EXTERNAL DERIVATIONS OF INTERNAL GROUPOIDS

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ABSTRACT. If H is a G-crossed module, the set of derivations of G in H is a monoid under Whitehead product of derivations. We interpret Whitehead product using the correspondence between crossed modules and internal groupoids in the category of groups. Working in the general context of internal groupoids in a finitely complete category, we relate derivations to holomorphisms, translations, affine transformations, and to the embedding category of a groupoid.

1. Introduction

Let G be a group and $\varphi \colon G \to AutH$ a G-group. A derivation of G in H is a map $d \colon G \to H$ such that $d(xy) = d(x) + x \cdot d(y)$. If H is a G-module, i.e. if H is abelian, the set $Der(G, H)$ of derivations is an abelian group w.r.t. the point-wise sum. If H is not abelian, in general $Der(G, H)$ is just a pointed set (the zero-morphism $0 \colon G \to H$ is a derivation). Whitehead [25] discovered the following fact.

1.1. THEOREM. *Let ($H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH$) be a crossed module of groups. The set $Der(G, H)$ is a monoid w.r.t. $(d_1 + d_2)(x) = d_1(\partial(d_2(x))x) + d_2(x)$.*

The aim of this note is to understand in a more conceptual way Whitehead product of derivations. The idea is to replace crossed modules of groups by the equivalent notion of internal groupoids in the category of groups. Using the language of internal groupoids, Whitehead product becomes clear: it is nothing but the composition in the internal category. The surprise is that, once expressed in terms of internal groupoids, Whitehead theorem, as well as some other basic properties of derivations (notably, the characterization of regular derivations and the left exactness of $Der(G, H)$ as a functor of the second variable), has nothing to do with groups, but holds in the very general context of internal groupoids in an arbitrary category $\mathcal{G}$ with finite limits [2, 7]. Just to quote some categories where internal crossed modules and internal groupoids are intensively studied, let us mention the category of Lie algebras [5] (in fact, Lie algebras as well as groups constitute semi-abelian categories [1, 14], and for semi-abelian categories, internal crossed modules and internal groupoids coincide [13]), the category of topological spaces and continuous maps [8, 11, 18], the category of topological spaces and local homeomorphisms (whose internal groupoids are called étale groupoids) [22], the category of smooth manifolds (whose internal groupoids are called Lie groupoids) [17, 18, 22], and of course the category of sets, which gives ordinary groupoids [3, 12, 24].

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In his paper [9], Gilbert explains Whitehead product of derivations replacing crossed modules by the equivalent notion of groups in the category of groupoids, whereas we use groupoids in the category of groups. Even if the equivalence between groups in groupoids and groupoids in groups is a trivial fact, the advantage of working with groupoids in groups is that this immediately suggests the more general context of internal groupoids in any finitely complete category. This gain of generality allows us, for example, to include in the same theory the holomorph of a group: this is possible because a group is a particular groupoid in sets. In fact, it is precisely this easy example the guiding example to describe derivations using holomorphisms and translations as in Sections 5 and 6. Moreover, it is a fact that several definitions, constructions and proofs become more transparent having in mind the set-theoretical case instead of the group-theoretical case. Finally, since internal groupoids are the objects of a 2-category, we can exploit some general 2-categorical facts to define derivations, translations, and the category of embeddings of a groupoids.

Warning : the composition of $f\colon x \to y$ and $g\colon y \to z$ is written $f \cdot g$.

2. The monoid of derivations

In this section, we construct the monoid of $\mathbb{C}$-derivations, for $\mathbb{C}$ an internal groupoid.

We fix, once for all, a category $\mathcal{G}$ with finite limits. The notation for an internal groupoid $\mathbb{C}$ in $\mathcal{G}$ is

$$\mathbb{C} = \left(C_0 \underset{\underset{cod}{\overset{u}{\rightrightarrows}}}{\overset{dom}{\leftleftarrows}} C_1 \xleftarrow{\;\;\circ\;\;} C_1 \times_{C_0} C_1 \;,\; C_1 \xrightarrow{\;()^{-1}\;} C_1 \right)$$

where $C_1 \times_{C_0} C_1$ is the object of "composable pairs", that is

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle dom} \\
C_1 & \xrightarrow{cod} & C_0
\end{array}
$$

is a pullback in $\mathcal{G}$. We also write $\circ^2\colon C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1$ for the diagonal of the commutative square

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{1\times\circ} & C_1 \times_{C_0} C_1 \\
{\scriptstyle \circ\times 1}\downarrow & & \downarrow{\scriptstyle \circ} \\
C_1 \times_{C_0} C_1 & \xrightarrow{\;\;\circ\;\;} & C_1
\end{array}
$$

where

$$
\begin{array}{ccccc}
C_1 & \xleftarrow{\pi_1} & C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_3} & C_1 \\
{\scriptstyle cod}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle dom} \\
C_0 & \xleftarrow{dom} & C_1 & \xrightarrow{cod} & C_0
\end{array}
$$

is a limit in $\mathcal{G}$.

We denote by $Grpd(\mathcal{G})$ the 2-category of internal groupoids, internal functors and internal natural transformations (which always are natural isomorphisms). For $\mathbb{C}, \mathbb{B}$ internal groupoids, we denote by $Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})$ the corresponding hom-category (which is a groupoid), and we write

$$[Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})]_1 \xrightarrow[cod]{dom} [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{B})]_0$$

for its sets of arrows and of objects, together with the domain and the codomain maps. In particular, $Grpd(\mathcal{G})(\mathbb{C}, \mathbb{C})$ is a strict monoidal groupoid : tensor product is composition of internal functors and horizontal composition of internal natural transformations, the unit object is the identity functor on $\mathbb{C}$. As with any strict monoidal category, the map

$$cod \colon [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0$$

is an homomorphism of monoids.

2.1. DEFINITION. *The monoid of $\mathbb{C}$-derivations is the kernel of the codomain map*

$$Der\mathbb{C} = Ker(cod) \to [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [Grpd(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0$$

Explicitly, a $\mathbb{C}$-derivation is a pair (D, d)

$$\mathbb{C} \underset{Id}{\overset{D}{\rightrightarrows}} \Downarrow d \; \mathbb{C}$$

with D an internal functor and d an internal natural transformation. Product of derivations is horizontal composition of natural transformations.

When $\mathcal{G}$ is the category of sets, to give a $\mathbb{C}$-derivation just means to choose, for each object x of $\mathbb{C}$, an arrow

$$d(x) \colon dom(d(x)) \to x$$

with codomain x. This suggests to describe derivations as sections of the codomain arrow.

2.2. PROPOSITION. *To give a $\mathbb{C}$-derivation amounts to give an arrow $d \colon C_0 \to C_1$ such that the diagram*

$$\begin{array}{ccc} & & C_1 \\ & \overset{d}{\nearrow} & \downarrow cod \\ & (1) & \\ C_0 & \underset{1}{\longrightarrow} & C_0 \end{array}$$

commutes.

