EXTERNAL DERIVATIONS OF INTERNAL GROUPOIDs
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ABSTRACT. If $H$ is a $G$-crossed module, the set of derivations of $G$ in $H$ is a monoid under Whitehead product of derivations. We interpret Whitehead product using the correspondence between crossed modules and internal groupoids in the category of groups. Working in the general context of internal groupoids in a finitely complete category, we relate derivations to holomorphisms, translations, affine transformations, and to the embedding category of a groupoid.

1. Introduction

Let $G$ be a group and $\varphi: G \rightarrow \text{Aut} H$ a $G$-group. A derivation of $G$ in $H$ is a map $d: G \rightarrow H$ such that $d(xy) = d(x) + x \cdot d(y)$. If $H$ is a $G$-module, i.e. if $H$ is abelian, the set $\text{Der}(G, H)$ of derivations is an abelian group w.r.t. the point-wise sum. If $H$ is not abelian, in general $\text{Der}(G, H)$ is just a pointed set (the zero-morphism $0: G \rightarrow H$ is a derivation). Whitehead [25] discovered the following fact.

1.1. Theorem. Let \((H \xrightarrow{\partial} G \xrightarrow{\varphi} \text{Aut} H)\) be a crossed module of groups. The set $\text{Der}(G, H)$ is a monoid w.r.t. $(d_1 + d_2)(x) = d_1(\partial(d_2(x))x) + d_2(x)$.

The aim of this note is to understand in a more conceptual way Whitehead product of derivations. The idea is to replace crossed modules of groups by the equivalent notion of internal groupoids in the category of groups. Using the language of internal groupoids, Whitehead product becomes clear: it is nothing but the composition in the internal category. The surprise is that, once expressed in terms of internal groupoids, Whitehead theorem, as well as some other basic properties of derivations (notably, the characterization of regular derivations and the left exactness of $\text{Der}(G, H)$ as a functor of the second variable), has nothing to do with groups, but holds in the very general context of internal groupoids in an arbitrary category $\mathcal{G}$ with finite limits [2, 7]. Just to quote some categories where internal crossed modules and internal groupoids are intensively studied, let us mention the category of Lie algebras [5] (in fact, Lie algebras as well as groups constitute semi-abelian categories [1, 14], and for semi-abelian categories, internal crossed modules and internal groupoids coincide [13]), the category of topological spaces and continuous maps [8, 11, 18], the category of topological spaces and local homeomorphisms (whose internal groupoids are called étale groupoids) [22], the category of smooth manifolds (whose internal groupoids are called Lie groupoids) [17, 18, 22], and of course the category of sets, which gives ordinary groupoids [3, 12, 24].

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In his paper [9], Gilbert explains Whitehead product of derivations replacing crossed modules by the equivalent notion of groups in the category of groupoids, whereas we use groupoids in the category of groups. Even if the equivalence between groups in groupoids and groupoids in groups is a trivial fact, the advantage of working with groupoids in groups is that this immediately suggests the more general context of internal groupoids in any finitely complete category. This gain of generality allows us, for example, to include in the same theory the holomorph of a group: this is possible because a group is a particular groupoid in sets. In fact, it is precisely this easy example the guiding example to describe derivations using holomorphisms and translations as in Sections 5 and 6. Moreover, it is a fact that several definitions, constructions and proofs become more transparent having in mind the set-theoretical case instead of the group-theoretical case. Finally, since internal groupoids are the objects of a 2-category, we can exploit some general 2-categorical facts to define derivations, translations, and the category of embeddings of a groupoids.

Warning: the composition of \( f: x \to y \) and \( g: y \to z \) is written \( f \cdot g \).

2. The monoid of derivations

In this section, we construct the monoid of \( C \)-derivations, for \( C \) an internal groupoid.

We fix, once for all, a category \( \mathcal{G} \) with finite limits. The notation for an internal groupoid \( C \) in \( \mathcal{G} \) is

\[
C = \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C_0 \xleftarrow{\text{dom}} \xrightarrow{\text{cod}} C_1 \xleftarrow{\circ} C_1 \times_{C_0} C_1, \ C_1 \xrightarrow{(-)^{-1}} C_1
\end{array}
\end{array}
\end{array}\right)
\]

where \( C_1 \times_{C_0} C_1 \) is the object of “composable pairs”, that is

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi_1 \downarrow \quad \pi_2 \downarrow \\
\pi \downarrow \quad \text{dom} \downarrow \quad \text{cod} \downarrow \\
C_1 \xrightarrow{\circ} C_1 \xleftarrow{\circ} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1
\end{array}
\end{array}
\end{array}
\]

is a pullback in \( \mathcal{G} \). We also write \( \circ^2: C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1 \) for the diagonal of the commutative square

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ \times 1 \downarrow \quad 1 \times \circ \downarrow \\
\circ \downarrow \quad \circ \downarrow \\
C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1
\end{array}
\end{array}
\end{array}
\]

where

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi_1 \downarrow \quad \pi_2 \downarrow \quad \pi_3 \downarrow \\
\pi_1 \downarrow \quad \pi_2 \downarrow \quad \pi_3 \downarrow \\
C_1 \xleftarrow{\circ} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{dom} \downarrow \quad \text{cod} \downarrow \quad \text{dom} \downarrow \\
\text{dom} \downarrow \quad \text{cod} \downarrow \quad \text{dom} \downarrow \\
C_1 \xrightarrow{\circ} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1
\end{array}
\end{array}
\end{array}
\]
is a limit in $\mathcal{G}$.

We denote by $\text{Grpd}(\mathcal{G})$ the 2-category of internal groupoids, internal functors and internal natural transformations (which always are natural isomorphisms). For $\mathcal{C}, \mathcal{B}$ internal groupoids, we denote by $\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{B})$ the corresponding hom-category (which is a groupoid), and we write

$$\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{B})$$

for its sets of arrows and of objects, together with the domain and the codomain maps. In particular, $\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})$ is a strict monoidal groupoid: tensor product is composition of internal functors and horizontal composition of internal natural transformations, the unit object is the identity functor on $\mathcal{C}$. As with any strict monoidal category, the map

$$\text{cod}: [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_1 \to [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_0$$

is an homomorphism of monoids.

2.1. **Definition.** The monoid of $\mathcal{C}$-derivations is the kernel of the codomain map

$$\text{Der}_{\mathcal{C}} = \text{Ker}(\text{cod}) \to [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_1 \to [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_0$$

Explicitly, a $\mathcal{C}$-derivation is a pair $(D, d)$

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{D} & \mathcal{C} \\
\downarrow{d} & \downarrow{Id} & \\
\mathcal{C} & \xrightarrow{\text{dom}} & \mathcal{C}
\end{array}$$

with $D$ an internal functor and $d$ an internal natural transformation. Product of derivations is horizontal composition of natural transformations.

When $\mathcal{G}$ is the category of sets, to give a $\mathcal{C}$-derivation just means to choose, for each object $x$ of $\mathcal{C}$, an arrow

$$d(x): \text{dom}(d(x)) \to x$$

with codomain $x$. This suggests to describe derivations as sections of the codomain arrow.

2.2. **Proposition.** To give a $\mathcal{C}$-derivation amounts to give an arrow $d: C_0 \to C_1$ such that the diagram

$$\begin{array}{ccc}
C_0 & \xrightarrow{d} & C_1 \\
\downarrow{\text{cod}} & \downarrow{1} & \\
C_0 & \xrightarrow{1} & C_0
\end{array}$$

commutes.
Proof. Such an arrow $d$ given, we have to construct an internal functor $D: \mathbb{C} \to \mathbb{C}$ in such a way that $d$ becomes an internal natural transformation $d: D \Rightarrow Id$. The following picture explains the set-theoretical idea behind the construction of $D$.

