FAITHFUL CALCULUS OF FRACTIONS

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Dedicated to René Guitart on the occasion of his 65th birthday

RÉSUMÉ. Dans cet article, nous développons un argument simple sur les bicatégories de fractions qui montre que, si Σ est la classe des équivalences faibles entre groupoides internes à une catégorie régulière $\mathcal{A}$ qui admet suffisamment d'objets projectifs réguliers, alors la description de $\mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}]$ peut être considérablement simplifiée.

RÉSUMÉ. The aim of this note is to develop a simple argument on bicategories of fractions showing that, if Σ is the class of weak equivalences between groupoids internal to a regular category $\mathcal{A}$ with enough regular projective objects, then the description of $\mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}]$ can be considerably simplified.

1. Introduction

Bicategories of fractions, the 2-dimensional analogue of Gabriel and Zisman's categories of fractions [9], have been introduced by D. Pronk [14] and used mainly to study fractions of 2-categories of internal functors between various kinds of internal structures (internal categories, internal groupoids, internal crossed modules, etc.), see for example [16] for recent applications. Recently, general results on bicategories of fractions of internal functors with respect to internal weak equivalences have been obtained in [1, 10, 15]. In particular, in [1] the bicategory of fractions of crossed modules internal to a semi-abelian category $\mathcal{A}$ has been described in terms of "butterflies". This description generalizes the case where the base category $\mathcal{A}$ is the category of groups, which

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is the case studied by B. Noohi in [11, 12] (see also [2]). It is interesting to notice that bicategories of fractions do not appear explicitly in [11, 12], where the main result is stated in terms of an equivalence between hom-categories

$$\mathcal{B}(A,B) \simeq \mathcal{C}(X,B)$$

where $\mathcal{C}$ is the 2-category of crossed modules of groups, $\mathcal{B}$ is the bicategory of butterflies in groups, and X is a cofibrant replacement of A. In [1, Proposition 8.1], we explain that this equivalence of hom-categories easily follows from the fact that $\mathcal{B}$ is indeed the bicategory of fractions of $\mathcal{C}$ and the fact that the category of groups has enough regular projective objects. Moreover, a general argument on bicategories of fractions, subsuming the previous equivalence, is announced [1, Remark 8.2].

The aim of this note is to fully develop such an argument: we will show that, if the class Σ of arrows to be inverted has a "faithful calculus of fractions", a condition stronger than Pronk's right calculus of fractions, and if $\mathcal{C}$ has enough Σ-projective objects, then the description of the bicategory of fractions $\mathcal{C}[\Sigma^{-1}]$ can be drastically simplified and the equivalence

$$\mathcal{C}[\Sigma^{-1}](A,B) \simeq \mathcal{C}(X,B)$$

becomes almost tautological. The surprise is that, despite the fact that the condition of having a faithful calculus of fractions is a very strong condition (so strong that its 1-dimensional version for categories of fractions is probably totally uninteresting), it is satisfied by the prominent example where $\mathcal{C}$ is the 2-category of groupoids and functors internal to a regular category, and Σ is the class of weak equivalences. Moreover, the fact that $\mathcal{C}$ has enough Σ-projective objects holds if the base category has enough regular projective objects. This covers the case of groups and of Lie algebras studied in [11, 12, 2, 17].

Notation: the composite of $f\colon A \to B$ and $g\colon B \to C$ is written $f \cdot g$.

2. Calculus of fractions

The reader can consult [4] or [6, Chapter 7] for an introduction to Bénabou's notion of bicategory. In this paper, bicategory means bicategory with invertible 2-cells. Moreover, for the sake of readability, we write diagrams and equations as in a 2-category. Let us start with a point of standard terminology

Definition 2.1 Let $f\colon X \to Y$ be a 1-cell in a bicategory $\mathcal{C}$. We say that f is

1. full (faithful) if, for every object $C \in \mathcal{C}$, the functor
$$\mathcal{C}(C,f)\colon \mathcal{C}(C,X) \to \mathcal{C}(C,Y)$$
is full (faithful); in other words, for every 2-cell $\beta\colon h \cdot f \Rightarrow k \cdot f$, there exists at least (at most) a 2-cell $\alpha\colon h \Rightarrow k$ such that $\alpha \cdot f = \beta$;

2. an equivalence if, for every object $C \in \mathcal{C}$, the functor
$$\mathcal{C}(C,f)\colon \mathcal{C}(C,X) \to \mathcal{C}(C,Y)$$
is an equivalence of categories; in other words, there exist a 1-cell $f^*\colon Y \to X$ and two 2-cells $\epsilon_f\colon f^* \cdot f \Rightarrow 1_Y$ and $\eta_f\colon 1_X \Rightarrow f \cdot f^*$.