Proof. Such an arrow d given, we have to construct an internal functor $D \colon \mathbb{C} \to \mathbb{C}$ in such a way that d becomes an internal natural transformation $d \colon D \Rightarrow Id$. The following picture explains the set-theoretical idea behind the construction of D.

$$
\begin{array}{ccc}
x & D_0(x) = dom(d(x)) \xrightarrow{\ d(x)\ } x \\
\Big\downarrow a \quad \longmapsto \quad \quad D_1(a) \Big\vdots \quad\quad\quad\quad \Big\downarrow a \\
y & D_0(y) = dom(d(y)) \xleftarrow{\ d(y)^{-1}\ } y
\end{array}
$$

It suffices now to internalize this idea:

- On objects, the functor $D \colon \mathbb{C} \to \mathbb{C}$ is defined by

$$ D_0 \colon \ C_0 \xrightarrow{\ d\ } C_1 \xrightarrow{\ dom\ } C_0 $$

- As far as arrows are concerned, we consider the diagram

$$
\begin{array}{ccccc}
C_0 & \xleftarrow{\ dom\ } & C_1 & \xrightarrow{\ cod\ } C_0 \xrightarrow{\ d\ } C_1 \\
\ \downarrow d & & \ \downarrow 1 & \quad\quad\quad \downarrow ()^{-1} \\
C_1 \xrightarrow[cod]{} C_0 & \xleftarrow[dom]{} & C_1 & \xrightarrow[cod]{} C_0 \xleftarrow[dom]{} C_1
\end{array}
$$

By equation (1), this diagram commutes, and we get a unique factorization of the projective cone through the object of composable triples, say

$$ \overline{d} = \langle dom \cdot d, 1, cod \cdot d \cdot (\ )^{-1} \rangle \colon C_1 \to C_1 \times_{C_0} C_1 \times_{C_0} C_1 $$

Finally, the functor $D \colon \mathbb{C} \to \mathbb{C}$ is defined on arrows by

$$ D_1 \colon \ C_1 \xrightarrow{\ \overline{d}\ } C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\ \circ^2\ } C_1 $$

$\blacksquare$

We wish now to describe explicitly the operations in $Der\mathbb{C}$ using Proposition 2.2:

- The unit in $Der\mathbb{C}$ is $u \colon C_0 \to C_1$.

- The multiplication in $Der\mathbb{C}$ is the internal version of

$$ z \xrightarrow{\ d_1(y)\ } y = dom(d_2(x)) \xrightarrow{\ d_2(x)\ } x $$

(In other words, Whitehead product of derivations is just the internal composition in $\mathbb{C}$.) This means that, given two derivations $d_1, d_2 \colon C_0 \to C_1$, we start constructing the arrow

$$ d_1 \star d_2 = \langle d_2 \cdot dom \cdot d_1, d_2 \rangle \colon C_0 \to C_1 \times_{C_0} C_1 $$

and we get the product of d_1 and d_2 by composing internally

$$ d_1 \otimes d_2 \colon \ C_0 \xrightarrow{\ d_1 \star d_2\ } C_1 \times_{C_0} C_1 \xrightarrow{\ \circ\ } C_1 $$

2.3. EXAMPLE. When $\mathcal{G}$ is the category of groups, we recapture the classical notion of derivation. Indeed, it is well-known that to a crossed module of groups

$$(H \xrightarrow{\ \partial\ } G \xrightarrow{\ \varphi\ } AutH)$$

we can associate an internal groupoid $\mathbb{C}$, with

$$C_0 = G, \ C_1 = H \rtimes_\varphi G, \ m((a,x),(b,y)) = (a+b,y)$$

$$cod(a,x) = x, \ dom(a,x) = \partial(a)x, \ u(x) = (0,x)$$

(see [4, 13, 15]). Moreover, $\mathbb{C}$-derivations in the sense of Definition 2.1 are in bijection with derivations of G in H : a morphism $d\colon C_0 \to C_1$ is a $\mathbb{C}$-derivation precisely when its second component is the identity on C_0 and its first component is a derivation of G in H. (Let us recall here also the converse construction, which is needed later. Given an internal groupoid $\mathbb{C}$ in groups, we get a crossed module with $G = C_0$ and $H = Ker(cod)$; the map ∂ is the restriction of dom to H, and the action of G on H is given by $x \cdot a = u(x) + a - u(x)$.)

3. The group of regular derivations

In this section we characterize the invertible (or regular) elements of the monoid $Der\mathbb{C}$.

From Definition 2.1, we get three morphisms of monoids:

- $\mathcal{U}\colon Der\mathbb{C} \to [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0 \quad (D,d) \mapsto (D\colon \mathbb{C} \to \mathbb{C})$

- $(\ )_0\colon Der\mathbb{C} \to EndC_0 \quad (D,d) \mapsto (D_0\colon C_0 \to C_0)$

- $(\ )_1\colon Der\mathbb{C} \to EndC_1 \quad (D,d) \mapsto (D_1\colon C_1 \to C_1)$

As with any morphism of monoids, these morphisms restrict to the groups of invertible elements:

$$
\begin{array}{ccc}
Der\mathbb{C} \xrightarrow{\ \mathcal{U}\ } [Grpd(\mathcal{G})(\mathbb{C},\mathbb{C})]_0 & \quad Der\mathbb{C} \xrightarrow{\ ()_0\ } EndC_0 & \quad Der\mathbb{C} \xrightarrow{\ ()_1\ } EndC_1 \\
\uparrow \qquad\qquad \uparrow & \uparrow \qquad\quad \uparrow & \uparrow \qquad\quad \uparrow \\
Der^*\mathbb{C} \dashrightarrow [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0 & Der^*\mathbb{C} \dashrightarrow Aut\,C_0 & Der^*\mathbb{C} \dashrightarrow Aut\,C_1
\end{array}
$$

where $Grpd(\mathcal{G})^*$ is the sub-2-category of $Grpd(\mathcal{G})$ of those internal functors which are isomorphisms. In fact, more is true: the previous diagrams are pullbacks. This is a corollary of the following general fact.

3.1. LEMMA. *Let*

$$\mathbb{C} \xrightarrow[G]{\overset{F}{\Downarrow \alpha}} \mathbb{B}$$

be a 2-cell in $Grpd(\mathcal{G})$. The following conditions are equivalent:

1. *α is invertible with respect to the horizontal composition;*

2. *$F, G \colon \mathbb{C} \to \mathbb{B}$ are in $Grpd(\mathcal{G})^*$;*

3. *$F_1, G_1 \colon C_1 \to B_1$ are isomorphisms in $\mathcal{G}$;*

4. *$F_1 \colon C_1 \to B_1$ and $G_0 \colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$;*

5. *$G_1 \colon C_1 \to B_1$ and $F_0 \colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$.*

Proof. Implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious.

$3 \Rightarrow 4$: If G_1 is an isomorphism, then G_0 also is an isomorphism, with $G_0^{-1} = u \cdot G_1^{-1} \cdot dom$. Same for $3 \Rightarrow 5$. This argument also gives implication $3 \Rightarrow 2$, because if (F_1, F_0) is an internal functor with F_1 and F_0 invertible, then (F_1^{-1}, F_0^{-1}) also is an internal functor.

$4 \Rightarrow 3$: One has to internalize the following set-theoretical argument : if $g \colon a \to b$ is in B_1, then one defines $G_1^{-1}(g) = F_1^{-1}(\alpha(x) \cdot g \cdot \alpha(y)^{-1})$, where $x = G_0^{-1}(a)$ and $y = G_0^{-1}(b)$.

$2 \Rightarrow 1$: This implication holds in any 2-category : assume that the 2-cell α is invertible w.r.t. vertical composition, and let $F \dashv F^*, G \dashv G^*$ be adjunctions, with units and counits given by $\eta \colon Id_{\mathbb{C}} \to F \circ F^*, \epsilon \colon F^* \circ F \to Id_{\mathbb{B}}, \gamma \colon Id_{\mathbb{C}} \to G \circ G^*, \beta \colon G^* \circ G \to Id_{\mathbb{B}}$. Using triangular identities, one checks that the following diagrams commute

$$
\begin{array}{ccc}
F \cdot F^* \xrightarrow{\alpha \circ (\alpha^{-1})^*} G \cdot G^* & \qquad & F^* \cdot F \xrightarrow{(\alpha^{-1})^* \circ \alpha} G^* \cdot G \\
\eta \uparrow \qquad \qquad \uparrow \gamma & & \epsilon \downarrow \qquad \qquad \downarrow \beta \\
Id_{\mathbb{C}} \xrightarrow{\quad 1 \quad} Id_{\mathbb{C}} & & Id_{\mathbb{B}} \xrightarrow{\quad 1 \quad} Id_{\mathbb{B}}
\end{array}
$$

where $(\alpha^{-1})^*$ is defined by the following composition

$$F^* = F^* \cdot Id_{\mathbb{C}} \xrightarrow{1 \circ \gamma} F^* \cdot G \cdot G^* \xrightarrow{1 \circ \alpha^{-1} \circ 1} F^* \cdot F \cdot G^* \xrightarrow{\epsilon \circ 1} Id_{\mathbb{B}} \cdot G^* = G^*$$

