ON THE SECOND COHOMOLOGY CATEGORICAL GROUP AND A HOCHSCHILD-SERRE 2-EXACT SEQUENCE

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ABSTRACT. We introduce the second cohomology categorical group of a categorical group $\mathbb{G}$ with coefficients in a symmetric $\mathbb{G}$-categorical group and we show that it classifies extensions of $\mathbb{G}$ with symmetric kernel and a functorial section. Moreover, from an essentially surjective homomorphism of categorical groups we get 2-exact sequences à la Hochschild-Serre connecting the categorical groups of derivations and the first and the second cohomology categorical groups.

Contents

1. Introduction

Let $E \to G$ be a surjective homomorphism of groups with kernel N, and let A be a G-module. In [22], Hochschild and Serre obtained a 5-term exact sequence involving the

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group of derivations and the second cohomology group of E and G with coefficients in A

$$0 \to \mathrm{Der}(G,A) \longrightarrow \mathrm{Der}(E,A) \longrightarrow \mathrm{Hom}_G(N_{\mathrm{ab}},A) \longrightarrow \mathrm{H}^2(G,A) \longrightarrow \mathrm{H}^2(E,A)$$

where the middle point $\mathrm{Hom}_G(N_{\mathrm{ab}},A)$ is the group of G-module homomorphisms from the abelianization of N to A. The Hochschild-Serre sequence can also be modified to get a 5-term exact sequence involving the first and the second cohomology group

$$0 \to \mathrm{H}^1(G,A) \longrightarrow \mathrm{H}^1(E,A) \longrightarrow \mathrm{Hom}_G(N_{\mathrm{ab}},A) \longrightarrow \mathrm{H}^2(G,A) \longrightarrow \mathrm{H}^2(E,A)$$

(These sequences are indeed part of long exact sequences, see [28, 3, 21, 23, 27, 1].) In the present paper, our first aim is to give a 2-dimensional version of these exact sequences, replacing groups with categorical groups.

In order to construct cohomology categorical groups, a first step has been done by Ulbrich in [33] (see also [32, 14]) : if G is a group, $[G]_0$ is the discrete categorical group associated with it and $\mathbb{A}$ is a symmetric $[G]_0$-categorical group (a $[G]_0$-module in the terminology of [33]), Ulbrich defined cohomology groups $\mathrm{H}^n(G,\mathbb{A})$ by considering the cocomplex of symmetric categorical groups $\mathbb{C}(G,\mathbb{A})$ that, in dimension n, consists of functors from G^n to $\mathbb{A}$, where $G^n = G \times ... \times G$ is seen as a discrete category, and ∂ is obtained by taking alternating sums (i.e., tensor) of the homomorphisms $d_i \colon \mathbb{C}^n(G,\mathbb{A}) \to \mathbb{C}^{n+1}(G,\mathbb{A})$, $i = 0, ..., n+1$ defined, for all $x_1, ..., x_n \in G$ and $F \in \mathbb{C}^n(G,\mathbb{A})$, by
- $d_0(F)(x_1, ..., x_{n+1}) = {}^{x_1}F(x_2, ..., x_{n+1})$,
- $d_i(F)(x_1, ..., x_{n+1}) = F(x_1, ..., x_i x_{i+1}, ..., x_{n+1})$, $1 \le i \le n$, and
- $d_{n+1}(F)(x_1, ..., x_{n+1}) = F(x_1, ..., x_n)$.

This process can be generalized replacing the group G by any categorical group $\mathbb{G}$ and using a corresponding cocomplex of symmetric categorical groups $\mathbb{C}(\mathbb{G},\mathbb{A})$. Moreover, instead of cohomology groups, using relative kernels and relative cokernels we can construct from $\mathbb{C}(\mathbb{G},\mathbb{A})$ all the cohomology categorical groups $\mathbb{H}^n(\mathbb{G},\mathbb{A})$ as explained in [15]. What we do in this paper is to give an explicit description in terms of cobords and factor sets of the first and the second cohomology categorical groups (and of the categorical group of derivations), that is, those categorical groups entering in the Hochschild-Serre sequences.

Since we dispose of an explicit description of the second cohomology categorical group, it is natural to look at what kind of extensions this categorical group classifies. This leads us to revisit categorical crossed modules (in the sense of [13]) and finally to a classification of extensions with symmetric kernel and functorial section which generalizes the classification obtained in [6] for extensions of symmetric categorical groups.

More in detail, the layout of this paper is as follows : in Section 2 we revisit the case of groups and we give a proof of the Hochschild-Serre exact sequence slightly different from the one available in the literature (see for instance [21]) and more convenient to prepare our 2-dimensional version. In particular, we separe the construction of the sequence, which is a special case of the Kernel-Cokernel Lemma and could be obtained even if the group homomorphism $E \to G$ is not surjective (see [35]), from the interpretation of the middle point of the sequence, which definitely depends on the fact that the homomorphism is surjective. Section 3 is a reminder on categorical groups. We fix notation and we recall convenient versions for categorical groups of actions, derivations, kernel and cokernel, and exactness. In Section 4 we discuss cobords and factor sets of a categorical group $\mathbb{G}$ with coefficients in a symmetric $\mathbb{G}$-categorical group $\mathbb{A}$, and define the cohomology categorical group $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$. Specializing in different ways $\mathbb{G}$ and $\mathbb{A}$ and taking the group $\pi_0(\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi))$ of connected components, we recover Eilenberg-Mac Lane and Ulbrich cohomology groups. Section 5 is devoted to the construction of two Hochschild-Serre exact sequences of categorical groups from an essentially surjective homomorphism of categorical groups $\mathbb{E} \to \mathbb{G}$ and a symmetric $\mathbb{G}$-categorical group $\mathbb{A}$. In Section 6 we concentrate on an interpretation of the middle point of the Hochschild-Serre exact sequences similar to the one establised in the group case. In order to introduce extensions with a symmetric kernel, in Section 7 we discuss categorical crossed modules, a notion introduced in [13] (see also [10]). The reason why we need categorical crossed modules deserves an explication : in the case of groups, an extension can be presented as a surjective homomorphism together with its kernel or, equivalently, as a normal subgroup together with the corresponding quotient group. Even for categorical groups the two approaches are equivalent ; this is due to some general results on categorical crossed modules established in [13] and summarized in the present paper as Proposition 7.9. Having this fact in mind, in Section 8 we define an extension as a normal sub-categorical group together with the corresponding quotient categorical group. The advantage with this definition is that, when the normal sub-categorical group is symmetric, it is easy to construct an action of the quotient on the kernel, much more easy than to do the same if one defines an extension as an essentially surjective homomorphism with a symmetric kernel. The price to pay is that (contrarily to what happens with groups) to be a normal sub-categorical group is a structure, and not a property, and the only reasonable way to handle such a structure is to see it as a special case of the general structure of categorical crossed module. Finally, Section 9 is devoted to the cohomological classification of extensions with symmetric kernel. We give only the constructive part of the proof and we omit the long diagrammatic arguments needed to check that the various constructions fulfill the requested coherence conditions.

Remark 1.1 The cohomology categorical group $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ introduced in Section 4 is one of two possible variants. In order to understand this point, let us look at the simpler case where the action φ is trivial, and let us start with the classical case of groups. For a group G and an abelian group A, to construct the cohomology groups $\mathrm{H}^n(G, A)$ one can start considering the chain cocomplex of maps (in any dimension) from the simplicial set $K(G, 1)$ (the Grothendieck's nerve of G seen as a category with only one object) to A, with differentials given by alternating sums of the induced cofaces. The same process can be carried out for a categorical group $\mathbb{G}$ and a symmetric categorical group $\mathbb{A}$. What changes in the case of categorical groups is that at the beginning of the story two different choices are possible, and these different choices lead to different cohomology categorical groups. Indeed, one can consider as starting point either $Ner(\mathbb{G})$, the simplicial set of a categorical group $\mathbb{G}$ (see [9]), or the pseudo-simplicial category that $\mathbb{G}$ defines by the familiar bar construction, where face and degeneracy functors are defined in a standard way and where simplicial identities hold up to isomorphism (see [8, 11] and the references therein). When the action φ is trivial, our cohomology categorical group $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ is the one coming from this pseudo-simplicial nerve. The $Ner(\mathbb{G})$ version is studied in a separate paper [20] and is related to singular extensions of categorical groups.