![Diagram]

It suffices now to internalize this idea:

- On objects, the functor $D: \mathbb{C} \to \mathbb{C}$ is defined by
  $$D_0: \mathbb{C}_0 \xrightarrow{d} \mathbb{C}_1 \xrightarrow{\text{dom}} \mathbb{C}_0$$

- As far as arrows are concerned, we consider the diagram
  $$\begin{array}{ccc}
  \mathbb{C}_0 & \xrightarrow{\text{dom}} & \mathbb{C}_1 \\
  d & \downarrow & 1 \\
  \mathbb{C}_1 & \xleftarrow{\text{cod}} & \mathbb{C}_0 \\
  \end{array}$$
  $$\begin{array}{ccc}
  \mathbb{C}_0 & \xleftarrow{\text{dom}} & \mathbb{C}_1 \\
  \downarrow & & \downarrow \text{(1)} \text{-}^{-1} \\
  \mathbb{C}_1 & \xrightarrow{\text{cod}} & \mathbb{C}_0 \\
  \end{array}$$

By equation (1), this diagram commutes, and we get a unique factorization of the projective cone through the object of composable triples, say
  $$\overline{d} = \langle \text{dom} \cdot d, 1, \text{cod} \cdot d \cdot (\cdot)^{-1} \rangle: \mathbb{C}_1 \to \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1$$

Finally, the functor $D: \mathbb{C} \to \mathbb{C}$ is defined on arrows by
  $$\begin{array}{ccc}
  \mathbb{C}_1 & \xrightarrow{\overline{d}} & \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \\
  \odot & \downarrow & 2 \downarrow \text{cod} \\
  \mathbb{C}_1 & \xrightarrow{\text{dom}} & \mathbb{C}_0 \\
  \end{array}$$

We wish now to describe explicitly the operations in $Der\mathbb{C}$ using Proposition 2.2:

- The unit in $Der\mathbb{C}$ is $u: \mathbb{C}_0 \to \mathbb{C}_1$.

- The multiplication in $Der\mathbb{C}$ is the internal version of
  $$z \xrightarrow{d_1(y)} y = \text{dom}(d_2(x)) \xrightarrow{d_2(x)} x$$
  (In other words, Whitehead product of derivations is just the internal composition in $\mathbb{C}$.) This means that, given two derivations $d_1, d_2: \mathbb{C}_0 \to \mathbb{C}_1$, we start constructing the arrow
  $$d_1 \ast d_2 = \langle d_2 \cdot \text{dom} \cdot d_1, d_2 \rangle: \mathbb{C}_0 \to \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1$$
  and we get the product of $d_1$ and $d_2$ by composing internally
  $$d_1 \otimes d_2: \mathbb{C}_0 \xrightarrow{d_1 \ast d_2} \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \xrightarrow{\odot} \mathbb{C}_1$$
2.3. Example. When $\mathcal{G}$ is the category of groups, we recapture the classical notion of derivation. Indeed, it is well-known that to a crossed module of groups

$$\begin{align*}
(H & \xrightarrow{\partial} G \xrightarrow{\varphi} \text{Aut}H) \\
\end{align*}$$

we can associate an internal groupoid $\mathcal{C}$, with

$$C_0 = G, \quad C_1 = H \times_\varphi G, \quad m((a, x), (b, y)) = (a + b, y)$$

$$\text{cod}(a, x) = x, \quad \text{dom}(a, x) = \partial(a)x, \quad u(x) = (0, x)$$

(see [4, 13, 15]). Moreover, $\mathcal{C}$-derivations in the sense of Definition 2.1 are in bijection with derivations of $G$ in $H$: a morphism $d: C_0 \to C_1$ is a $\mathcal{C}$-derivation precisely when its second component is the identity on $C_0$ and its first component is a derivation of $G$ in $H$.

(Let us recall here also the converse construction, which is needed later. Given an internal groupoid $\mathcal{C}$ in groups, we get a crossed module with $G = C_0$ and $H = \text{Ker}(\text{cod})$; the map $\partial$ is the restriction of $\text{dom}$ to $H$, and the action of $G$ on $H$ is given by $x \cdot a = u(x) + a - u(x)$.)

3. The group of regular derivations

In this section we characterize the invertible (or regular) elements of the monoid $\text{Der}\mathcal{C}$.

From Definition 2.1, we get three morphisms of monoids:

- $U: \text{Der}\mathcal{C} \to [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_0 \quad (D, d) \mapsto (D: \mathcal{C} \to \mathcal{C})$

- $(\_)_0: \text{Der}\mathcal{C} \to \text{End}C_0 \quad (D, d) \mapsto (D_0: C_0 \to C_0)$

- $(\_)_1: \text{Der}\mathcal{C} \to \text{End}C_1 \quad (D, d) \mapsto (D_1: C_1 \to C_1)$

As with any morphism of monoids, these morphisms restrict to the groups of invertible elements:

$$\begin{align*}
\text{Der}\mathcal{C} & \xrightarrow{U} [\text{Grpd}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_0 & \text{Der}\mathcal{C} & \xrightarrow{(\_)_0} \text{End}C_0 & \text{Der}\mathcal{C} & \xrightarrow{(\_)_1} \text{End}C_1 \\
\text{Der}^*\mathcal{C} & \xrightarrow{\text{Aut}(\_)} [\text{Grpd}(\mathcal{G})^*(\mathcal{C}, \mathcal{C})]_0 & \text{Der}^*\mathcal{C} & \xrightarrow{\text{Aut}(\_)} \text{End}C_0 & \text{Der}^*\mathcal{C} & \xrightarrow{\text{Aut}(\_)} \text{End}C_1
\end{align*}$$

where $\text{Grpd}(\mathcal{G})^*$ is the sub-2-category of $\text{Grpd}(\mathcal{G})$ of those internal functors which are isomorphisms. In fact, more is true: the previous diagrams are pullbacks. This is a corollary of the following general fact.
3.1. **Lemma.** Let

\[
\begin{array}{c}
\mathbb{C} \\
\downarrow \alpha \\
\mathbb{B}
\end{array}
\xrightarrow{F}
\begin{array}{c}
\mathbb{C} \\
\downarrow \epsilon \\
\mathbb{B}
\end{array}
\]

be a 2-cell in \(\text{Grpd}(\mathcal{G})\). The following conditions are equivalent:

1. \(\alpha\) is invertible with respect to the horizontal composition;
2. \(F, G: \mathbb{C} \to \mathbb{B}\) are in \(\text{Grpd}(\mathcal{G})^*\);
3. \(F_1, G_1: C_1 \to B_1\) are isomorphisms in \(\mathcal{G}\);
4. \(F_1: C_1 \to B_1\) and \(G_0: C_0 \to B_0\) are isomorphisms in \(\mathcal{G}\);
5. \(G_1: C_1 \to B_1\) and \(F_0: C_0 \to B_0\) are isomorphisms in \(\mathcal{G}\).

**Proof.** Implications \(1 \Rightarrow 2 \Rightarrow 3\) are obvious.

\(3 \Rightarrow 4\): If \(G_1\) is an isomorphism, then \(G_0\) also is an isomorphism, with \(G_0^{-1} = u \cdot G_1^{-1} \cdot \text{dom}\). Same for \(3 \Rightarrow 5\). This argument also gives implication \(3 \Rightarrow 2\), because if \((F_1, F_0)\) is an internal functor with \(F_1\) and \(F_0\) invertible, then \((F_1^{-1}, F_0^{-1})\) also is an internal functor.