Remark 2.2

1. If f is full and faithful and there exists $\epsilon_f\colon f^* \cdot f \Rightarrow 1_Y$, then f is an equivalence.

2. If f is an equivalence, it is always possible to choose η_f and ϵ_f so that the usual triangular identities are satisfied:
$$\begin{tikzcd} & f \cdot f^* \cdot f \arrow[dr, "f \cdot \epsilon_f"] & \\ f \arrow[ur, "\eta_f \cdot f"] \arrow[rr, "f"'] & & f \end{tikzcd} \qquad \begin{tikzcd} & f^* \cdot f \cdot f^* \arrow[dr, "\epsilon_f \cdot f^*"] & \\ f^* \arrow[ur, "f^* \cdot \eta_f"] \arrow[rr, "f^*"'] & & f^* \end{tikzcd}$$

3. If $f, g\colon X \to Y$ are equivalences, $\beta\colon f \Rightarrow g$ is a 2-cell, and $(f^*, \eta_f, \epsilon_f)$ and $(g^*, \eta_g, \epsilon_g)$ satisfy the triangular identities, then there exists a unique $\beta^*\colon f^* \Rightarrow g^*$ making commutative the following diagrams:
$$\begin{tikzcd} & 1_X \arrow[dl, "\eta_f"'] \arrow[dr, "\eta_g"] & \\ f \cdot f^* \arrow[rr, "\beta \cdot \beta^*"'] & & g \cdot g^* \end{tikzcd} \qquad \begin{tikzcd} & 1_Y & \\ f^* \cdot f \arrow[ur, "\epsilon_f"] \arrow[rr, "\beta^* \cdot \beta"'] & & g^* \cdot g \arrow[ul, "\epsilon_g"'] \end{tikzcd}$$

4. If f is an equivalence, for every object C the functor

$$\mathcal{C}(f,C)\colon \mathcal{C}(Y,C) \to \mathcal{C}(X,C)$$

is an equivalence of categories (use the triangular identities to check that it is full).

5. If $f\colon X \to Y$ and $g\colon Y \to Z$ are full (faithful) (equivalences), then so is the composite $f \cdot g\colon X \to Z$.

Now we recall from [14] the general definition of bicategory of fractions and we introduce the notion of faithful calculus of fractions.

Definition 2.3 (Pronk) Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. The bicategory of fractions of $\mathcal{C}$ with respect to Σ is a homomorphism of bicategories

$$P_\Sigma\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ such that $\mathcal{F}(s)$ is an equivalence for all $s \in \Sigma$. In other words, for every bicategory $\mathcal{A}$,

$$P_\Sigma \cdot - \, : \, \mathit{Hom}(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \to \mathit{Hom}_\Sigma(\mathcal{C}, \mathcal{A})$$

is a biequivalence of bicategories, where $\mathit{Hom}_\Sigma(\mathcal{C},\mathcal{A})$ is the bicategory of those homomorphisms $\mathcal{F}$ such that $\mathcal{F}(s)$ is an equivalence for all $s \in \Sigma$.

Definition 2.4 Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. The class Σ has a faithful calculus of fractions if the following conditions hold:

FF1. Σ contains all equivalences;

FF2. Given 1-cells $f\colon X \to Y$ and $g\colon Y \to Z$ with $g \in \Sigma$, then $f \cdot g \in \Sigma$ iff $f \in \Sigma$;

FF3. For every
$$\begin{tikzcd} & C \arrow[d, "g\in\Sigma"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
there exists
$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'\in\Sigma"'] \arrow[dr, phantom, "\Rightarrow"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

FF4. If there exists a 2-cell $f \Rightarrow g$, then $f \in \Sigma$ iff $g \in \Sigma$;

FF5. Σ is contained in the class of full and faithful 1-cells.

Remark 2.5 In (FF3), if $f \in \Sigma$, then $f' \in \Sigma$. Indeed, $g', f \in \Sigma$, so that, by (FF2), $g' \cdot f \in \Sigma$ and then, by (FF4), $f' \cdot g \in \Sigma$. Since $g \in \Sigma$, (FF2) implies now $f' \in \Sigma$.

It is easy to compare the conditions defining a faithful calculus of fractions with those defining a right calculus of fractions in the sense of [14].