2. The group case revisited

Let G be a group and $A = (A, \varphi)$ an abelian group together with a G-module structure $\varphi\colon G \times A \to A$. We write $\varphi(x, a) = {}^{x}a$. The starting point to construct the Hochschild-Serre sequence and the H^1-H^2 sequence is described by the following commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
A \arrow[r, "d_G"] & \mathrm{Der}(G,A) \arrow[r, "i_G"] \arrow[d, "\pi_G"'] & \mathrm{C}^1(G,A) \arrow[r, "\delta_G"] \arrow[d, "\underline{\pi}_G"'] & \mathrm{Z}^2(G,A) \arrow[r, "q_G"] \arrow[dr, "\underline{q}_G"'] & \mathrm{H}^2(G,A) \arrow[d, "s_G"] \\
& \mathrm{H}^1(G,A) \arrow[r, "\underline{i}_G"'] & \underline{\mathrm{C}}^1(G,A) \arrow[ur, "\underline{\delta}_G"'] & & \underline{\mathrm{H}}^2(G,A)
\end{tikzcd}$$

where

- $\mathrm{C}^1(G, A)$ is the group of 1-cochains, i.e. maps $g\colon G \to A$ such that $g(1) = 0$;
- $\mathrm{Z}^2(G, A)$ is the group of 2-cocycles, i.e. maps $f\colon G \times G \to A$ such that

$$f(x, 1) = 0 = f(1, y)\ ,\ \ {}^{x}f(y, z) + f(x, yz) = f(x, y) + f(xy, z)$$

- $\delta_G\colon \mathrm{C}^1(G, A) \to \mathrm{Z}^2(G, A)$ is defined by $\delta_G(g)(x, y) = {}^{x}g(y) - g(xy) + g(x)$;
- $\mathrm{Der}(G, A)$ and $\mathrm{H}^2(G, A)$ are, respectively, the kernel and the cokernel of δ_G (the element of $\mathrm{Der}(G, A)$ are called derivations) ;
- $d_G\colon A \to \mathrm{Der}(G, A)$ is defined by the inner derivation $d_G(a)(x) = {}^{x}a - a$;
- $\mathrm{H}^1(G, A)$ is the cokernel of d_G, and $\underline{C}^1(G, A)$ is the cokernel of $i_G \cdot d_G$;
- $\underline{i}_G$ and $\underline{\delta}_G$ are induced by the universal properties of $\mathrm{H}^1(G, A)$ and $\underline{\mathrm{C}}^1(G, A)$;
- $\underline{\mathrm{H}}^2(G, A)$ is the cokernel of $\underline{\delta}_G$;
- s_G is induced by the universal property of $\mathrm{H}^2(G, A)$.

Lemma 2.1 *In the previous diagram*

1. $\underline{i}_G\colon \mathrm{H}^1(G, A) \to \underline{\mathrm{C}}^1(G, A)$ *is the kernel of* $\underline{\delta}_G\colon \underline{\mathrm{C}}^1(G, A) \to \mathrm{Z}^2(G, A)$;
2. $s_G\colon \mathrm{H}^2(G, A) \to \underline{\mathrm{H}}^2(G, A)$ *is an isomorphism.*

Proof. Apply the Kernel-Cokernel Lemma to the diagram

$$\begin{tikzcd}
A \arrow[r, "i_G \cdot d_G"] \arrow[d, "!"'] & \mathrm{C}^1(G,A) \arrow[r, "\pi_G"] \arrow[d, "\delta_G"'] & \underline{\mathrm{C}}^1(G,A) \arrow[d, "\underline{\delta}_G"] \\
0 \arrow[r, "!"'] & \mathrm{Z}^2(G,A) \arrow[r, "\mathrm{id}"'] & \mathrm{Z}^2(G,A)
\end{tikzcd}$$

In the resulting exact sequence

$$\begin{tikzcd}
A \arrow[r, "d_G"] & \mathrm{Der}(G,A) \arrow[r] & \mathrm{Ker}(\underline{\delta}_G) \arrow[r] & 0 \arrow[r] & \mathrm{H}^2(G,A) \arrow[r, "s_G"] & \underline{\mathrm{H}}^2(G,A) \arrow[r] & 0
\end{tikzcd}$$

the exactness in $\mathrm{Der}(G,A)$ and in $\mathrm{Ker}(\underline{\delta}_G)$ gives that $\mathrm{H}^1(G,A) \simeq \mathrm{Ker}(\underline{\delta}_G)$, and the exactness in $\mathrm{H}^2(G,A)$ and in $\underline{\mathrm{H}}^2(G,A)$ gives that s_G is an isomorphism. ■

Consider now an epimorphism of groups $p\colon E \to G$ and the induced E-module structure $E \times A \to G \times A \to A$ on A. Composition with $p\colon E \to G$ induces also a monomorphism $p_1\colon \mathrm{C}^1(G,A) \to \mathrm{C}^1(E,A)$, and composition with $p \times p\colon E \times E \to G \times G$ induces a monomorphism $p_2\colon \mathrm{Z}^2(G,A) \to \mathrm{Z}^2(E,A)$. We have all the ingredients to construct the following commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[d, "d_G"'] & & A \arrow[d, "d_E"'] & & & \\
\mathrm{Der}(G,A) \arrow[rr, "a"] \arrow[dr, "\pi_G"] \arrow[dd, "i_G"'] & & \mathrm{Der}(E,A) \arrow[rr, "b"] \arrow[dr, "\pi_E"] \arrow[dd, "i_E"'] & & \mathrm{Ker}(\delta_p) \arrow[dr, "\pi_p"] \arrow[dd, "i_p"'] & \\
& \mathrm{H}^1(G,A) \arrow[rr, "\underline{a}"] \arrow[dd, "\underline{i}_G"'] & & \mathrm{H}^1(E,A) \arrow[rr, "\underline{b}"] \arrow[dd, "\underline{i}_E"'] & & \mathrm{Ker}(\underline{\delta}_p) \arrow[dd, "\underline{i}_p"] \\
\mathrm{C}^1(G,A) \arrow[rr, "p_1"] \arrow[dr, "\underline{\pi}_G"] \arrow[dd, "\delta_G"'] & & \mathrm{C}^1(E,A) \arrow[rr, "q_1"] \arrow[dr, "\underline{\pi}_E"] \arrow[dd, "\delta_E"'] & & \mathrm{C}^1(p,A) \arrow[dr, "\underline{\pi}_p"] \arrow[dd, "\delta_p"'] & \\
& \underline{\mathrm{C}}^1(G,A) \arrow[rr, "\underline{p}_1"] \arrow[dl, "\underline{\delta}_G"'] & & \underline{\mathrm{C}}^1(E,A) \arrow[rr, "\underline{q}_1"] \arrow[dl, "\underline{\delta}_E"'] & & \underline{\mathrm{C}}^1(p,A) \arrow[dl, "\underline{\delta}_p"] \\
\mathrm{Z}^2(G,A) \arrow[rr, "p_2"] \arrow[dr, "\underline{q}_G"] \arrow[dd, "q_G"'] & & \mathrm{Z}^2(E,A) \arrow[rr, "q_2"] \arrow[dr, "\underline{q}_E"] \arrow[dd, "q_E"'] & & \mathrm{Z}^2(p,A) \arrow[dr, "\underline{q}_p"] \arrow[dd, "q_p"'] & \\
& \underline{\mathrm{H}}^2(G,A) \arrow[rr, "\underline{c}"] & & \underline{\mathrm{H}}^2(E,A) \arrow[rr, "\underline{d}"] & & \mathrm{Coker}(\underline{\delta}_p) \\
\mathrm{H}^2(G,A) \arrow[rr, "c"'] \arrow[ur, "s_G"] & & \mathrm{H}^2(E,A) \arrow[rr, "d"'] \arrow[ur, "s_E"] & & \mathrm{Coker}(\delta_p) \arrow[ur, "s_p"'] &
\end{tikzcd}$$

where

- $\underline{p}_1 \colon \underline{\mathrm{C}}^1(G,A) \to \underline{\mathrm{C}}^1(E,A)$ is induced by the universal property of $\underline{\mathrm{C}}^1(G,A)$;
- $\mathrm{C}^1(p,A)$ is the cokernel of p_1, $\underline{\mathrm{C}}^1(p,A)$ is the cokernel of $\underline{p}_1$, and $\mathrm{Z}^2(p,A)$ is the cokernel of p_2;
- δ_p and $\underline{\delta}_p$ are induced by the universal properties of $\mathrm{C}^1(p,A)$ and $\underline{\mathrm{C}}^1(p,A)$, π_p and $\underline{\pi}_p$ are induced by the universal properties of $\mathrm{Ker}(\underline{\delta}_p)$ and $\mathrm{C}^1(p,A)$, and s_p is induced by the universal property of $\mathrm{Coker}(\delta_p)$.