\(4 \Rightarrow 3\): One has to internalize the following set-theoretical argument: if \(g: a \to b\) is in \(B_1\), then one defines \(G_1^{-1}(g) = F_1^{-1}(\alpha(x) \cdot g \cdot \alpha(y)^{-1})\), where \(x = G_0^{-1}(a)\) and \(y = G_0^{-1}(b)\).

\(2 \Rightarrow 1\): This implication holds in any 2-category: assume that the 2-cell \(\alpha\) is invertible w.r.t. vertical composition, and let \(F \dashv F^*, G \dashv G^*\) be adjunctions, with units and counits given by \(\eta: \text{Id}_{\mathbb{C}} \to F \circ F^*, \epsilon: F^* \circ F \to \text{Id}_{\mathbb{B}}, \gamma: \text{Id}_{\mathbb{C}} \to G \circ G^*, \beta: G^* \circ G \to \text{Id}_{\mathbb{B}}\). Using triangular identities, one checks that the following diagrams commute

\[
\begin{array}{ccc}
F \cdot F^* \xrightarrow{\alpha \circ \alpha^{-1}} G \cdot G^* & & F^* \cdot F \xrightarrow{(\alpha^{-1}) \circ \alpha} G^* \cdot G \\
\downarrow \eta & & \downarrow \gamma & & \downarrow \epsilon \\
\text{Id}_{\mathbb{C}} & \xrightarrow{1} & \text{Id}_{\mathbb{C}} & \xrightarrow{1} & \text{Id}_{\mathbb{B}}
\end{array}
\]

where \((\alpha^{-1})^*\) is defined by the following composition

\[
F^* = F^* \cdot \text{Id}_{\mathbb{C}} \xrightarrow{1 \circ \gamma} F^* \cdot G \cdot G^* \xrightarrow{1 \circ \alpha^{-1} \circ 1} F^* \cdot F \cdot G^* \xrightarrow{\epsilon \circ 1} \text{Id}_{\mathbb{B}} \cdot G^* = G^*
\]

In particular, if \(F\) and \(G\) are isomorphisms, one can chose \(\eta, \epsilon, \gamma\) and \(\beta\) to be identities, and the proof is complete.

3.2. **Corollary.** Let \((D, d)\) be a \(\mathbb{C}\)-derivation. The following conditions are equivalent:

1. \((D, d)\) is a regular derivation;
2. \(D: \mathbb{C} \to \mathbb{C}\) is in \(\text{Grpd}(\mathcal{G})^*\);
3. \(D_0: C_0 \to C_0\) is an isomorphism (i.e. \(C_0 \xrightarrow{d} C_0 \xrightarrow{\text{dom}} C_1\) is an isomorphism);
4. \(D_1: C_1 \to C_1\) is an isomorphism.
3.3. Example. When \( \mathcal{G} \) is the category of groups and \( \mathcal{C} \) is the internal groupoid associated with a crossed module \( H \to G \to AutH \) as in Example 2.3, the previous corollary extends the following characterization of regular derivations, due to Whitehead [25]:

There are morphisms of monoids \( \sigma : Der(G,H) \to EndG : \sigma_d(x) = \partial(d(x))x \) and \( \theta : Der(G,H) \to EndH : \theta_d(a) = d(\partial(a)) + a \). Moreover, a derivation \( d \) is invertible iff \( \sigma_d \in AutG \) iff \( \theta_d \in AutH \).

Our definition of derivation also explains why the group of regular derivations \( Der^*(G,H) \) enters in the construction of Norrie’s actor of a crossed module (cf. [23], see also Theorem 3.3 in [9]). In fact, for any internal groupoid \( \mathcal{C} \) in any finitely complete category \( \mathcal{G} \), the data

\[
\text{Act}\mathcal{C} : \begin{cases}
\text{Der}^*\mathcal{C} \to [\text{Grpd}(\mathcal{G})^* (\mathcal{C}, \mathcal{C})]_0 & (D, d) \mapsto (D : \mathcal{C} \to \mathcal{C}) \\
[\text{Grpd}(\mathcal{G})^* (\mathcal{C}, \mathcal{C})]_0 \times \text{Der}^*\mathcal{C} \to \text{Der}^*\mathcal{C} & F, (D, d) \mapsto F_0 \cdot d \cdot F_1^{-1}
\end{cases}
\]

define a crossed module of groups: \( \text{Act}\mathcal{C} \) precisely is the crossed module associated with \( \text{Grpd}(\mathcal{G})^* (\mathcal{C}, \mathcal{C}) \), which is an internal groupoid in groups. Recall now that the actor \( \text{Act}(G,H) \) of a crossed module is a new crossed module intended to recapture, in the category of crossed modules, the idea of “group of automorphisms of a group”. If we look at the crossed module \( H \to G \to AutH \) as an internal groupoid \( \mathcal{C} \) in groups, then the group of automorphisms must be replaced by \( \text{Grpd}(\mathcal{G})^* (\mathcal{C}, \mathcal{C}) \), and \( \text{Act}(G,H) \) is nothing but \( \text{Act}\mathcal{C} \).

4. Left exactness of \( Der\mathcal{C} \)

If \( ( H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH ) \) is a crossed module of groups, one of the main properties of the group of regular derivations \( Der^*(G,H) \) is that, when it is seen as a functor of the second variable, it preserves kernels. Indeed, this allows one to apply the kernel-cokernel lemma for groups, and obtaining in this way the fundamental exact sequence in nonabelian group cohomology. The aim of this section is to study the main properties of \( Der\mathcal{C} \) and \( Der^*\mathcal{C} \) as functors.

Consider two internal groupoids \( \mathcal{C} \) and \( \mathcal{C}' \) in \( \mathcal{G} \) having the same object of objects, and an internal functor \( F : \mathcal{C} \to \mathcal{C}' \) which is the identity on objects

\[
\begin{array}{ccc}
C_0 & \xrightarrow{F_1} & C_1' \\
\downarrow\text{dom} & & \downarrow\text{cod} \\
C_0 & \xrightarrow{F_0=1} & C_0
\end{array}
\]

Composing with \( F_1 \) gives a morphism of monoids

\[
\begin{array}{ccc}
DerF : Der\mathcal{C} & \to & Der\mathcal{C}' \\
C_0 \xrightarrow{d} C_1 & \mapsto & C_0 \xrightarrow{d} C_1 \xrightarrow{F_1} C_1'
\end{array}
\]
and its restrictions to the groups of regular derivations $\text{Der}^* F : \text{Der}^* C \to \text{Der}^* C'$.

In fact, this construction is a functor

$$
\text{Der} : \mathcal{F}_{C_0} \to \text{Mon}
$$

where $\text{Mon}$ is the category of monoids, and $\mathcal{F}_{C_0}$ is the fibre over $C_0$ of the functor

$$
\text{Grpd}(\mathcal{G}) \to \mathcal{G}
$$

which associate to an internal groupoid $\mathcal{C}$ its object of objects $C_0$. Moreover, the functor $\text{Der}$ factorizes through the comma category $\text{Mon}/\text{End}_{C_0}$

because $\text{Der}\mathcal{C}$ is equipped with a canonical morphism

$$
\text{Der}\mathcal{C} \to \text{End}_{C_0} \quad C_0 \xrightarrow{d} C_1 \quad \Rightarrow \quad \text{(dom) } C_0 \xrightarrow{d} C_1 \xrightarrow{\text{dom}} C_0
$$

(cf. Proposition 2.2). In the same way, using regular derivations instead of arbitrary derivations, we obtain two functors

$$
\text{Der}^* : \mathcal{F}_{C_0} \to \text{Grp}/\text{Aut}_{C_0} \quad \text{Der}^* : \mathcal{F}_{C_0} \to \text{Grp}
$$

where $\text{Grp}$ is the category of groups.