Proposition 2.6 *Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. If Σ has a faithful calculus of fractions, then it has a right calculus of fractions.*

Proof. We have to check the following condition:

RF. For every $\alpha \colon f \cdot w \Rightarrow g \cdot w$ with $w \in \Sigma$, there exist $v \in \Sigma$ and $\beta \colon v \cdot f \Rightarrow v \cdot g$ such that $v \cdot \alpha = \beta \cdot w$, and for any other $v' \in \Sigma$ and $\beta' \colon v' \cdot f \Rightarrow v' \cdot g$ such that $v' \cdot \alpha = \beta' \cdot w$, there exist u, u' and $\varepsilon \colon u \cdot v \Rightarrow u' \cdot v'$ such that $u \cdot v \in \Sigma$ and

$$\begin{tikzcd} u \cdot v \cdot f \arrow[r, "u \cdot \beta"] \arrow[d, "\varepsilon \cdot f"'] & u \cdot v \cdot g \arrow[d, "\varepsilon \cdot g"] \\ u' \cdot v' \cdot f \arrow[r, "u' \cdot \beta'"'] & u' \cdot v' \cdot g \end{tikzcd}$$

commutes.

As far as the existence of (v, β) is concerned, we can take $v = 1_X \in \Sigma$ and, since w is full, there exists $\beta \colon f \Rightarrow g$ such that $\beta \cdot w = \alpha$.
Let now (v, β) and (v', β') be as in condition (RF); by (FF3), there exists $\epsilon \colon u \cdot v \Rightarrow u' \cdot v'$ with $u \in \Sigma$ and then $u \cdot v \in \Sigma$. It remains to show that the diagram in condition (RF) commutes. Since w is faithful, it is enough to check the commutativity of the diagram obtained by composing with w

$$\begin{tikzcd} u \cdot v \cdot f \cdot w \arrow[r, "u \cdot \beta \cdot w"] \arrow[d, "\varepsilon \cdot f \cdot w"'] & u \cdot v \cdot g \cdot w \arrow[d, "\varepsilon \cdot g \cdot w"] \\ u' \cdot v' \cdot f \cdot w \arrow[r, "u' \cdot \beta' \cdot w"'] & u' \cdot v' \cdot g \cdot w \end{tikzcd}$$

and this is obvious because we can replace $u \cdot \beta \cdot w$ by $u \cdot v \cdot \alpha$ and $u' \cdot \beta' \cdot w$ by $u' \cdot v' \cdot \alpha$. ∎

3. Σ-projective objects and Σ-covers

Definition 3.1 Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$.

1. An object X is Σ-projective if, for every 1-cell $s\colon A \to B$ in Σ, the functor
$$\mathcal{C}(X, s)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B)$$
is essentially surjective; in other words, for every
$$\begin{tikzcd} & X \arrow[d, "f"] \\ A \arrow[r, "s\in\Sigma"'] & B \end{tikzcd}$$
there exists
$$\begin{tikzcd} & X \arrow[dl, "f'"'] \arrow[d, "f"] \arrow[d, phantom, shift right=3, "\Rightarrow"] \\ A \arrow[r, "s"'] & B \end{tikzcd}$$

2. A Σ-cover of an object A is a 1-cell $a\colon X \to A$ in Σ with X a Σ-projective object.

3. We say that $\mathcal{C}$ has enough Σ-projective objects if each object has a Σ-cover.

Remark 3.2 Assume that Σ is contained in the class of full and faithful 1-cells.

1. If $s\colon A \to X$ is in Σ and X is a Σ-projective object, then s is an equivalence. Indeed, use condition 3.1.1 with $f = 1_X$ to get s^* and ϵ_s, and conclude by Remark 2.2.1.

2. If a Σ-cover of an object exists, then it is unique up to an essentially unique equivalence.

In Example 3.5, we will state that the class of weak equivalences between groupoids internal to a regular category has a faithful calculus of fractions. The reader can consult [7, Chapter 2] for an introduction to regular categories (in the sense of M. Barr [3]), and [6, Chapter 8] for basic facts about internal category theory. If $\mathcal{A}$ is a category with finite limits, we denote by $\mathrm{Grpd}(\mathcal{A})$ the 2-category of groupoids, functors and natural transformations internal to $\mathcal{A}$. The notions of essentially surjective and of weak equivalence for internal functors come from [8].

Definition 3.3 (Bunge-Paré) Let $\mathcal{A}$ be a regular category and let

$$\begin{tikzcd} A_1 \arrow[r, "F_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ A_0 \arrow[r, "F_0"'] & B_0 \end{tikzcd}$$

be a functor between groupoids in $\mathcal{A}$. The functor (F_1, F_0) is:

1. essentially surjective (on objects) if

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

is a regular epimorphism, where t_2 is defined by the following pullback

$$\begin{tikzcd} A_0 \times_{F_0,d} B_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] & B_1 \arrow[d, "d"] \\ A_0 \arrow[r, "F_0"'] & B_0 \end{tikzcd}$$

2. a weak equivalence if it is full and faithful and essentially surjective.