In particular, if F and G are isomorphisms, one can chose η, ϵ, γ and β to be identities, and the proof is complete. ∎

3.2. COROLLARY. *Let (D, d) be a $\mathbb{C}$-derivation. The following conditions are equivalent:*

1. *(D, d) is a regular derivation;*

2. *$D \colon \mathbb{C} \to \mathbb{C}$ is in $Grpd(\mathcal{G})^*$;*

3. *$D_0 \colon C_0 \to C_0$ is an isomorphism (i.e. $C_0 \xrightarrow{d} C_0 \xrightarrow{dom} C_1$ is an isomorphism);*

4. *$D_1 \colon C_1 \to C_1$ is an isomorphism.*

3.3. EXAMPLE. When $\mathcal{G}$ is the category of groups and $\mathbb{C}$ is the internal groupoid associated with a crossed module $H \to G \to AutH$ as in Example 2.3, the previous corollary extends the following characterization of regular derivations, due to Whitehead [25]:

There are morphisms of monoids $\sigma\colon Der(G, H) \to EndG : \sigma_d(x) = \partial(d(x))x$ and $\theta\colon Der(G, H) \to EndH : \theta_d(a) = d(\partial(a)) + a$. Moreover, a derivation d is invertible iff $\sigma_d \in AutG$ iff $\theta_d \in AutH$.

Our definition of derivation also explains why the group of regular derivations $Der^*(G, H)$ enters in the construction of Norrie's actor of a crossed module (cf. [23], see also Theorem 3.3 in [9]). In fact, for any internal groupoid $\mathbb{C}$ in any finitely complete category $\mathcal{G}$, the data

$$Act\mathbb{C}\colon \begin{cases} Der^*\mathbb{C} \to [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 & (D, d) \mapsto (D\colon \mathbb{C} \to \mathbb{C}) \\ \\ [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 \times Der^*\mathbb{C} \to Der^*\mathbb{C} & F, (D, d) \mapsto F_0 \cdot d \cdot F_1^{-1} \end{cases}$$

define a crossed module of groups: $Act\mathbb{C}$ precisely is the crossed module associated with $Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$, which is an internal groupoid in groups. Recall now that the actor $Act(G, H)$ of a crossed module is a new crossed module intended to recapture, in the category of crossed modules, the idea of "group of automorphisms of a group". If we look at the crossed module $H \to G \to AutH$ as an internal groupoid $\mathbb{C}$ in groups, then the group of automorphisms must be replaced by $Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$, and $Act(G, H)$ is nothing but $Act\mathbb{C}$.

4. Left exactness of $Der\mathbb{C}$

If ($H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH$) is a crossed module of groups, one of the main properties of the group of regular derivations $Der^*(G, H)$ is that, when it is seen as a functor of the second variable, it preserves kernels. Indeed, this allows one to apply the kernel-cokernel lemma for groups, and obtaining in this way the fundamental exact sequence in nonabelian group cohomology. The aim of this section is to study the main properties of $Der\mathbb{C}$ and $Der^*\mathbb{C}$ as functors.

Consider two internal groupoids $\mathbb{C}$ and $\mathbb{C}'$ in $\mathcal{G}$ having the same object of objects, and an internal functor $F\colon \mathbb{C} \to \mathbb{C}'$ which is the identity on objects

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & C_1' \\ {\scriptstyle dom}\Big\downarrow\Big\downarrow{\scriptstyle cod} & & {\scriptstyle dom'}\Big\downarrow\Big\downarrow{\scriptstyle cod'} \\ C_0 & \xrightarrow{F_0 = 1} & C_0 \end{array}$$

Composing with F_1 gives a morphism of monoids

$$DerF\colon Der\mathbb{C} \to Der\mathbb{C}' \qquad C_0 \xrightarrow{d} C_1 \mapsto C_0 \xrightarrow{d} C_1 \xrightarrow{F_1} C_1'$$

and its restrictions to the groups of regular derivations $Der^*F \colon Der^*\mathbb{C} \to Der^*\mathbb{C}'$.

In fact, this construction is a functor

$$Der \colon \mathcal{F}_{C_0} \to Mon$$

where Mon is the category of monoids, and $\mathcal{F}_{C_0}$ is the fibre over C_0 of the functor

$$Grpd(\mathcal{G}) \to \mathcal{G}$$

which associate to an internal groupoid $\mathbb{C}$ its object of objects C_0. Moreover, the functor Der factorizes through the comma category

$$Mon/EndC_0$$

because $Der\mathbb{C}$ is equipped with a canonical morphism

$$Der\mathbb{C} \to EndC_0 \qquad C_0 \xrightarrow{d} C_1 \;\mapsto\; C_0 \xrightarrow{d} C_1 \xrightarrow{dom} C_0$$

(cf. Proposition 2.2). In the same way, using regular derivations instead of arbitrary derivations, we obtain two functors

$$Der^* \colon \mathcal{F}_{C_0} \to Grp/AutC_0 \qquad Der^* \colon \mathcal{F}_{C_0} \to Grp$$

where Grp is the category of groups.

4.1. PROPOSITION.

1. The functor $Der \colon \mathcal{F}_{C_0} \to Mon/EndC_0$ preserves finite limits;

2. The functor $Der \colon \mathcal{F}_{C_0} \to Mon$ preserves equalizers;

3. The functor $Der^* \colon \mathcal{F}_{C_0} \to Grp/AutC_0$ preserves finite limits;

4. The functor $Der^* \colon \mathcal{F}_{C_0} \to Grp$ preserves equalizers.

Proof. The functor $Mon \to Grp$ which associates to a monoid the group of its invertible elements preserves limits, so that points 3 and 4 follow from points 1 and 2. Moreover, the canonical forgetful functor from a comma category to the base category preserves equalizers, so that point 2 follows from point 1. As far as point 1 is concerned, it is enough to give a glance to finite limits in the fibre $\mathcal{F}_{C_0}$.

- The object of arrows of the terminal object in $\mathcal{F}_{C_0}$ is the product $C_0 \times C_0$. Domain and codomain are the projections. Composition $C_0 \times C_0 \times C_0 \to C_0 \times C_0$ is the projection on the first and third components. The inverse $C_0 \times C_0 \to C_0 \times C_0$ is the twist.

- The object of arrows of the equalizer in $\mathcal{F}_{C_0}$ of $F, G\colon \mathbb{C} \to \mathbb{C}'$ is the equalizer in $\mathcal{G}$

$$E \xrightarrow{\ e\ } C_1 \overset{F_1}{\underset{G_1}{\rightrightarrows}} C_1'$$

with domain and codomain given by $dom \cdot e, cod \cdot e$. The rest of the structure is inherited from that of $\mathbb{C}$ using the universal property of E.

- The object of arrows of the product in $\mathcal{F}_{C_0}$ of $\mathbb{C}$ and $\mathbb{C}'$ is the limit L as in the following diagram

$$
\begin{array}{ccc}
& L & \\
{\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\
C_1 \underset{cod}{} & & \underset{dom'}{} C_1' \\
{\scriptstyle dom}\Big\downarrow & & \Big\downarrow{\scriptstyle cod'} \\
C_0 & & C_0
\end{array}
$$

The domain is $dom \cdot p_1 = dom' \cdot p_2$ and the codomain is $cod \cdot p_1 = cod' \cdot p_2$. The rest of the structure is inherited from those of $\mathbb{C}$ and $\mathbb{C}'$ via the universal property of L.

