a presheaf category;*

2. $\mathcal{C}$ *is equivalent to the free completion under finite limits of a small category;*

3. *Any object of $\mathcal{C}$ is a regular subobject of a finite product $\prod_I X_i$ such that, for all $i \in I$, the functor $\mathcal{C}(-,X_i): \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ preserves finite colimits.*

Proof: Once again we work with the dual of the theory. We write $\mathcal{B} \longrightarrow \mathcal{B}_{rc}$ for the completion under coequalizers of reflexive graphs of a category $\mathcal{B}$ with finite coproducts. Because of the general setting stated above, we have to prove that:

1'. $\mathcal{C} \simeq (Fam_f \mathcal{A})_{rc}$ for some small Cauchy complete category $\mathcal{A}$

if and only if (the dual of) condition 3 holds. (Indeed, the implication $2 \Rightarrow 1$ is obvious and the implication $1' \Rightarrow 2$ is proved in [9].)

$1' \Rightarrow 3$: Note that a functor defined on a finitely cocomplete category preserves finite colimits if it preserves coequalizers of reflexive graphs and finite coproducts. Each object of $\mathcal{C}$ is a regular quotient of a finite coproduct of objects in $\mathcal{A}$. Each object of $\mathcal{A}$ is finitely connected in $\mathcal{B} = Fam_f \mathcal{A}$ and it is effective projective in $\mathcal{C}$, so that we have only to prove that $\mathcal{B} \longrightarrow \mathcal{B}_{rc}$ preserves finitely connected objects. But this follows, via the commutativity of finite coproducts with coequalizers of reflexive graphs, from the fact that each object of $\mathcal{B}_{rc}$ is the coequalizer of a reflexive graph in $\mathcal{B}$.

$3 \Rightarrow 1'$: Let us write $\mathcal{A}$ for the full subcategory of $\mathcal{C}$ of the objects A such that $\mathcal{C}(A,-): \mathcal{C} \longrightarrow \mathbf{Set}$ preserves finite colimits, $\mathcal{B}$ for the full subcategory spanned by finite coproducts of objects of $\mathcal{A}$, and $\mathcal{E}$ for the full subcategory of effective projectives. Since each $A \in \mathcal{A}$ is effective projective and effective projectives are closed under finite coproducts, we have that $\mathcal{B}$ is contained in $\mathcal{E}$ and $\mathcal{C}$ has enough effective projectives, so that $\mathcal{C} \simeq \mathcal{E}_{rc}$. Moreover, since $\mathcal{C}$ has finite colimits, each object of $\mathcal{E}$ is regular projective and then it is a retract of an object of $\mathcal{B}$. By Lemma 20, $\mathcal{B}$ is equivalent to $Fam_f \mathcal{A}$ and then it is Cauchy complete because $\mathcal{A}$ is Cauchy complete (same argument as in the proof of Lemma 14). Finally, $\mathcal{E} \simeq \mathcal{B}$ and $\mathcal{C} \simeq (Fam_f \mathcal{A})_{rc}$. ∎

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