4.1. PROPOSITION.

1. The functor $\text{Der} : \mathcal{F}_{C_0} \to \text{Mon}/\text{End}_{C_0}$ preserves finite limits;

2. The functor $\text{Der} : \mathcal{F}_{C_0} \to \text{Mon}$ preserves equalizers;

3. The functor $\text{Der}^* : \mathcal{F}_{C_0} \to \text{Grp}/\text{Aut}_{C_0}$ preserves finite limits;

4. The functor $\text{Der}^* : \mathcal{F}_{C_0} \to \text{Grp}$ preserves equalizers.

Proof. The functor $\text{Mon} \to \text{Grp}$ which associates to a monoid the group of its invertible elements preserves limits, so that points 3 and 4 follow from points 1 and 2. Moreover, the canonical forgetful functor from a comma category to the base category preserves equalizers, so that point 2 follows from point 1. As far as point 1 is concerned, it is enough to give a glance to finite limits in the fibre $\mathcal{F}_{C_0}$.

- The object of arrows of the terminal object in $\mathcal{F}_{C_0}$ is the product $C_0 \times C_0$. Domain and codomain are the projections. Composition $C_0 \times C_0 \times C_0 \to C_0 \times C_0$ is the projection on the first and third components. The inverse $C_0 \times C_0 \to C_0 \times C_0$ is the twist.
- The object of arrows of the equalizer in $\mathcal{F}_{C_0}$ of $F, G : \mathbb{C} \to \mathbb{C}'$ is the equalizer in $\mathcal{G}$

$$E \xrightarrow{e} C_1 \xrightarrow{F_1} C_1'$$

with domain and codomain given by $\text{dom} \cdot e, \text{cod} \cdot e$. The rest of the structure is inherited from that of $\mathbb{C}$ using the universal property of $E$.

- The object of arrows of the product in $\mathcal{F}_{C_0}$ of $\mathbb{C}$ and $\mathbb{C}'$ is the limit $\mathcal{L}$ as in the following diagram

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The domain is $\text{dom} \cdot p_1 = \text{dom}' \cdot p_2$ and the codomain is $\text{cod} \cdot p_1 = \text{cod}' \cdot p_2$. The rest of the structure is inherited from those of $\mathbb{C}$ and $\mathbb{C}'$ via the universal property of $\mathcal{L}$.

It is now easy to verify that the functor $\text{Der} : \mathcal{F}_{C_0} \to \text{Mon}/\text{End}C_0$ preserves finite limits. Let us look, for example, at the case of products. Consider a pair $(d_1, d_2) \in \text{Der}\mathbb{C} \times_{\text{End}C_0} \text{Der}\mathbb{C}'$

(which is the product in the comma category $\text{Mon}/\text{End}C_0$). Since $\text{cod} \cdot d_1 = 1 = \text{cod}' \cdot d_2$ and $\text{dom} \cdot d_1 = \text{dom}' \cdot d_2$, there is a unique arrow $d : C_0 \to \mathcal{L}$ such that $p_1 \cdot d = d_1$ and $p_2 \cdot d = d_2$. Moreover, $\text{cod} \cdot p_1 \cdot d = \text{cod} \cdot d_1 = 1$, so that $d$ is a derivation. Conversely, if $d : C_0 \to \mathcal{L}$ is a derivation, then $\text{cod} \cdot p_1 \cdot d = 1 = \text{cod}' \cdot p_2 \cdot d$ and $\text{dom} \cdot p_1 \cdot d = \text{dom}' \cdot p_2 \cdot d$, so that the pair $(p_1 \cdot d, p_2 \cdot d)$ is an element of the pullback $\text{Der}\mathbb{C} \times_{\text{End}C_0} \text{Der}\mathbb{C}'$.  

5. The 2-category of holomorphisms

In this section, we give a different description of 2-cells in $\text{Grpd}(\mathcal{G})$. For this, we introduce the notion of holomorphism between two groupoids. Our terminology is justified by Example 5.5.

The set-theoretical idea behind the notion of holomorphism is quite easy: given a 2-cell

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in $\text{Grpd}(\mathcal{G})$, then $F, G$ and $\alpha$ itself are completely determined by the map associating to an internal arrow $(a : x \to y) \in C_1$ the diagonal $(Fx \to Gy) \in B_1$ of the commutative
To make this more precise, we need some preliminary work. Consider two internal groupoids $C, B$ and let $h: C_1 \to B_1$ be an arrow making commutative the following diagrams

$$
\begin{array}{ccc}
C_1 & \xrightarrow{h} & B_1 \\
\downarrow & & \downarrow \\
C_0 & \xrightarrow{u} & C_1
\end{array} \quad \text{dom} \quad \begin{array}{ccc}
B_1 & \xrightarrow{\text{dom}} & B_0 \\
\downarrow & & \downarrow \\
B_0 & \xrightarrow{\text{dom}} & B_1
\end{array} \quad \text{cod}
$$

Thanks to conditions (2) and (3), we get two arrows

$$
\widehat{h} = \langle \pi_1 \cdot h, \pi_1 \cdot \text{dom} \cdot u \cdot h \cdot ( )^{-1}, \pi_2 \cdot h \rangle: C_1 \times_{C_0} C_1 \to B_1 \times_{B_0} B_1 \times_{B_0} B_1
$$

$$
\tilde{h} = \langle \pi_1 \cdot h, \pi_2 \cdot h, \pi_3 \cdot h \rangle: P \to Q
$$

where $P$ and $Q$ are defined by the following limits in $G$

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\pi_1} & P \\
\downarrow & \downarrow & \downarrow \\
C_1 & \xrightarrow{\pi_3} & C_1 \\
\downarrow & \downarrow & \downarrow \\
C_0 & \xrightarrow{\text{cod}} & C_0
\end{array} \quad \begin{array}{ccc}
B_1 & \xrightarrow{\pi_1} & Q \\
\downarrow & \downarrow & \downarrow \\
B_1 & \xrightarrow{\text{dom}} & B_1 \\
\downarrow & \downarrow & \downarrow \\
B_0 & \xrightarrow{\text{cod}} & B_0
\end{array}
$$

5.1. Lemma.

1. Diagram (4) commutes iff diagram (4') commutes

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{o} & C_1 \\
\downarrow & \downarrow & \downarrow \\
B_1 \times_{B_0} B_1 \times_{B_0} B_1 & \xrightarrow{o^2} & B_1
\end{array} \quad \text{(4)} \quad \begin{array}{ccc}
P & \xrightarrow{\widehat{h}} & Q \\
\downarrow & \downarrow & \downarrow \\
\langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle & \xrightarrow{(4')} & \langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle
\end{array}
$$

$$
\begin{array}{ccc}
\langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle & \xrightarrow{(4')} & \langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle \\
\downarrow & \downarrow & \downarrow \\
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{o^2} & B_1 \times_{B_0} B_1 \times_{B_0} B_1
\end{array} \quad \begin{array}{ccc}
C_1 & \xrightarrow{h} & B_1 \\
\downarrow & \downarrow & \downarrow \\
C_1 & \xrightarrow{h} & B_1
\end{array} \quad \begin{array}{ccc}
P & \xrightarrow{\tilde{h}} & Q \\
\downarrow & \downarrow & \downarrow \\
\langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle & \xrightarrow{(4')} & \langle \pi_1, \pi_2 \cdot ( )^{-1}, \pi_3 \rangle
\end{array}
$$
2. Diagram (5) commutes iff diagram (5’) commutes

\[
\begin{array}{ccc}
C_0 \xrightarrow{u} C_1 & \xrightarrow{h} B_1 \\
\downarrow & & \downarrow \\
C_1 & \xrightarrow{h} B_1 & \text{dom} B_0
\end{array}
\]

\[
\begin{array}{ccc}
C_0 \xrightarrow{u} C_1 & \xrightarrow{h} B_1 \\
\downarrow & & \downarrow \\
C_1 & \xrightarrow{h} B_1 & \text{cod} B_0
\end{array}
\]