It is now easy to verify that the functor $Der\colon \mathcal{F}_{C_0} \to Mon/EndC_0$ preserves finite limits. Let us look, for example, at the case of products. Consider a pair

$$(d_1, d_2) \in Der\mathbb{C} \times_{EndC_0} Der\mathbb{C}'$$

(which is the product in the comma category $Mon/EndC_0$). Since $cod \cdot d_1 = 1 = cod' \cdot d_2$ and $dom \cdot d_1 = dom' \cdot d_2$, there is a unique arrow $d\colon C_0 \to L$ such that $p_1 \cdot d = d_1$ and $p_2 \cdot d = d_2$. Moreover, $cod \cdot p_1 \cdot d = cod \cdot d_1 = 1$, so that d is a derivation. Conversely, if $d\colon C_0 \to L$ is a derivation, then $cod \cdot p_1 \cdot d = 1 = cod' \cdot p_2 \cdot d$ and $dom \cdot p_1 \cdot d = dom' \cdot p_2 \cdot d$, so that the pair $(p_1 \cdot d, p_2 \cdot d)$ is an element of the pullback $Der\mathbb{C} \times_{EndC_0} Der\mathbb{C}'$. ∎

5. The 2-category of holomorphisms

In this section, we give a different description of 2-cells in $Grpd(\mathcal{G})$. For this, we introduce the notion of holomorphism between two groupoids. Our terminology is justified by Example 5.5.

The set-theoretical idea behind the notion of holomorphism is quite easy: given a 2-cell

$$
\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B} \qquad \Downarrow \alpha
$$

in $Grpd(\mathcal{G})$, then F, G and α itself are completely determined by the map associating to an internal arrow $(a\colon x \to y) \in C_1$ the diagonal $(Fx \to Gy) \in B_1$ of the commutative

square

$$Fx \xrightarrow{\alpha(x)} Gx$$
$$\downarrow{\scriptstyle Fa} \qquad \downarrow{\scriptstyle Ga}$$
$$Fy \xrightarrow[\alpha(y)]{} Gy$$

To make this more precise, we need some preliminary work. Consider two internal groupoids $\mathbb{C}, \mathbb{B}$ and let $h \colon C_1 \to B_1$ be an arrow making commutative the following diagrams

$$
\begin{array}{ccc}
C_1 \xrightarrow{h} B_1 \xrightarrow{dom} B_0 & \qquad & C_1 \xrightarrow{h} B_1 \xrightarrow{cod} B_0 \\
\downarrow{\scriptstyle dom} \quad (2) \quad \uparrow{\scriptstyle dom} & & \downarrow{\scriptstyle cod} \quad (3) \quad \uparrow{\scriptstyle cod} \\
C_0 \xrightarrow{u} C_1 \xrightarrow{h} B_1 & & C_0 \xrightarrow{u} C_1 \xrightarrow{h} B_1
\end{array}
$$

Thanks to conditions (2) and (3), we get two arrows

$$\widehat{h} = \langle \pi_1 \cdot h, \pi_1 \cdot cod \cdot u \cdot h \cdot (\ )^{-1}, \pi_2 \cdot h \rangle \colon C_1 \times_{C_0} C_1 \to B_1 \times_{B_0} B_1 \times_{B_0} B_1$$

$$\widetilde{h} = \langle \pi_1 \cdot h, \pi_2 \cdot h, \pi_3 \cdot h \rangle \colon P \to Q$$

where P and Q are defined by the following limits in $\mathcal{G}$

$$
\begin{array}{ccc}
C_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_3} C_1 & \qquad & B_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_3} B_1 \\
\downarrow{\scriptstyle cod} \quad \downarrow{\scriptstyle \pi_2} \quad \downarrow{\scriptstyle dom} & & \downarrow{\scriptstyle cod} \quad \downarrow{\scriptstyle \pi_2} \quad \downarrow{\scriptstyle dom} \\
C_0 \xleftarrow{cod} C_1 \xrightarrow{dom} C_0 & & B_0 \xleftarrow{cod} B_1 \xrightarrow{dom} B_0
\end{array}
$$

5.1. LEMMA.

1. *Diagram (4) commutes iff diagram (4') commutes*

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\circ} & C_1 \\
\downarrow{\scriptstyle \widehat{h}} & (4) & \downarrow{\scriptstyle h} \\
B_1 \times_{B_0} B_1 \times_{B_0} B_1 & \xrightarrow{\circ^2} & B_1
\end{array}
$$

$$
\begin{array}{ccc}
P & \xrightarrow{\widetilde{h}} & Q \\
{\scriptstyle \langle \pi_1, \pi_2 \cdot (\ )^{-1}, \pi_3 \rangle} \downarrow & (4') & \downarrow {\scriptstyle \langle \pi_1, \pi_2 \cdot (\ )^{-1}, \pi_3 \rangle} \\
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & & B_1 \times_{B_0} B_1 \times_{B_0} B_1 \\
{\scriptstyle \circ^2} \downarrow & & \downarrow {\scriptstyle \circ^2} \\
C_1 & \xrightarrow{h} & B_1
\end{array}
$$

2. Diagram (5) commutes iff diagram (5') commutes

$$
\begin{array}{ccccc}
C_0 & \xrightarrow{u} & C_1 & \xrightarrow{h} & B_1 \\
\downarrow{\scriptstyle u} & & (5) & & \uparrow{\scriptstyle u} \\
C_1 & \xrightarrow{h} & B_1 & \xrightarrow{dom} & B_0
\end{array}
\qquad
\begin{array}{ccccc}
C_0 & \xrightarrow{u} & C_1 & \xrightarrow{h} & B_1 \\
\downarrow{\scriptstyle u} & & (5') & & \uparrow{\scriptstyle u} \\
C_1 & \xrightarrow{h} & B_1 & \xrightarrow{cod} & B_0
\end{array}
$$

Proof. Part 2 being quite obvious, let us concentrate on part 1. If the base category $\mathcal{G}$ is the category of sets and $a\colon x \to y, b\colon z \to y, c\colon z \to w$ are elements of C_1, condition (4') means that $h(a \cdot b^{-1} \cdot c) = h(a) \cdot h(b)^{-1} \cdot h(c)$. Condition (4) expresses the special case of condition (4') where $z = y$ and $b = 1_y$. It is therefore clear that $(4') \Rightarrow (4)$. Conversely, assume that h satisfies (4) and put

$$
\delta_h\colon C_1 \to B_1 \qquad \delta_h(\, x \xrightarrow{a} y \,) = h(a) \cdot h(1_y)^{-1}\,.
$$

Clearly δ_h preserves units. Moreover, (4) immediately implies that δ_h preserves composition too. Therefore,

$$
h(a \cdot b^{-1} \cdot c) = \delta_h(a \cdot b^{-1} \cdot c) \cdot h(1_w) = \delta_h(a) \cdot \delta_h(b)^{-1} \cdot \delta_h(c) \cdot h(1_w) =
$$

$$
= h(a) \cdot h(1_y)^{-1} \cdot (h(b) \cdot h(1_y)^{-1})^{-1} \cdot h(c) \cdot h(1_w)^{-1} \cdot h(1_w) = h(a) \cdot h(b)^{-1} \cdot h(c)\,.
$$

This concludes the proof when $\mathcal{G}$ is the category of sets. Following Metatheorem 0.1.3 in [1], the result holds for any finitely complete category $\mathcal{G}$. ∎

5.2. DEFINITION. *Consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$.*

1. *An holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is an arrow $h\colon C_1 \to B_1$ making commutative diagram (2), diagram (3), and diagram (4).*

2. *An holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is pointed if it makes commutative diagram (5).*

5.3. LEMMA.

1. *Holomorphisms and pointed holomorphisms are stable under composition in $\mathcal{G}$.*

2. *If $h\colon \mathbb{C} \to \mathbb{B}$ is an holomorphism, then the arrows*

$$
\delta_h\colon\ C_1 \xrightarrow{\langle h,\, cod \cdot u \cdot h \cdot (\,)^{-1}\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1
$$

$$
\gamma_h\colon\ C_1 \xrightarrow{\langle dom \cdot u \cdot h \cdot (\,)^{-1},\, h\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1
$$

   are pointed holomorphisms from $\mathbb{C}$ to $\mathbb{B}$. We call δ_h the domain of h and γ_h the codomain of h.

Proof. The proof is straightforward if $\mathcal{G}$ is the category of sets. One concludes using once again Metatheorem 0.1.3 in [1]. ∎

We are ready to describe the 2-category $Hol(\mathcal{G})$ of holomorphisms:

- Objects are internal groupoids in $\mathcal{G}$.