Remark 3.4 With the notation of Definition 3.3. A functor (F_1, F_0) is:

1. full and faithful iff the following diagram is a limit diagram

$$\begin{tikzcd} & & A_1 \arrow[dll, "d"'] \arrow[d, "F_1"] \arrow[drr, "c"] & & \\ A_0 \arrow[dr, "F_0"'] & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dl, "F_0"] \\ & B_0 & & B_0 & \end{tikzcd}$$

2. an equivalence iff it is full and faithful and

$$A_0 \times_{F_0,d} B_1 \xrightarrow{t_2} B_1 \xrightarrow{c} B_0$$

is a split epimorphism.

Example 3.5 Let $\mathcal{A}$ be a regular category and Σ the class of weak equivalences in the 2-category Grpd($\mathcal{A}$).

1. Σ has a faithful calculus of fractions.
   The proof can be reconstructed by examining the proofs of Proposition 4.5 and Proposition 5.5 in [17]. For the reader's convenience we reproduce here some points; we refer to [17] for more details.
   - Condition (FF1) immediately follows from Remark 3.4.2.
   - Condition (FF2): consider two internal functors $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$

$$\begin{tikzcd} A_1 \arrow[r, "F_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & B_1 \arrow[r, "G_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & C_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ A_0 \arrow[r, "F_0"'] & B_0 \arrow[r, "G_0"'] & C_0 \end{tikzcd}$$

   • If F and G are essentially surjective, so is the composite $F \cdot G$: consider the following pullbacks

$$\begin{tikzcd} A_0 \times_{F_0,d} B_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] & B_1 \arrow[d, "d"] \\ A_0 \arrow[r, "F_0"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} B_0 \times_{G_0,d} C_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] & C_1 \arrow[d, "d"] \\ B_0 \arrow[r, "G_0"'] & C_0 \end{tikzcd}$$

$$\begin{tikzcd} A_0 \times_{F_0 \cdot G_0,d} C_1 \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & C_1 \arrow[d, "d"] \\ A_0 \arrow[r, "F_0 \cdot G_0"'] & C_0 \end{tikzcd}$$

   and the commutative diagram (where m is the internal composition in $\mathbb{C}$)

$$\begin{tikzcd}
A_0 \times_{F_0,d} B_1 \times_{G_1 \cdot c,d} C_1 \arrow[rr, "t_2 \times 1"] \arrow[d, "1 \times G_1 \times 1"'] & & B_1 \times_{G_1 \cdot c,d} C_1 \arrow[r, "c \times 1"] & B_0 \times_{G_0,d} C_1 \arrow[d, "t_2"] \\
A_0 \times_{F_0 \cdot G_0,d} C_1 \times_{c,d} C_1 \arrow[d, "1 \times m"'] & & & C_1 \arrow[d, "c"] \\
A_0 \times_{F_0 \cdot G_0,d} C_1 \arrow[rr, "\tau_2"'] & & C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

In a regular category, regular epimorphisms are closed under composition and finite products; moreover, if a composite is a regular epimorphism then the last component is a regular epimorphism. Therefore, from the previous diagram we deduce that $\tau_2 \cdot c$ is a regular epimorphism, as needed.

• If $F \cdot G$ is essentially surjective and G is full and faithful, then F is essentially surjective: consider one more pullback

$$\begin{tikzcd}
Q \arrow[rr, "\lambda_2"] \arrow[d, "\lambda_1"'] & & B_0 \arrow[d, "G_0"] \\
A_0 \times_{F_0 \cdot G_0,d} C_1 \arrow[r, "\tau_2"'] & C_1 \arrow[r, "c"'] & C_0
\end{tikzcd}$$

We have that λ_2 is a regular epimorphism because, by assumption, $\tau_2 \cdot c$ is a regular epimorphism and regular epimorphisms are pullback stable in any regular category. Since G is full and faithful, by Remark 3.4.1 we get $\lambda \colon Q \to B_1$ such that $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$, $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$ and $\lambda \cdot c = \lambda_2$. From the first equation on λ, we get $\mu \colon Q \to A_0 \times_{F_0,d} B_1$ such that $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$ and $\mu \cdot t_2 = \lambda$. Finally, $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$, so that $t_2 \cdot c$ is a regular epimorphism, as needed.

- The stability of regular epimorphisms under pullbacks gives also that Σ is stable under bipullbacks (in the sense of bilimits introduced in [5]). This immediately implies condition (FF3).
- Condition (FF4) is a simple exercise and condition (FF5) is obvious by definition of weak equivalence.