Proof. Part 2 being quite obvious, let us concentrate on part 1. If the base category \( G \) is the category of sets and \( a: x \rightarrow y, b: z \rightarrow y, c: z \rightarrow w \) are elements of \( C_1 \), condition (4’) means that \( h(a \cdot b^{-1} \cdot c) = h(a) \cdot h(b)^{-1} \cdot h(c) \). Condition (4) expresses the special case of condition (4’) where \( z = y \) and \( b = 1_y \). It is therefore clear that \( (4’) \Rightarrow (4) \). Conversely, assume that \( h \) satisfies (4) and put

\[ \delta_h: C_1 \rightarrow B_1 \quad \delta_h(x \xrightarrow{a} y) = h(a) \cdot h(1_y)^{-1} . \]

Clearly \( \delta_h \) preserves units. Moreover, (4) immediately implies that \( \delta_h \) preserves composition too. Therefore,

\[
h(a \cdot b^{-1} \cdot c) = \delta_h(a \cdot b^{-1} \cdot c) \cdot h(1_w) = \delta_h(a) \cdot \delta_h(b)^{-1} \cdot \delta_h(c) \cdot h(1_w) = h(a) \cdot h(1_y)^{-1} \cdot (h(b) \cdot h(1_y)^{-1})^{-1} \cdot h(c) \cdot h(1_w)^{-1} \cdot h(1_w) = h(a) \cdot h(b)^{-1} \cdot h(c) .
\]

This concludes the proof when \( G \) is the category of sets. Following Metatheorem 0.1.3 in [1], the result holds for any finitely complete category \( G \).

5.2. Definition. Consider two groupoids \( \mathbb{C}, \mathbb{B} \) in \( G \).

1. An holomorphism \( h: \mathbb{C} \rightarrow \mathbb{B} \) is an arrow \( h: C_1 \rightarrow B_1 \) making commutative diagram (2), diagram (3), and diagram (4).

2. An holomorphism \( h: \mathbb{C} \rightarrow \mathbb{B} \) is pointed if it makes commutative diagram (5).

5.3. Lemma.

1. Holomorphisms and pointed holomorphisms are stable under composition in \( G \).

2. If \( h: \mathbb{C} \rightarrow \mathbb{B} \) is an holomorphism, then the arrows

\[ \delta_h: C_1 \xrightarrow{(h, \text{cod} u \cdot h(\cdot)^{-1})} B_1 \times B_0 \xrightarrow{\circ} B_1 \]

\[ \gamma_h: C_1 \xrightarrow{(\text{dom} u \cdot h(\cdot)^{-1}, h)} B_1 \times B_0 \xrightarrow{\circ} B_1 \]

are pointed holomorphisms from \( \mathbb{C} \) to \( \mathbb{B} \). We call \( \delta_h \) the domain of \( h \) and \( \gamma_h \) the codomain of \( h \).

Proof. The proof is straightforward if \( G \) is the category of sets. One concludes using once again Metatheorem 0.1.3 in [1].
We are ready to describe the 2-category $\text{Hol}(\mathcal{G})$ of holomorphisms:
- Objects are internal groupoids in $\mathcal{G}$.
- If $\mathbb{C}$ and $\mathbb{B}$ are internal groupoids, 1-cells $\mathbb{C} \to \mathbb{B}$ are pointed holomorphisms.
- Composition of 1-cells $f: \mathbb{C} \to \mathbb{B}, g: \mathbb{B} \to \mathbb{D}$ is the composition of $f: C_1 \to B_1$ and $g: B_1 \to D_1$ in $\mathcal{G}$.
- If $f, g: \mathbb{C} \to \mathbb{B}$ are pointed holomorphisms, 2-cells $f \Rightarrow g$ are holomorphisms $h: \mathbb{C} \to \mathbb{B}$ such that $\delta h = f$ and $\gamma h = g$.
- Horizontal composition of 2-cells $h: f \Rightarrow g: \mathbb{C} \to \mathbb{B}, k: f' \Rightarrow g': \mathbb{B} \to \mathbb{D}$ is the composition of $h: C_1 \to B_1$ and $k: B_1 \to D_1$ in $\mathcal{G}$.
- If $h, k: \mathbb{C} \to \mathbb{B}$ are holomorphisms with $\gamma h = \delta k$, their vertical composition is given by $C_1 \xrightarrow{(\text{dom-u,h,k})} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$
or, equivalently, by $C_1 \xrightarrow{(h,\text{cod-u-k})} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$
- The identity 2-cell on a 1-cell $f: \mathbb{C} \to \mathbb{B}$ is $f$ itself.

5.4. Example. We can consider a group $G$ as a groupoid (in sets) with just one object, and having the elements of $G$ as arrows. Group homomorphisms correspond then to internal functors. If $f, g: G \to H$ are group homomorphisms, a natural transformation $h: f \Rightarrow g$ is just an element $h \in H$ such that, for all $a \in G$, one has $f(a) + h = h + g(a)$.
We can therefore define a map $h: G \to H$ by $h(a) = f(a) + h$, so that $h(0) = h$. Such a map satisfies the equation $h(a + c) = h(a) - h(0) + h(c)$, which is also equivalent to the equation $h(a - b + c) = h(a) - h(b) + h(c)$ (compare with Lemma 5.1). A map satisfying these equivalent conditions is called a group holomorphism (see, for example, Section IV.1 in [20]). Conversely, an holomorphism $h: G \to H$ is an homomorphism precisely when it is pointed, that is when $h(0) = 0$. We can therefore construct two homomorphisms from an holomorphism $h$:

$$\delta_h: G \to H, \quad \delta_h(a) = h(a) - h(0)$$
$$\gamma_h: G \to H, \quad \gamma_h(a) = -h(0) + h(a)$$

The element $h(0)$ gives then a natural transformation $h(0): \delta_h \Rightarrow \gamma_h$ (compare with Lemma 5.3).

Because of the way holomorphisms compose, we have the following fact.

5.5. Corollary. An holomorphism $h: \mathbb{C} \to \mathbb{B}$ is invertible with respect to horizontal composition iff $h: C_1 \to B_1$ is an isomorphism in $\mathcal{G}$.

As announced at the beginning of this section, $\text{Hol}(\mathcal{G})$ provides an equivalent description of $\text{Grpd}(\mathcal{G})$. In fact, we have the following result.
5.6. Proposition. There is a 2-functor $\epsilon: \text{Hol}(\mathcal{G}) \to \text{Grpd}(\mathcal{G})$ which is the identity on objects and an isomorphism on hom-categories. The 2-functor $\epsilon$ restricts to the sub-2-categories of isomorphisms $\text{Hol}(\mathcal{G})^* \to \text{Grpd}(\mathcal{G})^*$.

Proof. If $f: C \to B$ is a pointed holomorphism, we get an internal functor $\epsilon(f) = (F_1, F_0): C \to B$ by $F_1 = f: C_1 \to B_1$ and $F_0 = u \cdot f \cdot \text{dom}: C_0 \to B_0$.