- If $\mathbb{C}$ and $\mathbb{B}$ are internal groupoids, 1-cells $\mathbb{C} \to \mathbb{B}$ are pointed holomorphisms.

- Composition of 1-cells $f\colon \mathbb{C} \to \mathbb{B}, g\colon \mathbb{B} \to \mathbb{D}$ is the composition of $f\colon C_1 \to B_1$ and $g\colon B_1 \to D_1$ in $\mathcal{G}$.

- If $f, g\colon \mathbb{C} \to \mathbb{B}$ are pointed holomorphisms, 2-cells $f \Rightarrow g$ are holomorphisms $h\colon \mathbb{C} \to \mathbb{B}$ such that $\delta_h = f$ and $\gamma_h = g$.

- Horizontal composition of 2-cells $h\colon f \Rightarrow g\colon \mathbb{C} \to \mathbb{B}, k\colon f' \Rightarrow g'\colon \mathbb{B} \to \mathbb{D}$ is the composition of $h\colon C_1 \to B_1$ and $k\colon B_1 \to D_1$ in $\mathcal{G}$.

- If $h, k\colon \mathbb{C} \to \mathbb{B}$ are holomorphisms with $\gamma_h = \delta_k$, their vertical composition is given by

$$C_1 \xrightarrow{\langle dom \cdot u \cdot h, \, k \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle h, \, cod \cdot u \cdot k \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\circ} B_1$$

- The identity 2-cell on a 1-cell $f\colon \mathbb{C} \to \mathbb{B}$ is f itself.

5.4. EXAMPLE. We can consider a group G as a groupoid (in sets) with just one object, and having the elements of G as arrows. Group homomorphisms correspond then to internal functors. If $f, g\colon G \to H$ are group homomorphisms, a natural transformation $h\colon f \Rightarrow g$ is just an element $h_* \in H$ such that, for all $a \in G$, one has $f(a) + h_* = h_* + g(a)$. We can therefore define a map $h\colon G \to H$ by $h(a) = f(a) + h_*$, so that $h(0) = h_*$. Such a map satisfies the equation $h(a + c) = h(a) - h(0) + h(c)$, which is also equivalent to the equation $h(a - b + c) = h(a) - h(b) + h(c)$ (compare with Lemma 5.1). A map satisfying these equivalent conditions is called a group holomorphism (see, for example, Section IV.1 in [20]). Conversely, an holomorphism $h\colon G \to H$ is an homomorphism precisely when it is pointed, that is when $h(0) = 0$. We can therefore construct two homomorphisms from an holomorphism h :

$$\delta_h\colon G \to H , \quad \delta_h(a) = h(a) - h(0) \quad ; \quad \gamma_h\colon G \to H , \quad \gamma_h(a) = -h(0) + h(a)$$

The element $h(0)$ gives then a natural transformation $h(0)\colon \delta_h \Rightarrow \gamma_h$ (compare with Lemma 5.3).

Because of the way holomorphisms compose, we have the following fact.

5.5. COROLLARY. *An holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is invertible with respect to horizontal composition iff $h\colon C_1 \to B_1$ is an isomorphism in $\mathcal{G}$.*

As announced at the beginning of this section, $Hol(\mathcal{G})$ provides an equivalent description of $Grpd(\mathcal{G})$. In fact, we have the following result.

5.6. PROPOSITION. *There is a 2-functor $\epsilon\colon Hol(\mathcal{G}) \to Grpd(\mathcal{G})$ which is the identity on objects and an isomorphism on hom-categories. The 2-functor ϵ restricts to the sub-2-categories of isomorphisms $Hol(\mathcal{G})^* \to Grpd(\mathcal{G})^*$.*

Proof. If $f\colon \mathbb{C} \to \mathbb{B}$ is a pointed holomorphism, we get an internal functor $\epsilon(f) = (F_1, F_0)\colon \mathbb{C} \to \mathbb{B}$ by $F_1 = f\colon C_1 \to B_1$ and $F_0 = u \cdot f \cdot dom\colon C_0 \to B_0$.

If $h\colon \mathbb{C} \to \mathbb{B}$ is an holomorphism, we get an internal natural transformation $\epsilon(h)\colon \epsilon(\delta_h) \Rightarrow \epsilon(\gamma_h)$ by $\epsilon(h) = u \cdot h\colon C_0 \to C_1 \to B_1$.

Conversely, if $\alpha\colon F = (F_1, F_0) \Rightarrow G = (G_1, G_0)\colon \mathbb{C} \to \mathbb{B}$ is an internal natural transformation (with $\alpha\colon C_0 \to B_1$), we get an holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ by

$$C_1 \xrightarrow{\langle F_1, \, cod \cdot \alpha \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle dom \cdot \alpha, \, G_1 \rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

Details are routine and are left to the reader. ∎

6. Translations

In this section, we specialize the notion of holomorphism to get a different description of derivations in terms of what we call translations. This name is justified by Example 6.4.

6.1. DEFINITION. *The monoid of $\mathbb{C}$-translations is the kernel of the codomain map*

$$Tr\mathbb{C} = Ker(cod) \to [Hol(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [Hol(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0$$

As we did in Proposition 2.2 with derivations, we give now a more geometrical description of translations. Fix an arrow $t\colon C_1 \to C_1$ such that the diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\ \ t\ \ } & C_1 \\
& {\scriptstyle cod} \searrow \ {\scriptstyle (6)} \ \swarrow {\scriptstyle cod} & \\
& C_1 &
\end{array}
$$

commutes, and consider the factorizations

$$\widehat{t} = \langle dom \cdot u \cdot t, 1 \rangle\colon C_1 \to C_1 \times_{C_0} C_1 \quad , \quad \widetilde{t} = \langle \pi_1 \cdot t, \pi_2 \rangle\colon C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$$

6.2. LEMMA. *Diagram (7) commutes iff diagram (7') commutes*

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\ \ \ t\ \ \ } & C_1 \\
{\scriptstyle \widehat{t}} \searrow & {\scriptstyle (7)} & \nearrow {\scriptstyle \circ} \\
& C_1 \times_{C_0} C_1 &
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\ \ \circ\ \ } & C_1 \\
{\scriptstyle \widetilde{t}} \downarrow & {\scriptstyle (7')} & \downarrow {\scriptstyle t} \\
C_1 \times_{C_0} C_1 & \xrightarrow{\ \ \circ\ \ } & C_1
\end{array}
$$

Proof. Let us sketch the proof in the case $\mathcal{G}$ is the category of sets. The first condition means that, for any arrow $a\colon x \to y$, the diagram

$$
\xymatrix{
& & y \\
& x \ar[ur]_{a} & \\
\ar[uur]^{t(a)} \ar[ur]_{t(1_x)} & &
}
$$

commutes; the second condition means that, for any composable pair of arrows

$$
z \xrightarrow{\ b\ } x \xrightarrow{\ a\ } y
$$

the diagram

$$
\xymatrix{
& & y \\
& x \ar[ur]_{a} & \\
\ar[uur]^{t(ba)} \ar[ur]_{t(b)} & &
}
$$

commutes. Clearly, the first condition is a special case of the second one (take $b = 1_x$). Conversely, using the first condition, both sides of the diagram expressing the second condition are equal to

$$
\xrightarrow{t(1_x)} z \xrightarrow{\ b\ } y \xrightarrow{\ a\ } x
$$

The general case follows using 0.1.3 in [1]. ∎

6.3. PROPOSITION. *To give a $\mathbb{C}$-translation amounts to give an arrow $t\colon C_1 \to C_1$ such that diagram (6) and diagram (7) commute.*

Proof. Routine. ∎

6.4. EXAMPLE. Consider once again a group G as a groupoid $\mathbb{C}$ with a single object. Thanks to Proposition 6.3, a $\mathbb{C}$-translation in the sense of Definition 6.1 is nothing but a map $t\colon G \to G$ such that $t(a) = t(0) + a$ for all $a \in G$. That is, t is the right translation by $t(0)$. Therefore, in this case $Tr\mathbb{C}$ is a group isomorphic to G.