Lemma 2.2 *In the previous diagram*

1. $\underline{p}_1 \colon \underline{\mathrm{C}}^1(G,A) \to \underline{\mathrm{C}}^1(E,A)$ *is a monomorphism;*
2. $\underline{\pi}_p \colon \mathrm{C}^1(p,A) \to \underline{\mathrm{C}}^1(p,A)$ *is an isomorphism and, therefore,*
3. π_p *and* s_p *are isomorphisms.*

Proof. Consider

$$\begin{tikzcd}
A \arrow[rr, "i_G \cdot d_G"] \arrow[dr, "q^G"'] \arrow[dd, "\mathrm{id}"'] & & \mathrm{C}^1(G,A) \arrow[r, "\underline{\pi}_G"] \arrow[dd, "p_1"] & \underline{\mathrm{C}}^1(G,A) \arrow[dd, "\underline{p}_1"] \\
& \mathrm{Ker}(\underline{\pi}_G) \arrow[ur] \arrow[dd, "h"] & & \\
A \arrow[rr, "i_E \cdot d_E"] \arrow[dr, "q^E"'] & & \mathrm{C}^1(E,A) \arrow[r, "\underline{\pi}_E"'] & \underline{\mathrm{C}}^1(E,A) \\
& \mathrm{Ker}(\underline{\pi}_E) \arrow[ur] & &
\end{tikzcd}$$

Observe that h is an epimorphism because q^E is an epimorphism. Now apply the Kernel-Cokernel Lemma to the diagram

$$\begin{tikzcd}
\mathrm{Ker}(\underline{\pi}_G) \arrow[r] \arrow[d, "h"'] & \mathrm{C}^1(G,A) \arrow[r, "\underline{\pi}_G"] \arrow[d, "p_1"'] & \underline{\mathrm{C}}^1(G,A) \arrow[d, "\underline{p}_1"] \\
\mathrm{Ker}(\underline{\pi}_E) \arrow[r] & \mathrm{C}^1(E,A) \arrow[r, "\underline{\pi}_E"'] & \underline{\mathrm{C}}^1(E,A)
\end{tikzcd}$$

The resulting exact sequence

$$0 \longrightarrow \mathrm{Ker}(h) \longrightarrow 0 \longrightarrow \mathrm{Ker}(\underline{p}_1) \longrightarrow 0 \longrightarrow \mathrm{C}^1(p,A) \xrightarrow{\underline{\pi}_p} \underline{\mathrm{C}}^1(p,A) \longrightarrow 0$$

gives that $\mathrm{Ker}(\underline{p}_1) = 0$, so that $\underline{p}_1$ is a monomorphism, and that $\underline{\pi}_p$ is an isomorphism. ∎

Corollary 2.3 *Consider an epimorphism of groups $p\colon E \to G$ and a G-module A. There are two exact sequences of abelian groups connected by a morphism of complexes*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}(G,A) \arrow[r, "a"] \arrow[d, "\pi_G"'] & \mathrm{Der}(E,A) \arrow[r, "b"] \arrow[d, "\pi_E"'] & \mathrm{Ker}(\delta_p) \arrow[r, "\Delta"] \arrow[d, "\pi_p"'] & \mathrm{H}^2(G,A) \arrow[r, "c"] \arrow[d, "s_G"] & \mathrm{H}^2(E,A) \arrow[r, "d"] \arrow[d, "s_E"] & \mathrm{Coker}(\delta_p) \arrow[r] \arrow[d, "s_p"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(G,A) \arrow[r, "\underline{a}"'] & \mathrm{H}^1(E,A) \arrow[r, "\underline{b}"'] & \mathrm{Ker}(\underline{\delta}_p) \arrow[r, "\underline{\Delta}"'] & \underline{\mathrm{H}}^2(G,A) \arrow[r, "\underline{c}"'] & \underline{\mathrm{H}}^2(E,A) \arrow[r, "\underline{d}"'] & \mathrm{Coker}(\underline{\delta}_p) \arrow[r] & 0
\end{tikzcd}$$

Moreover, π_G and π_E are epimorphisms, and π_p, s_G, s_E and s_p are isomorphisms.

Proof. Apply the Kernel-Cokernel Lemma to the diagrams

$$\begin{tikzcd}
\mathrm{C}^1(G,A) \arrow[r, "p_1"] \arrow[d, "\delta_G"'] & \mathrm{C}^1(E,A) \arrow[r, "q_1"] \arrow[d, "\delta_E"] & \mathrm{C}^1(p,A) \arrow[d, "\delta_p"] \\
\mathrm{Z}^2(G,A) \arrow[r, "p_2"'] & \mathrm{Z}^2(E,A) \arrow[r, "q_2"'] & \mathrm{Z}^2(p,A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{C}}^1(G,A) \arrow[r, "\underline{p}_1"] \arrow[d, "\underline{\delta}_G"'] & \underline{\mathrm{C}}^1(E,A) \arrow[r, "\underline{q}_1"] \arrow[d, "\underline{\delta}_E"] & \underline{\mathrm{C}}^1(p,A) \arrow[d, "\underline{\delta}_p"] \\
\mathrm{Z}^2(G,A) \arrow[r, "p_2"'] & \mathrm{Z}^2(E,A) \arrow[r, "q_2"'] & \mathrm{Z}^2(p,A)
\end{tikzcd}$$

and use Lemma 2.1 and Lemma 2.2. ∎

We look now for a different description of the group $\mathrm{Ker}(\delta_p)$. Let N be the kernel of $p\colon E \to G$ and $N_{\mathrm{ab}} = N/[N,N]$ its abelianization. Since p is surjective, N_{ab} has a G-module structure induced by conjugation in E:

$${}^{x}\overline{n} = \overline{ene^{-1}} \ \text{ for } n \in N, x \in G \ \text{ and } \ p(e) = x$$

In the next result, we denote by $\mathrm{Hom}_G(N_{\mathrm{ab}}, A)$ the abelian group of G-module homomorphisms $N_{\mathrm{ab}} \to A$, and by $\underline{\mathrm{Hom}}(N,A)$ the abelian group of equivariant homomorphisms, that is, group homomorphisms $h\colon N \to A$ such that $h(ene^{-1}) = {}^{p(e)}h(n)$ for all $e \in E, n \in N$.

Proposition 2.4 *Let $p\colon E \to G$ be a group epimorphism with kernel N. The abelian groups* $\mathrm{Hom}_G(N_{\mathrm{ab}}, A), \underline{\mathrm{Hom}}(N,A)$ *and* $\mathrm{Ker}(\delta_p)$ *are isomorphic.*

Proof. 1) The isomorphism $\underline{\mathrm{Hom}}(N,A) \simeq \mathrm{Hom}_G(N_{\mathrm{ab}}, A)$ is just the restriction of the natural isomorphism $\mathbf{Grp}(N,A) \simeq \mathbf{Ab}(N_{\mathrm{ab}}, A)$ given by the universal property of N_{ab}.
2) Consider now the kernel of $\delta_p\colon \mathrm{C}^1(p,A) \to \mathrm{Z}^2(p,A)$. Explicitly

$$\mathrm{Ker}(\delta_p) = \{[g\colon E \to A] \mid \exists f\colon G \times G \to A : \delta_E(g) = p_2(f)\}$$

with $[g] = [g']$ if there exists $u\colon G \to A$ such that $g = p_1(u) + g'$, and with $\delta_E(g) = p_2(f)$ meaning that ${}^{p(e_1)}g(e_2) - g(e_1e_2) + g(e_1) = f(p(e_1), p(e_2))$ for all $e_1, e_2 \in E$. We put

$$\Sigma\colon \mathrm{Ker}(\delta_p) \to \underline{\mathrm{Hom}}(N,A) \quad [g] \mapsto (g \cdot i\colon N \to E \to A)$$

where $i\colon N \to G$ is the inclusion.
- Σ is well-defined : if $[g]=[g']$, then condition $g = p_1(u) + g'$ gives for all $n \in N$

$$g(n) = u(p(n)) + g'(n) = u(1) + g'(n) = g'(n)$$

- $\Sigma[g]\colon N \to A$ is a group homomorphism : condition $\delta_E(g) = p_2(f)$ gives for all $n \in N$ and $e \in E$

$$g(ne) = {}^{p(n)}g(e) + g(n) - f(p(n), p(e)) = {}^{1}g(e) + g(n) - f(1, p(e)) = g(e) + g(n) \quad (1)$$