If $h: C \to B$ is an holomorphism, we get an internal natural transformation $\epsilon(h): \epsilon(\delta_h) \Rightarrow \epsilon(\gamma_h)$ by $\epsilon(h) = u \cdot h: C_0 \to C_1$.

Conversely, if $\alpha: F = (F_1, F_0) \Rightarrow G = (G_1, G_0): C \to B$ is an internal natural transformation (with $\alpha: C_0 \to B_1$), we get an holomorphism $h: C \to B$ by

$$
\begin{array}{c}
C_1 \\
\Downarrow \langle F_1, \text{cod} \cdot \alpha \rangle
\end{array}
\xrightarrow{(F_1, \text{cod} \cdot \alpha)}
\begin{array}{c}
B_1 \\
\Downarrow \circ
\end{array}
\xrightarrow{\circ}
B_1
$$

or, equivalently, by

$$
\begin{array}{c}
C_1 \\
\Downarrow \langle \text{dom} \cdot \alpha, G_1 \rangle
\end{array}
\xrightarrow{(\text{dom} \cdot \alpha, G_1)}
\begin{array}{c}
B_1 \\
\Downarrow \circ
\end{array}
\xrightarrow{\circ}
B_1
$$

Details are routine and are left to the reader. □

6. Translations

In this section, we specialize the notion of holomorphism to get a different description of derivations in terms of what we call translations. This name is justified by Example 6.4.

6.1. Definition. The monoid of $\mathcal{C}$-translations is the kernel of the codomain map

$$Tr\mathcal{C} = \text{Ker}(\text{cod}) \to [\text{Hol}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_1 \to [\text{Hol}(\mathcal{G})(\mathcal{C}, \mathcal{C})]_0$$

As we did in Proposition 2.2 with derivations, we give now a more geometrical description of translations. Fix an arrow $t: C_1 \to C_1$ such that the diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{t} & C_1 \\
\downarrow \text{cod} & \qquad & \downarrow \text{cod} \\
C_1 & \xrightarrow{\circ} & C_1
\end{array}
$$

commutes, and consider the factorizations

$$\tilde{t} = \langle \text{dom} \cdot u \cdot t, 1 \rangle: C_1 \to C_1 \times_{C_0} C_1 \quad \tilde{t} = \langle \pi_1 \cdot t, \pi_2 \rangle: C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$$

6.2. Lemma. Diagram (7) commutes iff diagram (7') commutes

$$
\begin{array}{ccc}
C_1 & \xrightarrow{t} & C_1 \\
\downarrow \tilde{t} & \qquad & \downarrow \tilde{t} \\
C_1 \times_{C_0} C_1 & \xrightarrow{(\pi_1 \cdot t, \pi_2) \circ} & C_1
\end{array}
$$
Proof. Let us sketch the proof in the case \( \mathcal{G} \) is the category of sets. The first condition means that, for any arrow \( a: x \to y \), the diagram

\[
\begin{array}{c}
\text{t}(a) \downarrow \\
\text{t}(1_x) \downarrow \\
x \downarrow \downarrow \\
\end{array}
\begin{array}{c}
y \\
a \\
x
\end{array}
\]

commutes; the second condition means that, for any composable pair of arrows

\[
\begin{array}{c}
z \xrightarrow{b} x \xrightarrow{a} y \\
\end{array}
\]

the diagram

\[
\begin{array}{c}
\text{t}(ba) \downarrow \\
\text{t}(b) \downarrow \\
x \downarrow \downarrow \\
\end{array}
\begin{array}{c}
y \\
a \\
x
\end{array}
\]

commutes. Clearly, the first condition is a special case of the second one (take \( b = 1_x \)). Conversely, using the first condition, both sides of the diagram expressing the second condition are equal to

\[
\begin{array}{c}
\text{t}(1_x) \downarrow \\
z \xrightarrow{b} y \xrightarrow{a} x \\
\end{array}
\]

The general case follows using 0.1.3 in [1].

6.3. Proposition. To give a \( \mathbb{C} \)-translation amounts to give an arrow \( t: C_1 \to C_1 \) such that diagram (6) and diagram (7) commute.

Proof. Routine.

6.4. Example. Consider once again a group \( G \) as a groupoid \( \mathbb{C} \) with a single object. Thanks to Proposition 6.3, a \( \mathbb{C} \)-translation in the sense of Definition 6.1 is nothing but a map \( t: G \to G \) such that \( t(a) = t(0) + a \) for all \( a \in G \). That is, \( t \) is the right translation by \( t(0) \). Therefore, in this case \( Tr\mathbb{C} \) is a group isomorphic to \( G \).

In contrast with the situation described in Example 6.4, the monoid \( Tr\mathbb{C} \) in general is not a group.

6.5. Corollary. The group of regular translation \( Tr^*\mathbb{C} \) is given by \( Tr\mathbb{C} \cap \text{Aut} C_1 \).

By Proposition 4.7, we get the following corollary.

6.6. Corollary. The 2-functor \( \epsilon: \text{Hol}(\mathcal{G}) \to \text{Grpd}(\mathcal{G}) \) induces two isomorphisms of monoids \( Tr\mathbb{C} \simeq Der\mathbb{C} \) and \( Tr^*\mathbb{C} \simeq Der^*\mathbb{C} \).

We can describe the isomorphism \( Tr\mathbb{C} \simeq Der\mathbb{C} \) using Propositions 2.2 and 6.3:

- Given \( (t: C_1 \to C_1) \in Tr\mathbb{C} \), we get \( (u \cdot t: C_0 \to C_1 \to C_1) \in Der\mathbb{C} \).

- Given \( (d: C_0 \to C_1) \in Der\mathbb{C} \), we get \( C_1 \xrightarrow{(\text{dom}, \text{dom}, \text{dom})} C_1 \times C_0 \xrightarrow{(0, 1)} C_1 \in Tr\mathbb{C} \).
Let us summarize the situation we have so far with the following picture, where the unlabelled vertical arrows are the inclusion of the kernel.

\[
\begin{array}{ccc}
Tr^*C & \cong & Der^*C \\
\downarrow & & \downarrow \\
[Hol(G)^*(\mathbb{C}, \mathbb{C})]_1 & \cong & [Grpd(G)^*(\mathbb{C}, \mathbb{C})]_1 \\
\downarrow_{\text{dom}} & & \downarrow_{\text{dom}} \\
[Hol(G)^*(\mathbb{C}, \mathbb{C})]_0 & \cong & [Grpd(G)^*(\mathbb{C}, \mathbb{C})]_0
\end{array}
\]

6.7. Example. Since \( Hol(G)^*(\mathbb{C}, \mathbb{C}) \) is a groupoid in groups, using the constructions described in Example 2.3 we can pass to a crossed module of groups, and then come back to a groupoid isomorphic to \( Hol(G)^*(\mathbb{C}, \mathbb{C}) \). Using also the isomorphisms of the previous picture, we get a group isomorphism

\[
[Hol(G)^*(\mathbb{C}, \mathbb{C})]_1 \cong Tr^*C \rtimes [Grpd(G)^*(\mathbb{C}, \mathbb{C})]_0
\]

If we specialize this isomorphism to the case where \( G \) is the category of sets and \( \mathbb{C} \) is the one-object groupoid associated to a group \( G \) (Examples 5.4 and 6.4), we get the classical isomorphism

\[
HolG \simeq G \rtimes AutG
\]

where \( HolG \) is the group of bijective holomorphisms from \( G \) to \( G \) (see [20], Section IV.1).