In contrast with the situation described in Example 6.4, the monoid $Tr\mathbb{C}$ in general is not a group.

6.5. COROLLARY. *The group of regular translation $Tr^*\mathbb{C}$ is given by $Tr\mathbb{C} \cap Aut\, C_1$.*

By Proposition 4.7, we get the following corollary.

6.6. COROLLARY. *The 2-functor $\epsilon\colon Hol(\mathcal{G}) \to Grpd(\mathcal{G})$ induces two isomorphisms of monoids $Tr\mathbb{C} \simeq Der\mathbb{C}$ and $Tr^*\mathbb{C} \simeq Der^*\mathbb{C}$.*

We can describe the isomorphism $Tr\mathbb{C} \simeq Der\mathbb{C}$ using Propositions 2.2 and 6.3:

-   Given $(t\colon C_1 \to C_1) \in Tr\mathbb{C}$, we get $(u \cdot t\colon C_0 \to C_1 \to C_1) \in Der\mathbb{C}$.

-   Given $(d\colon C_0 \to C_1) \in Der\mathbb{C}$, we get $C_1 \xrightarrow{\langle dom \cdot d,\, 1 \rangle} C_1 \times_{C_0} C_1 \xrightarrow{\ \circ\ } C_1 \in Tr\mathbb{C}$.

Let us summarize the situation we have so far with the following picture, where the unlabelled vertical arrows are the inclusion of the kernel.

$$
\begin{array}{ccc}
Tr^*\mathbb{C} & \xrightarrow{\ \simeq\ } & Der^*\mathbb{C} \\
\downarrow & & \downarrow \\
[Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 & \xrightarrow{\ \simeq\ } & [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 \\
dom\,\big\downarrow\big\downarrow\,cod & & dom\,\big\downarrow\big\downarrow\,cod \\
[Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0 & \xrightarrow{\ \simeq\ } & [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0
\end{array}
$$

6.7. EXAMPLE. Since $Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})$ is a groupoid in groups, using the constructions described in Example 2.3 we can pass to a crossed module of groups, and then come back to a groupoid isomorphic to $Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})$. Using also the isomorphisms of the previous picture, we get a group isomorphism

$$[Hol(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 \simeq Tr^*\mathbb{C} \rtimes [Grpd(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0$$

If we specialize this isomorphism to the case where $\mathcal{G}$ is the category of sets and $\mathbb{C}$ is the one-object groupoid associated to a group G (Examples 5.4 and 6.4), we get the classical isomorphism

$$Hol\,G \simeq G \rtimes Aut\,G$$

where $Hol\,G$ is the group of bijective holomorphisms from G to G (see [20], Section IV.1).

6.8. EXAMPLE. Let us consider a K-vector space $\mathcal{V}$ as a 1-object (commutative) groupoid with an additional multiplicative structure. It is easy to verify that

$$[Hol^*(\mathcal{V})]_0 \simeq GL(\mathcal{V})$$

i.e. the general linear group of $\mathcal{V}$. It is less obvious to notice that $[Hol^*(\mathcal{V})]_1$ corresponds to the group of affine transformations of $\mathbb{A}$, the affine space of the "points" of $\mathcal{V}$. Moreover, this group is canonically isomorphic to the semidirect product

$$Tr^*\mathcal{V} \rtimes GL(\mathcal{V})$$

where $Tr^*\mathcal{V}\ (\simeq \mathcal{V})$ is indeed the group of translations of $\mathbb{A}$.
More precisely, if a is the affine structure of $\mathbb{A}$ (*i.e.* $a(P,Q) = Q - P$), an affine transformation is a pair of bijections

$$(F : \mathbb{A} \to \mathbb{A}, \varphi : \mathcal{V} \to \mathcal{V})$$

resp. in **Set** and in **Vect$_\mathbf{K}$**, such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{A} \times \mathbb{A} & \xrightarrow{\ a\ } & \mathcal{V} \\
F\times F\,\big\downarrow & & \big\downarrow\,\varphi \\
\mathbb{A} \times \mathbb{A} & \xrightarrow[\ a\ ]{} & \mathcal{V}
\end{array}
$$

The linear map φ is determined by F in the following way: fix a point P and set

$$\varphi_P : \mathbf{v} \mapsto F(P + \mathbf{v}) - F(P).$$

Now, given a bijection $F : \mathbb{A} \to \mathbb{A}$, F induces an affine transformation iff φ_P is linear. This expresses exactly condition of diagram (4). In fact

$$
\begin{aligned}
\varphi_P(\mathbf{v} + \mathbf{w}) &= \varphi_P(\mathbf{v}) + \varphi_P(\mathbf{w}) \\
F(P + \mathbf{v} + \mathbf{w}) - F(P) &= F(P + \mathbf{v}) - F(P) + F(P + \mathbf{w}) - F(P) \\
F(P + \mathbf{v} + \mathbf{w}) &= F(P + \mathbf{v}) - F(P) + F(P + \mathbf{w}) \\
F(\mathbf{v} + \mathbf{w}) &= F(\mathbf{v}) - F(0) + F(\mathbf{w})
\end{aligned}
$$

where in the last line we identify $\mathbb{A}$ with $\mathcal{V}$.

This example may be further generalized to modules, and to not-necessarily bijective maps. Under this perspective, group holomorphisms are group transformations that preserve the affine structure of a group, namely equivalent bi-points.

6.9. EXAMPLE. As in Example 6.7, we have a group isomorphism

$$[Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1 \simeq Der^*\mathbb{C} \rtimes [Grpd(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0$$

This generalizes the isomorphism established by Lue in the case where $\mathcal{G}$ is the category of groups, and groupoids are replaced by crossed modules (see [16], Theorem 9). It is interesting to observe that, in that case, only the analogue of our conditions (2) and (3) are used to define the analogue of $[Hol(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1$. This is because the category of groups is a Mal'cev category (see [1] for the notion of Mal'cev category). In fact, we have the following result.

6.10. LEMMA. *Let $\mathcal{G}$ be a Mal'cev category, and consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$. If an arrow $h \colon C_1 \to B_1$ satisfies conditions (2) and (3), then it is an holomorphism.*

Proof. We shall use set theoretic notations, using once again Metatheorem 0.1.3 in [1]. By results in [6], we know that, if $\mathbb{C}$ is a groupoid in a Mal'cev category, composition is uniquely determined and it corresponds to a (unique) partial associative Mal'cev operation p on C_1. This operation p is defined on those (a, b, c) in C_1 such that $cod(a) = cod(b)$ and $dom(b) = dom(c)$:

and, in this case,

$$p(a, b, c) = a \circ b^{-1} \circ c.$$

Now, given two groupoids $\mathbb{C}, \mathbb{B}$ and (a, b, c) in C_1 such that $cod(a) = cod(b)$ and $dom(b) = dom(c)$, if an arrow $h \colon C_1 \to B_1$ satisfies conditions (2) and (3), $h(a), h(b)^{-1}, h(c)$ are composable, that is $cod(h(a)) = cod(h(b))$ and $dom(h(b)) = dom(h(c))$. This means that

the (unique) partial associative Mal'cev operation p' on B_1 associated to composition in $\mathbb{B}$ is defined on $(h(a), h(b), h(c))$ and h satisfies condition (4') if and only if

$$h(p(a, b, c)) = p'((h(a), h(b), h(c)).$$

But this is true because in any Mal'cev category every relation $\mathcal{R}$ is difunctional, that is $\mathcal{R} \circ \mathcal{R}^{op} \circ \mathcal{R} = \mathcal{R}$. In fact, let $\mathcal{R}$ be the relation on $C_1^2 \times C_1$ given by

$$(a, b)\mathcal{R}c \iff cod(a) = cod(b), \, dom(b) = dom(c) \, \text{and} \, h(p(a, b, c)) = p'((h(a), h(b), h(c)).$$