- $\Sigma[g] \in \underline{\mathrm{Hom}}(N, A)$: using condition $\delta_E(g) = p_2(f)$ once again we have for all $e \in E$

$${}^{p(e)}g(e^{-1}) = g(ee^{-1}) - g(e) + f(p(e), p(e^{-1})) = -g(e) + f(p(e), p(e^{-1})) \quad (2)$$

Now if $n \in N$ and $e \in E$, from (2) and $\delta_E(g) = p_2(f)$ it follows that

$${}^{p(e)}g(ne^{-1}) = g(ene^{-1}) - g(e) + f(p(e), p(ne^{-1})) = g(ene^{-1}) + {}^{p(e)}g(e^{-1})$$

and therefore using (1) we get

$$g(ene^{-1}) = {}^{p(e)}g(ne^{-1}) - {}^{p(e)}g(e^{-1}) = {}^{p(e)}(g(n) + g(e^{-1})) - {}^{p(e)}g(e^{-1}) =$$

$$= {}^{p(e)}g(n) + {}^{p(e)}g(e^{-1}) - {}^{p(e)}g(e^{-1}) = {}^{p(e)}g(n)$$

- Σ is surjective : let $h\colon N \to A$ be in $\underline{\mathrm{Hom}}(N, A)$ and choose a set-theoretical section $s\colon G \to E$ of p such that $s(1) = 1$. We can construct $g_s \in \mathrm{C}^1(E, A)$ and $f_s \in \mathrm{Z}^2(G, A)$ as follows

$$g_s\colon E \to A\colon g_s(e) = h(es(p(e))^{-1}) \ , \quad f_s\colon G \times G \to A\colon f_s(x, y) = h(s(xy)s(y)^{-1}s(x)^{-1})$$

To check that $\delta_E(g_s) = p_2(f_s)$ is long but easy, so let us check that $\Sigma[g_s] = h$: for all $n \in N$ we have $g_s(n) = h(ns(p(n))^{-1}) = h(ns(1)^{-1}) = h(n)$.
- Σ is injective : let $[g] \in \mathrm{Ker}(\delta_p)$ such that the restriction of g to N is 0. Fix once again a set-theoretical section s of p such that $s(1) = 1$ and put $u_s = g \cdot s\colon G \to A$. We have to show that $g = u_s \cdot p$, so that $[g] = 0$. For this, observe that from $\delta_E(g) = p_2(f)$ we deduce

$$0 = g(1) = g(s(x)s(x)^{-1}) = g(s(x)) + {}^{x}g(s(x)^{-1}) - f(x, x^{-1}) \quad (3)$$

for all $x \in G$. Finally, since $es(p(e))^{-1} \in N$ for all $e \in E$, using condition $\delta_E(g) = p_2(f)$ one more time and putting $x = p(e)$ in (3) we have

$$0 = g(es(p(e))^{-1}) = g(e) + {}^{p(e)}g(s(p(e))^{-1}) - f(p(e), p(e)^{-1}) =$$

$$= g(e) - g(s(p(e))) = g(e) - u_s(p(e))$$

■

Putting together Corollary 2.3 and Proposition 2.4, we get the exact sequences originally due to Hochschild and Serre [22].

Corollary 2.5 *Consider an epimorphism of groups $p\colon E \to G$ with kernel N and a G-module A. There are exact sequences of abelian groups*

$$0 \to \mathrm{Der}(G,A) \to \mathrm{Der}(E,A) \to \mathrm{Hom}_G(N_{\mathrm{ab}},A) \to \mathrm{H}^2(G,A) \to \mathrm{H}^2(E,A)$$

$$0 \to \mathrm{H}^1(G,A) \to \mathrm{H}^1(E,A) \to \mathrm{Hom}_G(N_{\mathrm{ab}},A) \to \mathrm{H}^2(G,A) \to \mathrm{H}^2(E,A)$$

3. A reminder on categorical groups

In this section we collect some basic facts on categorical groups and on the corresponding notions of action, derivation, kernel, cokernel, and exactness.
Starting from the works of Deligne [16], Frohlich and Wall [17] and Sinh [31], categorical groups have been studied extensively in the literature (they already appear in the work of Whitehead [36] in their strict form of crossed modules of groups). We refer to [5] and [29] for an introduction to monoidal categories, and to [2] and [34] for basic facts on categorical groups.
Actions of a categorical group on a (symmetric) categorical group have been introduced in [7] and [12]. The corresponding notion of derivation has been introduced in [18] and [19].
Convenient notions of kernel, cokernel and exactness for homomorphisms of (symmetric) categorical groups have been studied in [26] and [34].

3.1 Categorical groups. A categorical group (sometimes called gr-category or 2-group in the literature) is a monoidal groupoid $\mathbb{G} = (\mathbb{G}, \otimes, I, a, l, r)$ such that every object X is invertible, that is, the functor

$$X \otimes (-)\colon \mathbb{G} \to \mathbb{G}\,, \quad Y \mapsto X \otimes Y$$

is an equivalence. It is then possible to choose, for each $X \in \mathbb{G}$, an object $X^* \in \mathbb{G}$ (called an inverse of X) and arrows $\eta_X\colon I \to X \otimes X^*$ and $\varepsilon_X\colon X^* \otimes X \to I$ such that the usual triangular identities are satisfied. The choice of a system of inverses $(X^*, \eta_X, \epsilon_X)$, $X \in \mathbb{G}$, induces a monoidal equivalence

$$(-)^*\colon \mathbb{G} \to \mathbb{G} \quad f\colon X \to Y \;\mapsto\; f^*\colon X^* \to Y^*$$

where f^* is defined as follows

$$X^* \simeq X^* \otimes I \xrightarrow{\mathrm{id}\otimes\eta_Y} X^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id}\otimes f^{-1}\otimes\mathrm{id}} X^* \otimes X \otimes Y^* \xrightarrow{\varepsilon_X\otimes\mathrm{id}} I \otimes Y^* \simeq Y^*$$

The following simple lemma is quite useful, it will be tacitly used in several calculations all along the paper.

Lemma 3.2 *Let $f\colon X \to Y$ be an arrow in a categorical group. The following diagrams commute*

$$\begin{tikzcd} X^* \otimes X \arrow[rr, "f^* \otimes f"] \arrow[dr, "\varepsilon_X"'] & & Y^* \otimes Y \arrow[dl, "\varepsilon_Y"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} X \otimes X^* \arrow[rr, "f \otimes f^*"] & & Y \otimes Y^* \\ & I \arrow[ul, "\eta_X"] \arrow[ur, "\eta_Y"'] & \end{tikzcd}$$

In fact, the commutativity of these diagrams characterizes $f^*\colon X^* \to Y^*$ once f is given.

(Note that, if in the definition of f^* we use f instead of f^{-1}, then the monoidal equivalence $(-)^*\colon \mathbb{G} \to \mathbb{G}$ is contravariant and the previous lemma holds in any monoidal category with duals.)
A categorical group $\mathbb{G}$ is said to be symmetric if it is symmetric as a monoidal category, the symmetry being usually denoted by $c_{X,Y}\colon X \otimes Y \to Y \otimes X$. We will denote by $\mathcal{CG}$ (respectively $\mathcal{SCG}$) the 2-category whose objects are categorical groups (respectively, symmetric categorical groups). The 1-arrows, called homomorphisms, are monoidal functors $T = (T, T_2)\colon \mathbb{G} \to \mathbb{H}$ (respectively, symmetric monoidal functors), and the 2-arrows, called morphisms, are monoidal natural transformations. Note that a canonical arrow $T_0\colon I \to TI$ can be constructed from the natural and coherent family of arrows $T_2^{X,Y}\colon TX \otimes TY \to T(X \otimes Y)$. Note also that in $\mathcal{CG}$ and $\mathcal{SCG}$, the 2-arrows are invertible.