Recall that an object X_0 of the base category $\mathcal{A}$ is regular projective if the functor $\mathcal{A}(X_0, -) \colon \mathcal{A} \to \mathrm{Set}$ preserves regular epimorphisms. The category $\mathcal{A}$ has enough regular projective objects if for every object $A_0 \in \mathcal{A}$ there exists a regular epimorphism $X_0 \to A_0$ with X_0 regular projective. Examples of regular categories with enough regular projective objects abound: monadic categories over a power of Set and their regular epireflective subcategories are of this kind. In particular, algebraic categories, varieties and quasi-varieties of universal algebras are of this kind (see for example [13]), as well as presheaf categories and categories of separated presheaves.

2. If $\mathcal{A}$ has enough regular projective objects, then $\mathrm{Grpd}(\mathcal{A})$ has enough Σ-projective objects.
   For this, start with an internal groupoid and a regular epimorphism S_0

$$\begin{tikzcd} & A_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ X_0 \arrow[r, "S_0"'] & A_0 \end{tikzcd}$$

   with X_0 a regular projective object. Consider the limit diagram

$$\begin{tikzcd} & & X_1 \arrow[dll, "d"'] \arrow[d, "S_1"] \arrow[drr, "c"] & & \\ X_0 \arrow[dr, "S_0"'] & & A_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & X_0 \arrow[dl, "S_0"] \\ & A_0 & & A_0 & \end{tikzcd}$$

   The graph $d, c\colon X_1 \rightrightarrows X_0$ inherits a structure of groupoid from that of $d, c\colon A_1 \rightrightarrows A_0$, and the functor (F_1, F_0) is a weak equivalence. Indeed, it is full and faithful by construction, and it is essentially surjective because in

$$X_0 \times_{S_0, d} A_1 \xrightarrow{t_2} A_1 \xrightarrow{c} A_0$$

   t_2 is a regular epimorphism (because S_0 is a regular epimorphism) and c is a split epimorphism. Finally, since X_0 is regular projective, by Remark 3.4.2 every weak equivalence with codomain

$X_1 \rightrightarrows X_0$ is an equivalence. Since weak equivalences are stable under bipullbacks, this is enough to ensure the Σ-projectivity of $X_1 \rightrightarrows X_0$.

4. The bicategory of fractions

4.1 Let $\mathcal{C}$ be a bicategory and Σ any class of 1-cells in $\mathcal{C}$. We can construct a new bicategory

$$\mathcal{C}[\Sigma^*]$$

having Σ-covers as objects and, as hom-categories,

$$\mathcal{C}[\Sigma^*](a\colon X \to A, b\colon Y \to B) = \mathcal{C}(X, Y)$$

with identities and horizontal and vertical compositions given by those of $\mathcal{C}$.

Remark 4.2

1. If $\mathcal{C}$ is a 2-category, then $\mathcal{C}[\Sigma^*]$ is a 2-category as well.

2. If $b\colon Y \to B$ is full and faithful, then the functor $\mathcal{C}(X, b)$ is full and faithful, and it is essentially surjective because X is Σ-projective, so that it induces an equivalence of categories

$$\mathcal{C}[\Sigma^*](a\colon X \to A, b\colon Y \to B) \simeq \mathcal{C}(X, B)$$

4.3 Under the assumption that the class Σ has a right calculus of fractions, the bicategory of fractions $\mathcal{C}[\Sigma^{-1}]$ has been described in [14]: objects are those of $\mathcal{C}$, 1-cells and pre-2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "{(w,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "{\Downarrow (u_1,u_2,\alpha_1,\alpha_2)}"] & B \end{tikzcd}$$

are depicted in the following diagram

$$\begin{tikzcd}
 & C \arrow[dl, "w"'] \arrow[dr, "f"] & \\
A \arrow[r, phantom, "\alpha_1\Downarrow"] & E \arrow[u, "u_1"] \arrow[d, "u_2"] \arrow[r, phantom, "\Downarrow\alpha_2"] & B \\
 & D \arrow[ul, "v"] \arrow[ur, "g"'] &
\end{tikzcd}$$

with $w, v, u_1 \cdot w \simeq u_2 \cdot v \in \Sigma$. Given another pre-2-cell

$$\begin{tikzcd}
A \arrow[r, bend left, "{(w,f)}"] \arrow[r, bend right, "{(v,g)}"'] \arrow[r, phantom, "\Downarrow (s_1,s_2,\beta_1,\beta_2)"] & B
\end{tikzcd}$$

then the pre-2-cells $(u_1, u_2, \alpha_1, \alpha_2)$ and $(s_1, s_2, \beta_1, \beta_2)$ are equivalent if there exists $(r_1, r_2, \gamma_1, \gamma_2)$ as in

$$\begin{tikzcd}
 & E \arrow[dl, "u_1"'] \arrow[dr, "u_2"] & \\
C \arrow[r, phantom, "\gamma_1\Uparrow"] & F \arrow[u, "r_1"] \arrow[d, "r_2"] \arrow[r, phantom, "\Downarrow\gamma_2"] & D \\
 & E' \arrow[ul, "s_1"] \arrow[ur, "s_2"'] &
\end{tikzcd}$$