Now, if $cod(a) = cod(b)$, $dom(b) = dom(c)$,

- $(a, b)\mathcal{R}b$, since $h(p(a, b, b)) = h(a) = p'((h(a), h(b), h(b))$

- $(b, b)\mathcal{R}b$, as above, with $a = b$

- $(b, b)\mathcal{R}cb$, since $h(p(b, b, c)) = h(c) = p'((h(b), h(b), h(c))$

then, by difuntionality, $(a, b)\mathcal{R}c$, that is $h(p(a, b, c)) = p'((h(a), h(b), h(c))$. $\blacksquare$

6.11. EXAMPLE. Observe that, by 2.2 and 3.2, if the domain and codomain maps of an internal groupoid $\mathbb{C}$ are equal, then $\mathbb{C}$-derivations are invertible. This is the case for $\mathbb{C}$ the groupoid in groups associated with a crossed module of the form $H \xrightarrow{0} G \xrightarrow{\varphi} AutH$, where $\varphi \colon G \to AutH$ is a G-module and $0 \colon H \to G$ is the zero-morphism. Indeed, in this case, both domain and codomain coincide with the second projection $\pi_2 \colon H \rtimes G \to G$. Moreover, in this case a classical result (see [20], Proposition IV.2.1) asserts that the group $Der^*\mathbb{C}$ is isomorphic to the group of isomorphisms $t \colon H \rtimes G \to H \rtimes G$ making commutative the following duagrams

$$
\begin{array}{ccc}
H \rtimes G \xrightarrow{\;t\;} H \rtimes G & \qquad & H \xrightarrow{\;i\;} H \rtimes G \\
\quad_{\pi_2}\searrow \quad \downarrow{\pi_2} & & \quad_i\searrow \quad \downarrow{t} \\
\qquad G & & \qquad H \rtimes G
\end{array}
$$

where $i(a) = (a, 1)$. Since the first diagram is precisely diagram (6), and the commutativity of the second diagram is equivalent to the commutativity of diagram (7), this description of $Der^*\mathbb{C}$ is a specialization of the isomorphism $Der^*\mathbb{C} \simeq Tr^*\mathbb{C}$ established in 6.6. Even for an arbitrary crossed module $H \to G \to AutH$, the group $Der^*(G, H)$ can be described as a suitable subgroup of $Aut(H \rtimes G)$, see Proposition 3.5 in [9]. Once again, this description is a particular case of the isomorphism $Der^*\mathbb{C} \simeq Tr^*\mathbb{C}$.

7. The embedding category of an internal groupoid

If ($H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH$) is a crossed module of groups and $F_0 \colon A_0 \to G$ is a group homomorphism, a derivation relative to F_0, or F_0-derivation, is a map $d \colon A_0 \to H$ such

that $d(xy) = d(x) + F_0(x) \cdot d(y)$. Equivalently, an F_0-derivation is a group homomorphism $d \colon A_0 \to H \rtimes G$ such that

$$
\begin{array}{ccc}
 & & H \rtimes G \\
 & \nearrow^{d} & \downarrow^{\pi} \\
A_0 & \xrightarrow[F_0]{} & G
\end{array}
$$

commutes. Relative derivations have been used in nonabelian cohomology of groups, see for example [10, 15, 16], and contain derivations as a special case (take as F_0 the identity morphism).

Relative derivations can be defined in the general context of internal groupoids in a finitely complete category $\mathcal{G}$ in much the same way as we did for derivations in Section 2. In this section, we develop a different approach : for a fixed groupoid $\mathbb{C}$ in $\mathcal{G}$, we construct the category of embeddings $Emb\mathbb{C}$ as the underlying category of a comma 2-category, and we show that derivations and relative derivations can be obtained as particular hom-sets from $Emb\mathbb{C}$. The guiding example, which justifies our terminology, is the category of embeddings of an étale groupoid, introduced in [21] to study the homotopy type of the groupoid.

7.1. Definition.

1. *Let $\mathbb{A}, \mathbb{C}$ be internal groupoids in $\mathcal{G}$. An internal functor $F \colon \mathbb{A} \to \mathbb{C}$ is full and faithful if, for every internal groupoid $\mathbb{X}$, the functor $hom(\mathbb{X}, F) \colon hom(\mathbb{X}, \mathbb{A}) \to hom(\mathbb{X}, \mathbb{C})$ is full and faithful in the usual sense.*

2. *The category of embeddings $Emb\mathbb{C}$ has full and faithful functors $F \colon \mathbb{A} \to \mathbb{C}$ as objects. An arrow from $F \colon \mathbb{A} \to \mathbb{C}$ to $G \colon \mathbb{B} \to \mathbb{C}$ is a pair*

$$
(D \colon \mathbb{A} \to \mathbb{B}, \ \ d \colon D \cdot G \Rightarrow F)
$$

   with D an internal functor and d an internal natural transformation. Composition and identities are the obvious ones.

Clearly, $Emb\mathbb{C}(Id_{\mathbb{C}}, Id_{\mathbb{C}}) = Der\mathbb{C}$. More is true :

7.2. Proposition. *If $F \colon \mathbb{A} \to \mathbb{C}$ is full and faithful, then*
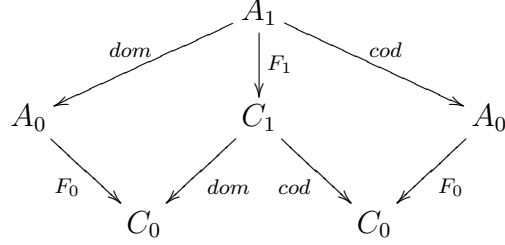
$$
- \cdot F \colon Der\mathbb{A} \to Emb\mathbb{C}(F, F) \qquad
\mathbb{C} \overset{D}{\underset{Id}{\Longrightarrow}} \mathbb{C} \ \mapsto \
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{D} & \mathbb{A} \\
{\scriptstyle F} \searrow & {\scriptstyle \Leftarrow d \circ F} & \swarrow {\scriptstyle F} \\
 & \mathbb{C} &
\end{array}
$$

is an isomorphism of monoids.

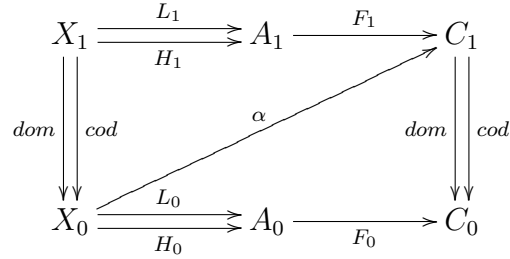Proof. This immediately follows from the definition of full and faithful internal functor. ∎

To recover relative derivations from the category of embeddings is less trivial.

7.3. LEMMA. *An internal functor $F \colon \mathbb{A} \to \mathbb{C}$ is full and faithful iff the diagram*

$$
\begin{array}{ccccc}
& & A_1 & & \\
& \overset{dom}{\swarrow} & \downarrow {\scriptstyle F_1} & \overset{cod}{\searrow} & \\
A_0 & & C_1 & & A_0 \\
{\scriptstyle F_0}\searrow & & \overset{dom}{\swarrow} \quad \overset{cod}{\searrow} & & \swarrow {\scriptstyle F_0} \\
& C_0 & & C_0 &
\end{array}
$$

is a limit diagram in $\mathcal{G}$.