If $F\colon \mathbb{G} \to \mathbb{H}$ is any functor between categorical groups, we denote by $F^*\colon \mathbb{G} \to \mathbb{H}$ the composite functor

$$\mathbb{G} \xrightarrow{F} \mathbb{H} \xrightarrow{(-)^*} \mathbb{H}$$

that is, $F^*(X) = (FX)^*$ and $F^*(f) = (Ff)^*$. In the same way, from a natural transformation $\alpha\colon F \Rightarrow G$ we get a natural tranformation $\alpha^*\colon F^* \Rightarrow G^*$ defined by $\alpha^*_X = (\alpha_X)^*$. When F is monoidal, we can assume that $F^*(X) = F(X^*)$ and $F^*(f) = F(f^*)$. Finally, if $\alpha\colon F \Rightarrow G$ is monoidal, then $\alpha^*_X = \alpha_{X^*}$ (to check this last fact, use Lemma 3.2).

If $\mathbb{G} \in \mathcal{CG}$, then the set $\pi_0(\mathbb{G})$ of its connected components is a group (abelian if $\mathbb{G}$ is symmetric) with operation induced by tensor product. We denote by $\pi_1(\mathbb{G})$ the abelian group $\mathrm{Aut}_{\mathbb{G}}(I)$ of automorphisms of the object I (to check that $\pi_1(\mathbb{G})$ is abelian, use the Eckmann-Hilton argument : in $\pi_1(\mathbb{G})$ composition coincide, up to the canonical isomorphism $I \simeq I \otimes I$, with tensor product). Finally, for a group G we denote by $[G]_0$ the discrete categorical group on G; if G is abelian, we denote by $[G]_1$ the symmetric categorical group with one object and such that $\pi_1([G]_1) = G$.

Remark 3.3 Throughout the paper we will use several variants of the following simple fact. If $\mathbb{G}$ and $\mathbb{H}$ are categorical groups with $\mathbb{H}$ symmetric, then $\mathcal{CG}(\mathbb{G}, \mathbb{H})$ is a symmetric categorical group with structure induced pointwise from that of $\mathbb{H}$. Moreover,

$$\pi_1(\mathcal{CG}(\mathbb{G}, \mathbb{H})) \simeq \mathbf{Grp}(\pi_0(\mathbb{G}), \pi_1(\mathbb{H}))$$

Indeed, let $0\colon \mathbb{G} \to \mathbb{H}$ be the constant homomorphism sending each arrow on the identity arrow on I, and let $\alpha\colon 0 \Rightarrow 0$ be a morphism. By naturality, we have that $\alpha_X = \alpha_Y : I \to I$ if $[X] = [Y]$ in $\pi_0(\mathbb{G})$, so that α induces a map $\pi_0(\mathbb{G}) \to \pi_1(\mathbb{H})$. By monoidality, we have that $\alpha_{X\otimes Y} = \alpha_X \otimes \alpha_Y$, so that such a map $\pi_0(\mathbb{G}) \to \pi_1(\mathbb{H})$ is a group homomorphism.

Notation 3.4 When no confusion arises, in order to simplify notation we will

- omit the associativity isomorphism $a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and the right and left unit isomorphisms $r_X\colon X \to X \otimes I$ and $l_X\colon X \to I \otimes X$
- omit indexes, writing for example $c\colon X \otimes Y \to Y \otimes X$ instead of $c_{X,Y}$
- denote canonical arrows as "can" or even as unlabelled arrows; for example, we write can: $X \to Y \otimes X \otimes Y^*$ or $X \to Y \otimes X \otimes Y^*$ or just $X \simeq Y \otimes X \otimes Y^*$ instead of

$$X \xrightarrow{r_X} X \otimes I \xrightarrow{\mathrm{id}\otimes\eta_Y} X \otimes Y \otimes Y^* \xrightarrow{c_{X,Y}\otimes\mathrm{id}} Y \otimes X \otimes Y^*$$

or of

$$X \xrightarrow{l_X} I \otimes X \xrightarrow{\eta_Y\otimes\mathrm{id}} Y \otimes Y^* \otimes X \xrightarrow{\mathrm{id}\otimes c_{Y^*,X}} Y \otimes X \otimes Y^*$$

3.5 Actions. If $\mathbb{G}$ and $\mathbb{A}$ are categorical groups, an action of $\mathbb{G}$ on $\mathbb{A}$ is a homomorphism of categorical groups $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A})$, where $\mathbb{E}\mathrm{q}(\mathbb{A})$ is the categorical group of monoidal autoequivalences of $\mathbb{A}$. When such a $\mathbb{G}$-action is given, we will say that $\mathbb{A} = (\mathbb{A}, \varphi)$ is a $\mathbb{G}$-categorical group. To give a $\mathbb{G}$-categorical group structure on $\mathbb{A}$ is equivalent to giving a functor

$$\varphi\colon \mathbb{G} \times \mathbb{A} \to \mathbb{A}\,, \quad (X, A) \xrightarrow{(f,u)} (Y, B) \;\mapsto\; \varphi(X)(A) = {}^{X}\!A \xrightarrow{\varphi(f)(u)={}^{f}u} \varphi(Y)(B) = {}^{Y}\!B$$

(when $f = \mathrm{id}_X$ or $u = \mathrm{id}_A$, we write respectively ${}^{X}u$ and ${}^{f}\!A$ instead of ${}^{f}u$) together with natural families of arrows

$$\varphi_2^{X,A,B}\colon\ {}^{X}\!A \otimes {}^{X}\!B \to {}^{X}(A \otimes B)\,,\quad \varphi_1^{X,Y,A}\colon\ {}^{X}({}^{Y}\!A) \to {}^{X\otimes Y}\!A\,,\quad \varphi_0^{A}\colon A \to {}^{I}\!A$$

satisfying the following coherence conditions (see Definition 2.1 in [12]) [1] :

(act1) $\varphi_2^{X,A,B\otimes C} \cdot (\mathrm{id} \otimes \varphi_2^{X,B,C}) \cdot a = {}^{X}a \cdot \varphi_2^{X,A\otimes B,C} \cdot (\varphi_2^{X,A,B} \otimes \mathrm{id})$

(act2) $\varphi_1^{X,Y\otimes Z,A} \cdot {}^{X}\varphi_1^{Y,Z,A} = {}^{a}\!A \cdot \varphi_1^{X\otimes Y,Z,A} \cdot \varphi_1^{X,Y,{}^{Z}\!A}$

(act3) $\varphi_1^{I,Y,A} \cdot \varphi_0^{{}^{Y}\!A} = {}^{l}\!A\,,\ \ \varphi_1^{X,I,A} \cdot {}^{X}\varphi_0^{A} = {}^{r}\!A$

(act4) $\varphi_1^{X,Y,A\otimes B} \cdot {}^{X}\varphi_2^{Y,A,B} \cdot \varphi_2^{X,{}^{Y}\!A,{}^{Y}\!B} = \varphi_2^{X\otimes Y,A,B} \cdot (\varphi_1^{X,Y,A} \otimes \varphi_1^{X,Y,B})$

(act5) $\varphi_2^{I,A,B} \cdot (\varphi_0^{A} \otimes \varphi_0^{B}) = \varphi_0^{A\otimes B}$

1. All along the paper, we will consider several coherence conditions. When they are already available in the literature, we give a precise reference and express the condition as an equation. This is the case in Subsections 3.5 and 3.11 (categorical actions and derivations) and in Subsections 7.2 and 7.3 (categorical precrossed and crossed modules). Otherwise, we express the condition in the more readable form of a commutative diagram.