such that $r_1 \cdot u_1 \cdot w \simeq r_2 \cdot s_1 \cdot w \in \Sigma$ and such that the following diagrams commute

$$\begin{tikzcd}
r_1 \cdot u_1 \cdot w \arrow[d, "r_1\cdot\alpha_1"'] \arrow[dr, phantom, "(i)"] & r_2 \cdot s_1 \cdot w \arrow[l, "\gamma_1\cdot w"'] \arrow[d, "r_2\cdot\beta_1"] \\
r_1 \cdot u_2 \cdot v \arrow[r, "\gamma_2\cdot v"'] & r_2 \cdot s_2 \cdot v
\end{tikzcd}
\qquad
\begin{tikzcd}
r_1 \cdot u_1 \cdot f \arrow[d, "r_1\cdot\alpha_2"'] \arrow[dr, phantom, "(ii)"] & r_2 \cdot s_1 \cdot f \arrow[l, "\gamma_1\cdot f"'] \arrow[d, "r_2\cdot\beta_2"] \\
r_1 \cdot u_2 \cdot g \arrow[r, "\gamma_2\cdot g"'] & r_2 \cdot s_2 \cdot g
\end{tikzcd}$$

Clearly, there is a homomorphism of bicategories $\mathcal{E} \colon \mathcal{C}[\Sigma^*] \to \mathcal{C}[\Sigma^{-1}]$

defined by

$$
\begin{tikzcd}
X \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\alpha\Downarrow"] \arrow[d, "a"'] & & Y \arrow[d, "b"] \\
A & & B
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
& X \arrow[dl, "a"'] \arrow[dr, "f\cdot b"] & \\
A \arrow[r, phantom, "a\Downarrow"] & X \arrow[u, "1"] \arrow[d, "1"] \arrow[r, phantom, "\Downarrow\alpha\cdot b"] & B \\
& X \arrow[ul, "a"] \arrow[ur, "g\cdot b"'] &
\end{tikzcd}
$$

Proposition 4.4 *Let Σ be a class of 1-cells in a bicategory $\mathcal{C}$. If Σ has a faithful calculus of fractions and $\mathcal{C}$ has enough Σ-projective objects, then $\mathcal{E}\colon \mathcal{C}[\Sigma^*] \to \mathcal{C}[\Sigma^{-1}]$ is a biequivalence.*

More precisely, we are going to prove the following statements:

1. If Σ has a faithful calculus of fractions, then $\mathcal{E}$ is locally an equivalence.

2. If $\mathcal{C}$ has enough Σ-projective objects, then $\mathcal{E}$ is surjective on objects.

Proof. 1. $\mathcal{E}$ is locally faithful: let

$$
\begin{tikzcd}
X \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\alpha\Downarrow\ \ \Downarrow\beta"] \arrow[d, "a"'] & & Y \arrow[d, "b"] \\
A & & B
\end{tikzcd}
$$

be 2-cells in $\mathcal{C}[\Sigma^*]$ and let

$$
\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[dr, "1"] & \\
X \arrow[r, phantom, "\gamma_1\Uparrow"] & F \arrow[u, "r_1"] \arrow[d, "r_2"] \arrow[r, phantom, "\Downarrow\gamma_2"] & X \\
& X \arrow[ul, "1"] \arrow[ur, "1"'] &
\end{tikzcd}
$$

be the datum attesting that $\mathcal{E}(\alpha) = \mathcal{E}(\beta)$ in $\mathcal{C}[\Sigma^{-1}]$. Since $a, r_2 \cdot a \in \Sigma$, then by (FF2) $r_2 \in \Sigma$, and then it is an equivalence because X is Σ-projective. The first condition on $(r_1, r_2, \gamma_1, \gamma_2)$ implies that $\gamma_1 = \gamma_2^{-1}$, the second condition gives then $r_2 \cdot \alpha \cdot b = r_2 \cdot \beta \cdot b$. Since r_2 is an equivalence and b is faithful, we have $\alpha = \beta$.
$\mathcal{E}$ is locally full: consider two 1-cells f, g in $\mathcal{C}[\Sigma^*]$ and a 2-cell $\mathcal{E}(f) \Rightarrow \mathcal{E}(g)$ as follows

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[d, "a"'] & Y \arrow[d, "b"] \\
A & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[dr, "f \cdot b"] & \\
A \arrow[r, phantom, "\alpha_1 \Downarrow"] & E \arrow[u, "u_1"'] \arrow[d, "u_2"] \arrow[r, phantom, "\Downarrow \alpha_2"] & B \\
 & X \arrow[ul, "a"] \arrow[ur, "g \cdot b"'] &
\end{tikzcd}$$