Proof. Assume that the diagram in the statement is a limit diagram. Consider internal functors $L, H \colon \mathbb{X} \to \mathbb{A}$ and an internal natural transformation $\alpha \colon L \cdot F \Rightarrow H \cdot F$. Explicitly

$$
\begin{array}{ccccc}
X_1 & \overset{L_1}{\underset{H_1}{\rightrightarrows}} & A_1 & \overset{F_1}{\longrightarrow} & C_1 \\
{\scriptstyle dom}\downdownarrows{\scriptstyle cod} & {\scriptstyle \alpha}\nearrow & & & {\scriptstyle dom}\downdownarrows{\scriptstyle cod} \\
X_0 & \overset{L_0}{\underset{H_0}{\rightrightarrows}} & A_0 & \overset{F_0}{\longrightarrow} & C_0
\end{array}
$$
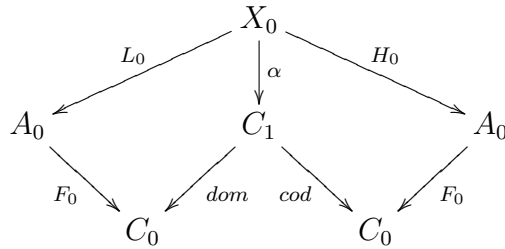
Therefore, $\alpha \cdot dom = L_0 \cdot F_0$ and $\alpha \cdot cod = H_0 \cdot F_0$. From the universal property of the limit A_1, we get a unique arrow $\beta \colon X_0 \to A_1$ such that $\beta \cdot dom = L_0, \beta \cdot cod = H_0$ and $\beta \cdot F_1 = \alpha$. This means that β is an internal (natural) transformation from L to H such that $\beta \circ F = \alpha$. (The naturality of β is expressed by the commutativity of

$$
\begin{array}{ccc}
X_1 & \overset{\langle dom \cdot \beta, H_1 \rangle}{\longrightarrow} & A_1 \times_{A_0} A_1 \\
{\scriptstyle \langle L_1, cod \cdot \beta \rangle}\downarrow & & \downarrow {\scriptstyle \circ} \\
A_1 \times_{A_0} A_1 & \overset{\circ}{\longrightarrow} & A_1
\end{array}
$$

which is easy to check composing with the projections of the limit A_1 and using the naturality of α.)

The converse implication follows from the fact that to give a factorization of a commutative diagram

$$
\begin{array}{ccccc}
& & X_0 & & \\
& \overset{L_0}{\swarrow} & \downarrow {\scriptstyle \alpha} & \overset{H_0}{\searrow} & \\
A_0 & & C_1 & & A_0 \\
{\scriptstyle F_0}\searrow & & \overset{dom}{\swarrow} \quad \overset{cod}{\searrow} & & \swarrow {\scriptstyle F_0} \\
& C_0 & & C_0 &
\end{array}
$$

through the diagram in the statement corresponds to give a (necessarily natural) transformation $\beta\colon L \Rightarrow H$ such that $\beta \circ F = \alpha\colon L \cdot F \Rightarrow H \cdot F$, where L and H are internal functors with discrete domain as in the following diagram

$$
\begin{array}{ccc}
X_0 & \underset{H_0 \cdot u}{\overset{L_0 \cdot u}{\rightrightarrows}} & A_1 \\
1 \Vert 1 & & dom \Vert cod \\
X_0 & \underset{H_0}{\overset{L_0}{\rightrightarrows}} & A_0
\end{array}
$$

∎

7.4. LEMMA. *Let $\mathbb{C}$ be an internal groupoid. The category $Emb\mathbb{C}$ can be described in the following way :*

- *Objects are arrows $F_0\colon A_0 \to C_0$ in $\mathcal{G}$.*

- *A morphism from $F_0\colon A_0 \to C_0$ to $G_0\colon B_0 \to C_0$ is a pair $(D_0\colon A_0 \to B_0, d\colon A_0 \to C_1)$ of arrows in $\mathcal{G}$ such that*

 *commute.*

Proof. Let us concentrate on objects (the argument for arrows is similar). Given an arrow $F_0\colon A_0 \to C_0$, the limit

produces an internal groupoid $\mathbb{A}$ and a full and faithful internal functor $F = (F_1, F_0)\colon \mathbb{A} \to \mathbb{C}$. Conversely, if $F\colon \mathbb{A} \to \mathbb{C}$ is full and faithful, than the unit $u\colon A_0 \to A_1$ and the composition $\circ\colon A_1 \times_{A_0} A_1 \to A_1$ of $\mathbb{A}$ are the unique factorizations through the previous

limit of the following commutative diagrams

$$
\begin{array}{ccccc}
 & & A_0 & & \\
 & {\scriptstyle 1}\swarrow & \downarrow{\scriptstyle F_0 \cdot u} & \searrow{\scriptstyle 1} & \\
A_0 & & C_1 & & A_0 \\
{\scriptstyle F_0}\searrow & \swarrow{\scriptstyle dom} & & {\scriptstyle cod}\searrow & \swarrow{\scriptstyle F_0} \\
 & C_0 & & C_0 &
\end{array}
\qquad
\begin{array}{ccccc}
 & & A_1 \times_{A_0} A_1 & & \\
 & {\scriptstyle \pi_1}\swarrow & \downarrow{\scriptstyle F_1 \times_{F_0} F_1} & \searrow{\scriptstyle \pi_2} & \\
A_1 & & C_1 \times_{C_0} C_1 & & A_1 \\
\downarrow{\scriptstyle dom} & & \downarrow{\scriptstyle \circ} & & \downarrow{\scriptstyle cod} \\
A_0 & & C_1 & & A_0 \\
{\scriptstyle F_0}\searrow & \swarrow{\scriptstyle dom} & & {\scriptstyle cod}\searrow & \swarrow{\scriptstyle F_0} \\
 & C_0 & & C_0 &
\end{array}
$$

∎
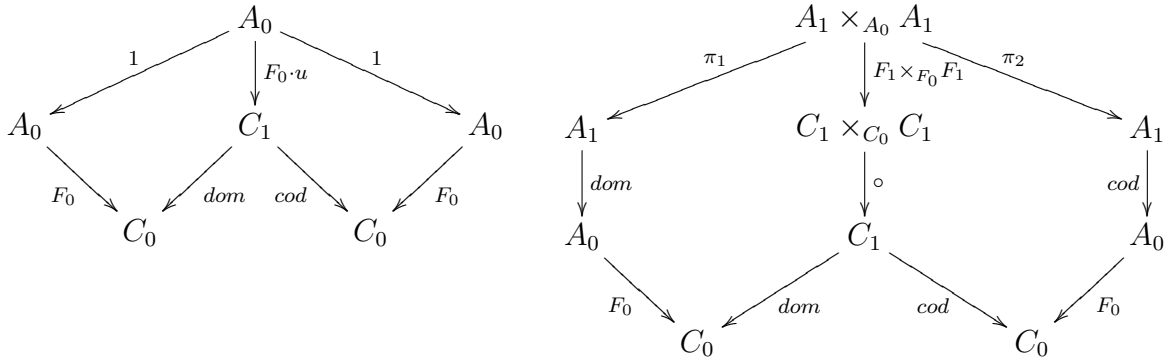
7.5. EXAMPLE. Let $\mathcal{G}$ be the category of topological spaces and local homeomorphisms, and let $\mathbb{C}$ be an internal groupoid in $\mathcal{G}$, that is $\mathbb{C}$ is an étale groupoid. Fix a basis $\mathcal{B}$ of contractible open sets for the space C_0. The category of embeddings $Emb_{\mathcal{B}}\mathbb{C}$ (see [21]) is the full subcategory of $Emb\mathbb{C}$ with objects the inclusions $U \to C_0$, $U \in \mathcal{B}$. Indeed, following 7.4, if G_0 is a monomorphism, then an arrow (D_0, d) is determined by d, in the sense that if D_0 exists, it is unique. (Note that in [21] the role of dom and cod is inverted: the equations defining an arrow (D_0, d) in 7.4 should be written as $d \cdot dom = F_0$ and $d \cdot cod = D_0 \cdot G_0$. Since $\mathbb{C}$ is a groupoid, this makes no difference.)

7.6. EXAMPLE. Consider a crossed module $H \to G \to AutH$, the corresponding internal groupoid $\mathbb{C}$, and a group homomorphism $F_0 \colon A_0 \to G$. The set of F_0-derivations is in bijection with the hom-set $Emb\mathbb{C}(F, Id_{\mathbb{C}})$, where $F \colon \mathbb{A} \to \mathbb{C}$ is the full and faithful internal functor corresponding to $F_0 \colon A_0 \to C_0$ as in the proof of Lemma 7.4. Indeed, following the description of $Emb\mathbb{C}$ given in 7.4, if $G = Id_{\mathbb{C}}$, then D_0 necessarily is $d \cdot dom$.

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