One can construct a unique natural family of arrows $\varphi_\bullet^X : I \to {}^X I$ such that

$$\varphi_2^{X,I,B} \cdot (\varphi_\bullet^X \otimes \mathrm{id}) \cdot l = {}^X l \ , \quad \varphi_2^{X,A,I} \cdot (\mathrm{id} \otimes \varphi_\bullet^X) \cdot r = {}^X r \ , \quad \varphi_1^{X,Y,I} \cdot {}^X\varphi_\bullet^Y \cdot \varphi_\bullet^X = \varphi_\bullet^{X \otimes Y} \ , \quad \varphi_0^I = \varphi_\bullet^I$$

We say that an action $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A})$ is symmetric (or that $\mathbb{A}$ is a symmetric $\mathbb{G}$-categorical group) if $\mathbb{A}$ is symmetric and φ factorizes through $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$, where $\mathbb{E}\mathrm{q}_s(\mathbb{A})$ is the categorical group of symmetric monoidal autoequivalences of $\mathbb{A}$. To express an action $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ as a functor $\varphi\colon \mathbb{G} \times \mathbb{A} \to \mathbb{A}$, we need one more condition (see Definition 2.1 in [12]) :

(act6) $\varphi_2^{X,B,A} \cdot c = {}^X c \cdot \varphi_2^{X,A,B}$

Remark 3.6 In Section 6 we will use the fact that any action $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A})$ induces two group actions

$$\pi_0(\varphi)\colon \pi_0(\mathbb{G}) \times \pi_0(\mathbb{A}) \to \pi_0(\mathbb{A}) \quad \text{and} \quad \underline{\varphi}\colon \pi_0(\mathbb{G}) \times \pi_1(\mathbb{A}) \to \pi_1(\mathbb{A})$$

The first one is defined in the obvious way because $\pi_0(\mathbb{G} \times \mathbb{A}) = \pi_0(\mathbb{G}) \times \pi_0(\mathbb{A})$. The second one sends a pair $([X], a) \in \pi_0(\mathbb{G}) \times \pi_1(\mathbb{A})$ on the composite

$$I \xrightarrow{\varphi_\bullet^X} {}^X I \xrightarrow{{}^X a} {}^X I \xrightarrow{(\varphi_\bullet^X)^{-1}} I$$

and is well-defined thanks to the naturality of $\varphi_\bullet$ and the functoriality of φ.

3.7 Kernel. Given a homomorphism $T\colon \mathbb{G} \to \mathbb{H}$ in $\mathcal{CG}$, its kernel is the following diagram in $\mathcal{CG}$

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow k(T)"] & \\ \mathbb{K}\mathrm{er}(T) \arrow[ur, "\mathrm{K}(T)"] \arrow[rr, "0"'] & {} & \mathbb{H} \end{tikzcd}$$

where

- an object of $\mathbb{K}\mathrm{er}(T)$ is a pair $(X \in \mathbb{G}, x\colon TX \to I)$;
- an arrow $f\colon (X,x) \to (Y,y)$ in $\mathbb{K}\mathrm{er}(T)$ is an arrow $f\colon X \to Y$ in $\mathbb{G}$ such that $y \cdot T(f) = x$;
- the faithful (but in general not full) homomorphism $\mathrm{K}(T)\colon \mathbb{K}\mathrm{er}(T) \to \mathbb{G}$ is defined by $\mathrm{K}(T)(f\colon (X,x) \to (Y,y)) = (f\colon X \to Y)$;
- the component at (X,x) of the morphism $k(T)$ is given by $x\colon T(\mathrm{K}(T)(X,x)) = TX \to I = 0(X,x)$.

The kernel of T is a bilimit in the sense of [4] and also a standard homotopy kernel. Bilimit means that for any other diagram in $\mathcal{CG}$ of the form

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow \rho"] & \\ \mathbb{A} \arrow[ur, "R"] \arrow[rr, "0"'] & {} & \mathbb{H} \end{tikzcd}$$

there exists a homomorphism $R' \colon \mathbb{A} \to \mathbb{K}\mathrm{er}(T)$ and a morphism $\rho' \colon \mathrm{K}(T) \cdot R' \Rightarrow R$ such that $\rho \cdot (T \circ \rho') = k(T) \circ R'$ (where $\circ$ is the horizontal or Godement composition of morphisms) ; moreover, if $R'' \colon \mathbb{A} \to \mathbb{K}\mathrm{er}(T)$ and $\rho'' \colon \mathrm{K}(T) \cdot R'' \Rightarrow R$ are such that $\rho \cdot (T \circ \rho'') = k(T) \circ R''$, then there exists a unique morphism $r \colon R' \Rightarrow R''$ such that $\rho'' \cdot (\mathrm{K}(T) \circ r) = \rho'$.
Standard homotopy kernel means that, in the same situation, there exists a unique homomorphism $R' \colon \mathbb{A} \to \mathbb{K}\mathrm{er}(T)$ such that $\mathrm{K}(T) \circ R' = R$ and $k(T) \cdot R' = \rho$.
Finally, if $T \colon \mathbb{G} \to \mathbb{H}$ and $R \colon \mathbb{A} \to \mathbb{G}$ are in $\mathcal{SCG}$, then $\mathbb{K}\mathrm{er}(T), \mathrm{K}(T)$ and R' also are in $\mathcal{SCG}$.
Note that, because of the double universal property of the kernel, we do not pay too much attention to the fact that a diagram in $\mathcal{CG}$ or in $\mathcal{SCG}$ involving kernels (or cokernels) commutes strictly or just up to a 2-arrow.

3.8 Cokernel. Let now $T \colon \mathbb{G} \to \mathbb{H}$ be a homomorphism in $\mathcal{SCG}$. Its cokernel is the following diagram in $\mathcal{SCG}$

$$\begin{tikzcd}
 & \mathbb{H} \arrow[dr, "\mathrm{C}(T)"] & \\
\mathbb{G} \arrow[ur, "T"] \arrow[rr, "0"'] & \arrow[u, phantom, "\Downarrow c(T)"] & \mathbb{C}\mathrm{oker}(T)
\end{tikzcd}$$

and it satisfies two universal properties dual to those of the kernel. It can be described as follows :

- the objects of $\mathbb{C}\mathrm{oker}(T)$ are those of $\mathbb{H}$;
- a prearrow from A to B is a pair $(X \in \mathbb{G}, f \colon A \to TX \otimes B)$;
- an arrow $[X, f] \colon A \mathrel{-\!\circ\!\!\to} B$ is an equivalence class of prearrows, where two prearrows $(X, f), (X', f')$ from$A$ to B are equivalent if there exists an arrow $x \colon X \to X'$ in $\mathbb{G}$ such that $(T(x) \otimes \mathrm{id}) \cdot f = f'$;
- the tensor product of two arrows $[X, f] \colon A \mathrel{-\!\circ\!\!\to} B$ and $[Y, g] \colon C \mathrel{-\!\circ\!\!\to} D$ is given by the class of the prearrow with object part $X \otimes Y$ and arrow part

$$A \otimes C \xrightarrow{f \otimes g} TX \otimes B \otimes TY \otimes D \xrightarrow{\mathrm{id} \otimes c \otimes \mathrm{id}} TX \otimes TY \otimes B \otimes D \simeq T(X \otimes Y) \otimes B \otimes D$$

- the essentially surjective on objects homomorphism $\mathrm{C}(T) \colon \mathbb{H} \to \mathbb{C}\mathrm{oker}(T)$ sends an arrow $f \colon A \to B$ to the class of the prearrow

$$(I,\ A \xrightarrow{f} B \longrightarrow TI \otimes B\ )$$

  $\mathrm{C}(T)$ in general is not a full functor ;
- the component at $X \in \mathbb{G}$ of the morphism $c(T)$ is given by the class of the prearrow $(X, TX \to I \otimes TX)$.

Exercise 3.9 We leave to the reader the following exercise, which is easy but meaningful to grasp the difference between groups and categorical groups when kernels and cokernels are involved.
Let $\mathbb{A}$ be a categorical group and consider the canonical homomorphisms $\mathbb{A} \to 0$ and $0 \to \mathbb{A}$. We have

$$\mathbb{K}\mathrm{er}(\mathbb{A} \to 0) = \mathbb{A} \ , \ \ \mathbb{K}\mathrm{er}(0 \to \mathbb{A}) = [\pi_1(\mathbb{A})]_0$$

If $\mathbb{A}$ is symmetric, we also have

$$\mathbb{C}\mathrm{oker}(\mathbb{A} \to 0) = [\pi_0(\mathbb{A})]_1 \ , \ \ \mathbb{C}\mathrm{oker}(0 \to \mathbb{A}) = \mathbb{A}$$

3.10 Exactness. A diagram

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\rho"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd}$$

in $\mathcal{CG}$ is 2-exact if the comparison homomorphism $F' \colon \mathbb{A} \to \mathbb{K}\mathrm{er}(G)$ is full and essentially surjective on objects. When the above diagram is in $\mathcal{SCG}$, its 2-exactness is equivalent to ask that the comparison homomorphism $G' \colon \mathbb{C}\mathrm{oker}(F) \to \mathbb{C}$ is full and faithful.
Obvious examples of 2-exact complexes are

$$\mathbb{K}\mathrm{er}(T) \xrightarrow{\mathrm{K}(T)} \mathbb{G} \xrightarrow{T} \mathbb{H} \quad \text{and} \quad \mathbb{G} \xrightarrow{T} \mathbb{H} \xrightarrow{\mathrm{C}(T)} \mathbb{C}\mathrm{oker}(T)$$

equipped respectively with $k(T) \colon T \cdot \mathrm{K}(T) \Rightarrow 0$ and $c(T) \colon \mathrm{C}(T) \cdot T \Rightarrow 0$. More interesting, if

$$\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C} \quad \text{with} \quad \rho \colon G \cdot F \Rightarrow 0$$

is 2-exact, then

$$\pi_0(\ \mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}\ ) \quad \text{and} \quad \pi_1(\ \mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}\ )$$

are exact sequences of groups (the converse implication is not true).