Since a is full and faithful, there exists a unique $\beta\colon u_1 \Rightarrow u_2$ such that $\beta \cdot a = \alpha_1$. Moreover, $a, u_1 \cdot a \in \Sigma$, so that $u_1 \in \Sigma$ by (FF2), and then u_1 is an equivalence because X is Σ-projective (the same argument holds for u_2). Since b also is full and faithful, there exists a unique $\alpha\colon f \Rightarrow g$ such that

$$\begin{tikzcd}
u_1 \cdot f \cdot b \arrow[r, "\alpha_2"] \arrow[dr, "u_1 \cdot \alpha \cdot b"'] & u_2 \cdot g \cdot b \\
 & u_1 \cdot g \cdot b \arrow[u, "\beta \cdot g \cdot b"']
\end{tikzcd}$$

commutes. To check that $\mathcal{E}(\alpha) = [u_1, u_2, \alpha_1, \alpha_2]$ we use the following datum, where $\beta^*\colon u_1^* \Rightarrow u_2^*$ corresponds to $\beta\colon u_1 \Rightarrow u_2$ as in Remark 2.2.3:

$$\begin{tikzcd}
 & E \arrow[dl, "u_1"'] \arrow[dr, bend left, "u_2"] \arrow[d, phantom, shift left=4ex, "\Downarrow \beta^* \cdot u_2"] & \\
X \arrow[r, phantom, "\epsilon_1^{-1} \Uparrow"] & X \arrow[u, "u_1^*"] \arrow[r, dotted, "u_2^* \cdot u_2"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \epsilon_2"] & X \\
 & X \arrow[ul, "1"] \arrow[ur, bend right, "1"'] &
\end{tikzcd}$$

Condition (i) easily follows from the definition of β and Remark 2.2.3. As far as condition (ii) is concerned, since u_1 is an equivalence, by

Remark 2.2.4 it is enough to check it precomposing with u_1. Using the definition of α, condition (ii) reduces now to the commutativity of

$$\begin{tikzcd}
u_1 \cdot u_1^* \cdot u_1 \cdot f \cdot b \arrow[rr, "u_1 \cdot \epsilon_1 \cdot f \cdot b"] \arrow[d, "\beta \cdot \beta^* \cdot u_1 \cdot f \cdot b"'] & & u_1 \cdot f \cdot b \arrow[d, "\alpha_2"] \\
u_2 \cdot u_2^* \cdot u_1 \cdot f \cdot b \arrow[r, "u_2 \cdot u_2^* \cdot \alpha_2"'] & u_2 \cdot u_2^* \cdot u_2 \cdot g \cdot b \arrow[r, "u_2 \cdot \epsilon_2 \cdot g \cdot b"'] & u_2 \cdot g \cdot b
\end{tikzcd}$$

To check this last equation, past on the left side the commutative triangle

$$\begin{tikzcd}
u_1 \cdot f \cdot b \arrow[r, "\eta_1 \cdot u_1 \cdot f \cdot b"] \arrow[dr, "\eta_2 \cdot u_1 \cdot f \cdot b"'] & u_1 \cdot u_1^* \cdot u_1 \cdot f \cdot b \arrow[d, "\beta \cdot \beta^* \cdot u_1 \cdot f \cdot b"] \\
 & u_2 \cdot u_2^* \cdot u_1 \cdot f \cdot b
\end{tikzcd}$$

and use the first triangular identity on η_1, ϵ_1 and on η_2, ϵ_2, so that both paths reduce to $\alpha_2 \colon u_1 \cdot f \cdot b \Rightarrow u_2 \cdot g \cdot g$.
$\mathcal{E}$ is locally essentially surjective: consider two objects $a \colon X \to A$ and $b \colon Y \to B$ in $\mathcal{C}[\Sigma^*]$ and a 1-cell

$$\begin{tikzcd} A & C \arrow[l, "w"'] \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathcal{C}[\Sigma^{-1}]$. Using twice that X is Σ-projective, we get

$$\begin{tikzcd}
 & X \arrow[dl, "h"'] \arrow[dr, "a"] \arrow[d, phantom, "\varphi\Rightarrow"] & \\
C \arrow[rr, "w"'] & {} & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "k"'] \arrow[dr, "h \cdot f"] \arrow[d, phantom, "\psi\Rightarrow"] & \\
Y \arrow[rr, "b"'] & {} & B
\end{tikzcd}$$

This gives a 1-cell in $\mathcal{C}[\Sigma^*]$ and a 2-cell in $\mathcal{C}[\Sigma^{-1}]$