6.8. Example. Let us consider a \( K \)-vector space \( V \) as a 1-object (commutative) groupoid with an additional multiplicative structure. It is easy to verify that

\[
[Hol^*(V)]_0 \simeq GL(V)
\]

i.e. the general linear group of \( V \). It is less obvious to notice that \( [Hol^*(V)]_1 \) corresponds to the group of affine transformations of \( A \), the affine space of the “points” of \( V \). Moreover, this group is canonically isomorphic to the semidirect product

\[
Tr^*V \rtimes GL(V)
\]

where \( Tr^*V \) (\( \simeq V \)) is indeed the group of translations of \( A \).

More precisely, if \( a \) is the affine structure of \( A \) (i.e. \( a(P, Q) = Q - P \)), an affine transformation is a pair of bijections

\[
(F : A \rightarrow A, \varphi : V \rightarrow V)
\]

resp. in \( \textbf{Set} \) and in \( \textbf{Vect}_K \), such that the following diagram commutes:

\[
\begin{array}{ccc}
A \times A & \xrightarrow{a} & V \\
\downarrow_{F \times F} & & \downarrow_{\varphi} \\
A \times A & \xrightarrow{a} & V
\end{array}
\]
The linear map $\varphi$ is determined by $F$ in the following way: fix a point $P$ and set

$$\varphi_P : v \mapsto F(P + v) - F(P).$$

Now, given a bijection $F : \mathbb{A} \to \mathbb{A}$, $F$ induces an affine transformation if and only if $\varphi_P$ is linear. This expresses exactly condition of diagram (4). In fact

$$\varphi_P(v + w) = \varphi_P(v) + \varphi_P(w)$$

$$F(P + v + w) - F(P) = F(P + v) - F(P) + F(P + w) - F(P)$$

$$F(P + v + w) = F(P + v) - F(P) + F(P + w)$$

$$F(v + w) = F(v) - F(0) + F(w)$$

where in the last line we identify $\mathbb{A}$ with $V$.

This example may be further generalized to modules, and to not-necessarily bijective maps. Under this perspective, group holomorphisms are group transformations that preserve the affine structure of a group, namely equivalent bi-points.

6.9. Example. As in Example 6.7, we have a group isomorphism

$$[\mathit{Hol}(\mathcal{G})^\ast(\mathbb{C}, \mathbb{C})]_1 \simeq \mathit{Der}^\ast \mathbb{C} \rtimes [\mathit{Grpd}(\mathcal{G})^\ast(\mathbb{C}, \mathbb{C})]_0$$

This generalizes the isomorphism established by Lue in the case where $\mathcal{G}$ is the category of groups, and groupoids are replaced by crossed modules (see [16], Theorem 9). It is interesting to observe that, in that case, only the analogue of our conditions (2) and (3) are used to define the analogue of $[\mathit{Hol}(\mathcal{G})^\ast(\mathbb{C}, \mathbb{C})]_1$. This is because the category of groups is a Mal’cev category (see [1] for the notion of Mal’cev category). In fact, we have the following result.

6.10. Lemma. Let $\mathcal{G}$ be a Mal’cev category, and consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$. If an arrow $h : C_1 \rightarrow B_1$ satisfies conditions (2) and (3), then it is an holomorphism.

Proof. We shall use set theoretic notations, using once again Metatheorem 0.1.3 in [1]. By results in [6], we know that, if $\mathbb{C}$ is a groupoid in a Mal’cev category, composition is uniquely determined and it corresponds to a (unique) partial associative Mal’cev operation $p$ on $C_1$. This operation $p$ is defined on those $(a, b, c)$ in $C_1$ such that $\text{cod}(a) = \text{cod}(b)$ and $\text{dom}(b) = \text{dom}(c)$:

$$a \quad b \quad c$$

and, in this case,

$$p(a, b, c) = a \circ b^{-1} \circ c.$$
the (unique) partial associative Mal’cev operation \( p' \) on \( B_1 \) associated to composition in \( \mathbb{B} \) is defined on \((h(a), h(b), h(c))\) and \( h \) satisfies condition (4') if and only if
\[
h(p(a, b, c)) = p'((h(a), h(b), h(c)).
\]

But this is true because in any Mal’cev category every relation \( \mathcal{R} \) is difunctional, that is \( \mathcal{R} \circ \mathcal{R}^{op} \circ \mathcal{R} = \mathcal{R} \). In fact, let \( \mathcal{R} \) be the relation on \( C_1 \times C_1 \) given by
\[
(a, b)\mathcal{R}c \iff cod(a) = cod(b), \ dom(b) = dom(c) \text{ and } h(p(a, b, c)) = p'((h(a), h(b), h(c)).
\]

Now, if \( cod(a) = cod(b), \ dom(b) = dom(c) \),
- \((a, b)\mathcal{R}b, \text{ since } h(p(a, b, b)) = h(a) = p'((h(a), h(b), h(b))\)
- \((b, b)\mathcal{R}b, \text{ as above, with } a = b\)
- \((b, b)\mathcal{R}cb, \text{ since } h(p(b, b, c)) = h(c) = p'((h(b), h(b), h(c))\)

then, by difunctionality, \((a, b)\mathcal{R}c, \text{ that is } h(p(a, b, c)) = p'((h(a), h(b), h(c)).\)

6.11. Example. Observe that, by 2.2 and 3.2, if the domain and codomain maps of an internal groupoid \( C \) are equal, then \( C \)-derivations are invertible. This is the case for \( C \) the groupoid in groups associated with a crossed module of the form \( H \overset{0}{\longrightarrow} G \overset{\varphi}{\longrightarrow} AutH \), where \( \varphi: G \to AutH \) is a \( G \)-module and \( 0: H \to G \) is the zero-morphism. Indeed, in this case, both domain and codomain coincide with the second projection \( \pi_2: H \times G \to G \). Moreover, in this case a classical result (see [20], Proposition IV.2.1) asserts that the group \( Der^*C \) is isomorphic to the group of isomorphisms \( \iota: H \times G \to H \times G \) making commutative the following diagrams

\[
\begin{align*}
H \times G \overset{\iota}{\longrightarrow} H \times G \\
\downarrow \pi_2 \\
G
\end{align*} \quad \begin{align*}
H \overset{i}{\longrightarrow} H \times G \\
\downarrow \pi_2 \\
H \times G
\end{align*}
\]

where \( i(a) = (a, 1) \). Since the first diagram is precisely diagram (6), and the commutativity of the second diagram is equivalent to the commutativity of diagram (7), this description of \( Der^*C \) is a specialization of the isomorphism \( Der^*C \simeq Tr^*C \) established in 6.6. Even for an arbitrary crossed module \( H \to G \to AutH \), the group \( Der^*(G, H) \) can be described as a suitable subgroup of \( Aut(H \times G) \), see Proposition 3.5 in [9]. Once again, this description is a particular case of the isomorphism \( Der^*C \simeq Tr^*C \).

7. The embedding category of an internal groupoid

If \( H \overset{\varphi}{\longrightarrow} G \overset{\varphi}{\longrightarrow} AutH \) is a crossed module of groups and \( F_0: A_0 \to G \) is a group homomorphism, a derivation relative to \( F_0 \), or \( F_0 \)-derivation, is a map \( d: A_0 \to H \) such
that \( d(xy) = d(x) + F_0(x) \cdot d(y) \). Equivalently, an \( F_0 \)-derivation is a group homomorphism \( d: A_0 \to H \rtimes G \) such that

\[
\begin{array}{c}
A_0 \\
\downarrow F_0 \\
\end{array} 
\begin{array}{c}
G \\
\downarrow \pi \\
H \rtimes G
\end{array}
\]

commutes. Relative derivations have been used in nonabelian cohomology of groups, see for example [10, 15, 16], and contain derivations as a special case (take as \( F_0 \) the identity morphism).