3.11 Derivations. If $\mathbb{G}$ is a categorical group and $(\mathbb{A}, \varphi)$ is a $\mathbb{G}$-categorical group, a derivation from $\mathbb{G}$ to $\mathbb{A}$ is a functor $D \colon \mathbb{G} \to \mathbb{A}$ together with a natural family of arrows

$$\beta_{X,Y} \colon DX \otimes {}^{X}DY \to D(X \otimes Y)$$

satisfying the following coherence condition (see Definition 3.1 in [19]) :

(der1) $D(a) \cdot \beta_{X \otimes Y, Z} \cdot (\beta_{X,Y} \otimes \mathrm{id}) \cdot a^{-1} \cdot (\mathrm{id} \otimes (\mathrm{id} \otimes \varphi_1^{X,Y,DZ})) =$
$= \beta_{X, Y \otimes Z} \cdot (\mathrm{id} \otimes {}^{X}\beta_{Y,Z}) \cdot (\mathrm{id} \otimes \varphi_2^{X,DY,{}^{Y}DZ})$

One can construct a unique arrow $\beta_0 \colon I \to DI$ such that

$$\beta_{X,I} \cdot (\mathrm{id} \otimes {}^{X}\beta_0) \cdot (\mathrm{id} \otimes \varphi_{\bullet}^{X}) \cdot r = D(r) \ , \ \ \beta_{I,X} \cdot (\mathrm{id} \otimes \varphi_0^{DX}) \cdot (\beta_0 \otimes \mathrm{id}) \cdot l = Dl$$

Given two derivations $(D, \beta), (D', \beta') \colon \mathbb{G} \to \mathbb{A}$, a morphism of derivations $\epsilon \colon (D, \beta) \Rightarrow (D', \beta')$ is a natural transformation $\epsilon \colon D \Rightarrow D'$ compatible with β and β': (see Definition 3.1 in [19]) (der2) $\epsilon_{X\otimes Y}\cdot\beta_{X,Y}=\beta'_{X,Y}\cdot(\epsilon_X\otimes{}^X\epsilon_Y)$

From the previous conditions, it follows that $\epsilon_I\cdot\beta_0=\beta'_0$.
When $(\mathbb{A},\varphi)$ is a symmetric $\mathbb{G}$-categorical group, the category of derivations $\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi)$ is a symmetric categorical group, with structure induced by that of $\mathbb{A}$. Moreover, in this case there exists a homomorphism $d_{\mathbb{G}}\colon\mathbb{A}\to\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi)$ defined, on an object A, by the inner derivation $d_{\mathbb{G}}(A)(X)={}^XA\otimes A^*$. For such a functor, the structure of derivation $d_{\mathbb{G}}(A)(X)\otimes{}^Xd_{\mathbb{G}}(A)(Y)\to d_{\mathbb{G}}(A)(X\otimes Y)$ is provided by

$$ {}^XA\otimes A^*\otimes{}^X({}^YA\otimes A^*)\xrightarrow{\mathrm{id}\otimes(\varphi_2^{X,{}^YA,A^*})^{-1}}(1)\xrightarrow{\varphi_1^{X,Y,A}\otimes\varphi_2^{X,A,A^*}}(2)\xrightarrow{\mathrm{id}\otimes\varphi_\bullet^X}{}^{X\otimes Y}A\otimes A^* $$

where

(1) is ${}^XA\otimes A^*\otimes{}^X({}^YA)\otimes{}^XA^*\simeq{}^X({}^YA)\otimes A^*\otimes{}^XA\otimes{}^XA^*$

(2) is ${}^{X\otimes Y}A\otimes A^*\otimes{}^X(A\otimes A^*)\simeq{}^{X\otimes Y}A\otimes A^*\otimes{}^XI$

For an arrow $f\colon A\to B$ in $\mathbb{A}$, the natural transformation $d_{\mathbb{G}}(f)\colon d_{\mathbb{G}}(A)\Rightarrow d_{\mathbb{G}}(B)$ has component at X given by ${}^Xf\otimes f^*\colon{}^XA\otimes A^*\to{}^XB\otimes B^*$. We denote the kernel and the cokernel of $d_{\mathbb{G}}$ by

$$ \mathbb{H}^0(\mathbb{G},\mathbb{A},\varphi)\xrightarrow{\mathrm{K}(d_{\mathbb{G}})}\mathbb{A}\xrightarrow{d_{\mathbb{G}}}\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi)\xrightarrow{\mathrm{C}(d_{\mathbb{G}})}\mathbb{H}^1(\mathbb{G},\mathbb{A},\varphi) $$

4. The second cohomology categorical group

In this section we introduce a possible second cohomology categorical group $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$. To avoid confusion with the second cohomology categorical group $\mathcal{H}^2(\mathbb{G},\mathbb{A})$ studied in [20], we adopt the terminology here of "cobord" and "factor set" to describe $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$, and we leave the terminology "cochain" and "cocycle" for $\mathcal{H}^2(\mathbb{G},\mathbb{A})$.
We fix a symmetric $\mathbb{G}$-categorical group $\mathbb{A}=(\mathbb{A},\varphi\colon\mathbb{G}\to\mathbb{E}\mathrm{q}_s(\mathbb{A}))$ as in Subsection 3.5. We start by describing the symmetric categorical group $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ of cobords and the symmetric categorical group $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ of factor sets. In both cases, the structure of symmetric categorical group is inherited from that of $\mathbb{A}$.

Definition 4.1 A cobord of $\mathbb{G}$ with coefficients in $\mathbb{A}$ is a pair $G=(G,G_0)$ with $G\colon\mathbb{G}\to\mathbb{A}$ a functor and $G_0\colon I\to GI$ an arrow. Given two cobords $G,G'\colon\mathbb{G}\to\mathbb{A}$, a morphism of cobords is a natural transformation $\beta\colon G\Rightarrow G'$ such that the following diagram commutes

(cob1)
$$\begin{tikzcd} GI \arrow[rr, "\beta_I"] & & G'I \\ & I \arrow[ul, "G_0"] \arrow[ur, "G'_0"'] & \end{tikzcd}$$

We denote by $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ the symmetric categorical group of cobords.

Definition 4.2 A factor set of $\mathbb{G}$ with coefficients in $\mathbb{A}$ is a 4-tuple $F = (F, a_F, r_F, l_F)$ where $F\colon \mathbb{G} \times \mathbb{G} \to \mathbb{A}$ is a functor and

$$a_F^{X,Y,Z}\colon F(X,Y) \otimes F(X \otimes Y, Z) \to {}^{X}F(Y,Z) \otimes F(X, Y \otimes Z)$$

$$r_F^X\colon I \to F(X,I) \qquad l_F^Y\colon I \to F(I,Y)$$

are natural families of arrows such that the following diagrams commute

(fs1)

$$\begin{tikzcd}
I \otimes F(X,Z) \arrow[r, "\varphi_\bullet^X \otimes \mathrm{id}"] \arrow[d, "r_F^X \otimes F(r_X,\mathrm{id})"'] & {}^{X}I \otimes F(X,Z) \arrow[d, "{}^{X}l_F^Z \otimes F(\mathrm{id}, l_Z)"] \\
F(X,I) \otimes F(X \otimes I, Z) \arrow[r, "a_F^{X,I,Z}"'] & {}^{X}F(I,Z) \otimes F(X, I \otimes Z)
\end{tikzcd}$$