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\
A & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[dr, "k \cdot b"] & \\
A \arrow[r, phantom, "\varphi^{-1}\Downarrow"] & X \arrow[u, "1"] \arrow[d, "h"] \arrow[r, phantom, "\Downarrow\psi"] & B \\
 & C \arrow[ul, "w"] \arrow[ur, "f"'] &
\end{tikzcd}$$

attesting that $\mathcal{E}$ is locally essentially surjective.
2. Obvious, just choose a Σ-cover $a \colon X \to A$ for every object A of $\mathcal{C}$. ■

Remark 4.5 Putting together Remark 4.2.2 and Proposition 4.4.1, we get an equivalence of hom-categories

$$\mathcal{C}[\Sigma^{-1}](A,B) \simeq \mathcal{C}[\Sigma^*](a\colon X \to A, b\colon Y \to B) \simeq \mathcal{C}(X,B),$$

as announced in the Introduction.

5. Extensions as fractions

In order to illustrate the difference between $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}[\Sigma^*]$, we discuss a special case of Example 3.5. We consider the bicategory $\mathrm{Grpd}(\mathcal{A})$ and we assume that $\mathcal{A}$ is semi-abelian, has split extension classifiers, and satisfies the "Huq = Smith" condition as in [1]. The typical examples of such an $\mathcal{A}$ are the category of groups (where the split extension classifier of a group H is the group of automorphisms of H) and the category of Lie algebras (where the split extension classifier of an algebra H is the Lie algebra of derivations of H).

Fix two objects G and H in $\mathcal{A}$. From [1, Section 7], we know that the groupoid of extensions $\mathrm{EXT}(G,H)$ is isomorphic to the hom-groupoid $\mathcal{B}(\mathcal{A})(D(G),[[H]])$, where $\mathcal{B}(\mathcal{A})$ is the bicategory of internal butterflies in $\mathcal{A}$ (since $\mathcal{A}$ is semi-abelian, we do not take care of the difference between internal groupoids and internal crossed modules), $D(G)$ is the discrete internal groupoid on G, and $[[H]]$ is the action groupoid, that is, the internal groupoid having the split extension classifier $[H]$ as object of objects and the holomorph $H \rtimes [H]$ as object of arrows. Since $\mathcal{B}(\mathcal{A})$ is biequivalent to the bicategory of fractions of $\mathrm{Grpd}(\mathcal{A})$ with respect to weak equivalences [1, Theorem 5.6], we have an equivalence

$$\mathrm{EXT}(G,H) \simeq \mathrm{Grpd}(\mathcal{A})[\Sigma^{-1}](D(G),[[H]])$$

and, by Remark 4.5, we also have an equivalence

$$\mathrm{EXT}(G,H) \simeq \mathrm{Grpd}(\mathcal{A})(\mathbb{X},[[H]])$$

Accordingly, we can describe an extension

$$H \xrightarrow{\iota} E \xrightarrow{\sigma} G$$

as a span of internal functors (with the left leg being a weak equivalence) or as a single internal functor. In the first case, we get the span

$$\begin{tikzcd}
G \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & R[\sigma] \arrow[l, "\sigma_i \cdot \sigma"'] \arrow[r, "\simeq"] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_2"] & H \rtimes E \arrow[r, "1 \rtimes \mathcal{I}"] & H \rtimes [H] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
G & E \arrow[l, "\sigma"] \arrow[rr, "\mathcal{I}"'] & & {[H]}
\end{tikzcd}$$

where $\sigma_1, \sigma_2 \colon R[\sigma] \rightrightarrows E$ is the kernel relation of σ, and $\mathcal{I} \colon E \to [H]$ is the action induced by the fact that $\iota \colon H \to E$ is normal. This is a "discrete fraction", in the sense that the right leg is a discrete fibration. To transform this span into a single internal functor, we fix a regular projective cover $s \colon X_0 \to G$ of G together with an extension σ_0 of s along σ as in the following commutative diagram

$$\begin{tikzcd}
 & X_0 \arrow[dl, "\sigma_0"'] \arrow[d, "s"] \\
E \arrow[r, "\sigma"'] & G
\end{tikzcd}$$

Composing with the discrete fibration above, we get the internal functor

$$\begin{tikzcd}
R[s] \arrow[r, "\overline{\sigma}"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & R[\sigma] \arrow[r] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_2"] & H \rtimes [H] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
X_0 \arrow[r, "\sigma_0"'] & E \arrow[r, "\mathcal{I}"'] & {[H]}
\end{tikzcd}$$

where $s_1, s_2 \colon R[s] \rightrightarrows X_0$ is the kernel relation of s, and $\overline{\sigma}$ is the canonical factorization of $R[s]$ through $R[\sigma]$.

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