Relative derivations can be defined in the general context of internal groupoids in a finitely complete category \( G \) in much the same way as we did for derivations in Section 2. In this section, we develop a different approach: for a fixed groupoid \( C \) in \( G \), we construct the category of embeddings \( \text{Emb} C \) as the underlying category of a comma 2-category, and we show that derivations and relative derivations can be obtained as particular hom-sets from \( \text{Emb} C \). The guiding example, which justifies our terminology, is the category of embeddings of an étale groupoid, introduced in [21] to study the homotopy type of the groupoid.

7.1. Definition.

1. Let \( A, C \) be internal groupoids in \( G \). An internal functor \( F: A \to C \) is full and faithful if, for every internal groupoid \( X \), the functor \( \text{hom}(X, F): \text{hom}(X, A) \to \text{hom}(X, C) \) is full and faithful in the usual sense.

2. The category of embeddings \( \text{Emb} C \) has full and faithful functors \( F: A \to C \) as objects. An arrow from \( F: A \to C \) to \( G: B \to C \) is a pair

\[
(D: A \to B, \ d: D \cdot G \Rightarrow F)
\]

with \( D \) an internal functor and \( d \) an internal natural transformation. Composition and identities are the obvious ones.

Clearly, \( \text{Emb} C(Id_C, Id_C) = \text{Der} C \). More is true:

7.2. Proposition. If \( F: A \to C \) is full and faithful, then

\[
- \cdot F: \text{Der} A \to \text{Emb} C(F, F)
\]

is an isomorphism of monoids.

Proof. This immediately follows from the definition of full and faithful internal functor.
To recover relative derivations from the category of embeddings is less trivial.

7.3. **Lemma.** An internal functor \( F : A \to C \) is full and faithful iff the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{F_1} & A_0 \\
\downarrow & & \downarrow \\
C_1 & \xleftarrow{\alpha} & C_0
\end{array}
\]

is a limit diagram in \( G \).

**Proof.** Assume that the diagram in the statement is a limit diagram. Consider internal functors \( L, H : X \to A \) and an internal natural transformation \( \alpha : L \cdot F \Rightarrow H \cdot F \). Explicitly

\[
\begin{array}{ccc}
X_1 & \xrightarrow{L_1} & A_1 \\
\downarrow & & \downarrow \\
X_0 & \xleftarrow{L_0} & A_0
\end{array}
\] \quad \begin{array}{ccc}
A_1 & \xrightarrow{F_1} & C_1 \\
\downarrow & & \downarrow \\
A_0 & \xleftarrow{F_0} & C_0
\end{array}
\]

Therefore, \( \alpha \cdot \text{dom} = L_0 \cdot F_0 \) and \( \alpha \cdot \text{cod} = H_0 \cdot F_0 \). From the universal property of the limit \( A_1 \), we get a unique arrow \( \beta : X_0 \to A_1 \) such that \( \beta \cdot \text{dom} = L_0, \beta \cdot \text{cod} = H_0 \) and \( \beta \cdot F_1 = \alpha \). This means that \( \beta \) is an internal (natural) transformation from \( L \) to \( H \) such that \( \beta \circ F = \alpha \). (The naturality of \( \beta \) is expressed by the commutativity of

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\langle \text{dom}, \beta, H_1 \rangle} & A_1 \times_{A_0} A_1 \\
\downarrow & & \downarrow \\
A_1 \times_{A_0} A_1 & \xrightarrow{\circ} & A_1
\end{array}
\]

which is easy to check composing with the projections of the limit \( A_1 \) and using the naturality of \( \alpha \).

The converse implication follows from the fact that to give a factorization of a commutative diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{L_0} & A_0 \\
\downarrow & & \downarrow \\
C_1 & \xleftarrow{\alpha} & C_0 \\
\downarrow & & \downarrow \\
A_0 & \xleftarrow{F_0} & C_0
\end{array}
\]
through the diagram in the statement corresponds to give a (necessarily natural) transformation $\beta: L \Rightarrow H$ such that $\beta \circ F = \alpha: L \cdot F \Rightarrow H \cdot F$, where $L$ and $H$ are internal functors with discrete domain as in the following diagram

$$
\begin{array}{c}
X_0 \xrightarrow{L_0 \cdot u} A_1 \\
\downarrow 1 \quad \downarrow 1 \\
X_0 \xrightarrow{H_0 \cdot u} A_0
\end{array}
$$

7.4. LEMMA. Let $\mathcal{C}$ be an internal groupoid. The category $\text{Emb}\mathcal{C}$ can be described in the following way:

- Objects are arrows $F_0: A_0 \rightarrow C_0$ in $\mathcal{G}$.

- A morphism from $F_0: A_0 \rightarrow C_0$ to $G_0: B_0 \rightarrow C_0$ is a pair $(D_0: A_0 \rightarrow B_0, d: A_0 \rightarrow C_1)$ of arrows in $\mathcal{G}$ such that

$$
\begin{array}{c}
A_0 \xrightarrow{F_0} C_0 \\
\downarrow d \quad \downarrow \text{cod} \\
A_0 \xrightarrow{G_0} B_0 \xrightarrow{d} C_0
\end{array}
$$

commute.

Proof. Let us concentrate on objects (the argument for arrows is similar). Given an arrow $F_0: A_0 \rightarrow C_0$, the limit

$$
\begin{array}{c}
A_0 \xrightarrow{F_0} C_0 \\
\downarrow \text{dom} \quad \downarrow \text{cod} \\
C_1 \xrightarrow{F_1} A_1
\end{array}
$$

produces an internal groupoid $\mathbb{A}$ and a full and faithful internal functor $F = (F_1, F_0): \mathbb{A} \rightarrow \mathcal{C}$. Conversely, if $F: \mathbb{A} \rightarrow \mathcal{C}$ is full and faithful, then the unit $u: A_0 \rightarrow A_1$ and the composition $\circ: A_1 \times_{A_0} A_1 \rightarrow A_1$ of $\mathbb{A}$ are the unique factorizations through the previous
7.5. Example. Let $G$ be the category of topological spaces and local homeomorphisms, and let $C$ be an internal groupoid in $G$, that is $C$ is an étale groupoid. Fix a basis $\mathcal{B}$ of contractible open sets for the space $C_0$. The category of embeddings $\text{Emb}_G C$ (see [21]) is the full subcategory of $\text{Emb} C$ with objects the inclusions $U \to C_0$, $U \in \mathcal{B}$. Indeed, following 7.4, if $G_0$ is a monomorphism, then an arrow $(D_0, d)$ is determined by $d$, in the sense that if $D_0$ exists, it is unique. (Note that in [21] the role of $\text{dom}$ and $\text{cod}$ is inverted: the equations defining an arrow $(D_0, d)$ in 7.4 should be written as $d \cdot \text{dom} = F_0$ and $d \cdot \text{cod} = D_0 \cdot G_0$. Since $C$ is a groupoid, this makes no difference.)

7.6. Example. Consider a crossed module $H \to G \to \text{Aut} H$, the corresponding internal groupoid $C$, and a group homomorphism $F_0 : A_0 \to G$. The set of $F_0$-derivations is in bijection with the hom-set $\text{Emb} C(F, \text{Id}_C)$, where $F : A \to C$ is the full and faithful internal functor corresponding to $F_0 : A_0 \to C_0$ as in the proof of Lemma 7.4. Indeed, following the description of $\text{Emb} C$ given in 7.4, if $G = \text{Id}_C$, then $D_0$ necessarily is $d \cdot \text{dom}$.

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