(fs2)

$$\begin{tikzcd}
(1) \arrow[r, "\mathrm{id} \otimes a_F^{X\otimes Y,Z,W}"] \arrow[d, "a_F^{X,Y,Z} \otimes F(a,\mathrm{id})"'] & (6) \arrow[r, "c \otimes \mathrm{id}"] & (7) \arrow[r, "\mathrm{id} \otimes a_F^{X,Y,Z\otimes W}"] & (8) \arrow[d, "(\varphi_1^{X,Y,F(Z,W)})^{-1} \otimes \mathrm{id}"] \\
(2) \arrow[d, "\mathrm{id} \otimes a_F^{X,Y\otimes Z,W}"'] & & & (9) \arrow[d, "\varphi_2^{X,\,{}^{Y}F(Z,W),F(Y,Z\otimes W)}"] \\
(3) \arrow[r, "\varphi_2^{X,F(Y,Z),F(Y\otimes Z,W)} \otimes \mathrm{id}"'] & (4) \arrow[rr, "{}^{X}a_F^{Y,Z,W} \otimes F(\mathrm{id},a)"'] & & (5)
\end{tikzcd}$$

where

$$\begin{aligned}
(1) &= F(X,Y) \otimes F(X \otimes Y, Z) \otimes F((X \otimes Y) \otimes Z, W) \\
(2) &= {}^{X}F(Y,Z) \otimes F(X, Y \otimes Z) \otimes F(X \otimes (Y \otimes Z), W) \\
(3) &= {}^{X}F(Y,Z) \otimes {}^{X}F(Y \otimes Z, W) \otimes F(X, (Y \otimes Z) \otimes W) \\
(4) &= {}^{X}(F(Y,Z) \otimes F(Y \otimes Z, W)) \otimes F(X, (Y \otimes Z) \otimes W) \\
(5) &= {}^{X}({}^{Y}F(Z,W) \otimes F(Y, Z \otimes W)) \otimes F(X, Y \otimes (Z \otimes W)) \\
(6) &= F(X,Y) \otimes {}^{X\otimes Y}F(Z,W) \otimes F(X \otimes Y, Z \otimes W) \\
(7) &= {}^{X\otimes Y}F(Z,W) \otimes F(X,Y) \otimes F(X \otimes Y, Z \otimes W) \\
(8) &= {}^{X\otimes Y}F(Z,W) \otimes {}^{X}F(Y, Z \otimes W) \otimes F(X, Y \otimes (Z \otimes W)) \\
(9) &= {}^{X}({}^{Y}F(Z,W)) \otimes {}^{X}F(Y, Z \otimes W) \otimes F(X, Y \otimes (Z \otimes W))
\end{aligned}$$

Given two factor sets $F, F'\colon \mathbb{G} \times \mathbb{G} \to \mathbb{A}$, a morphism of factor sets is a natural transformation $\alpha\colon F \Rightarrow F'$ such that the following diagrams commute

(fs3)

$$\begin{tikzcd}
F(X,Y) \otimes F(X \otimes Y, Z) \arrow[r, "a_F^{X,Y,Z}"] \arrow[d, "\alpha_{X,Y} \otimes \alpha_{X\otimes Y,Z}"'] & {}^{X}F(Y,Z) \otimes F(X, Y \otimes Z) \arrow[d, "{}^{X}\alpha_{Y,Z} \otimes \alpha_{X,Y\otimes Z}"] \\
F'(X,Y) \otimes F'(X \otimes Y, Z) \arrow[r, "a_{F'}^{X,Y,Z}"'] & {}^{X}F'(Y,Z) \otimes F'(X, Y \otimes Z)
\end{tikzcd}$$

(fs4)
$$\begin{tikzcd}
F(X,I) \arrow[dd, "\alpha_{X,I}"'] & & F(I,Y) \arrow[dd, "\alpha_{I,Y}"] \\
& I \arrow[ul, "r_F^X"'] \arrow[ur, "l_F^Y"] \arrow[dl, "r_{F'}^X"] \arrow[dr, "l_{F'}^Y"'] & \\
F'(X,I) & & F'(I,Y)
\end{tikzcd}$$

We denote by $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ the symmetric categorical group of factor sets.

Remark 4.3 The definition of factor set comes from the following fact. Given a functor $F\colon \mathbb{G}\times\mathbb{G}\to\mathbb{A}$ and three natural families $a_F^{X,Y,Z}, r_F^X, l_F^Y$, we can define a tensor product

$$\otimes_F\colon \mathbb{A}\times\mathbb{G}\times\mathbb{A}\times\mathbb{G}\to\mathbb{A}\times\mathbb{G}\ ,\quad (A,X)\otimes_F(B,Y)=(A\otimes {}^XB\otimes F(X,Y),X\otimes Y)$$

a unit objet (I,I), an inverse $(A,X)^*=({}^{X^*}A^*\otimes {}^{X^*}F^*(X,X^*),X^*)$ and three natural families

- $(A,X)\to(A,X)\otimes_F(I,I)$ with components

$$A\longrightarrow A\otimes I\otimes I\xrightarrow{\mathrm{id}\otimes\varphi_\bullet^X\otimes r_F^X} A\otimes {}^XI\otimes F(X,I)\ \ ,\quad X\longrightarrow X\otimes I$$

- $(B,Y)\to(I,I)\otimes_F(B,Y)$ with components

$$B\longrightarrow I\otimes B\otimes I\xrightarrow{\mathrm{id}\otimes\varphi_0^B\otimes l_F^Y} I\otimes {}^IB\otimes F(I,Y)\ \ ,\quad Y\longrightarrow I\otimes Y$$

- $((A,X)\otimes_F(B,Y))\otimes_F(C,Z)\to(A,X)\otimes_F((B,Y)\otimes_F(C,Z))$ with first component

$$(1)\xrightarrow{\mathrm{id}\otimes c\otimes\mathrm{id}}(2)\xrightarrow{\mathrm{id}\otimes(\varphi_1^{X,Y,C})^{-1}\otimes a_F^{X,Y,Z}}(3)\xrightarrow{\mathrm{id}\otimes\varphi_2^{X,B,{}^YC,F(Y,Z)}\otimes\mathrm{id}}(4)$$

  where

  $(1)\ = A\otimes {}^XB\otimes F(X,Y)\otimes {}^{X\otimes Y}C\otimes F(X\otimes Y,Z)$

  $(2)\ = A\otimes {}^XB\otimes {}^{X\otimes Y}C\otimes F(X,Y)\otimes F(X\otimes Y,Z)$

  $(3)\ = A\otimes {}^XB\otimes {}^X({}^YC)\otimes {}^XF(Y,Z)\otimes F(X,Y\otimes Z)$

  $(4)\ = A\otimes {}^X(B\otimes {}^YC\otimes F(Y,Z))\otimes F(X,Y\otimes Z)$

  and second component $(X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)$.

Conditions (fs1) and (fs2) in Definition 4.2 are precisely what is needed to make such families coherent in the usual sense of monoidal categories, so that $(\mathbb{A}\times\mathbb{G},\otimes_F,(I,I),\ldots)$ is a monoidal category (in fact, a categorical group). In Section 9 we will denote such a categorical group by $\mathbb{A}\times_F\mathbb{G}$. Observe that if the factor set F is the 0-functor, then $\mathbb{A}\times_F\mathbb{G}$ is the semi-direct product $\mathbb{A}\rtimes_\varphi\mathbb{G}$, see [19, 25].

Lemma 4.4 *There exists a homomorphism* $\delta = \delta_{\mathbb{G}} \colon \mathbb{C}^1(\mathbb{G}, \mathbb{A}) \to \mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi)$ *defined, on an object* G, *by*

$$\delta(G)(X, Y) = {}^X GY \otimes G^*(X \otimes Y) \otimes GX$$

$$r^X_{\delta G} \colon\ I \xrightarrow{\varphi^X_\bullet} {}^X I \xrightarrow{{}^X G_0} {}^X GI \longrightarrow {}^X GI \otimes G^*(X \otimes I) \otimes GX$$

$$l^Y_{\delta G} \colon\ I \longrightarrow GY \otimes G^*(I \otimes Y) \otimes I \xrightarrow{\varphi_0^{GY} \otimes \mathrm{id} \otimes G_0} {}^I GY \otimes G^*(I \otimes Y) \otimes GI$$

$$a^{X,Y,Z}_{\delta G} \colon\ (1) \longrightarrow (2) \xrightarrow{(\varphi_1^{X,Y,GZ})^{-1} \otimes \mathrm{can}} (3) \xrightarrow{\varphi_2^{X,\,{}^Y GZ, G^*(Y \otimes Z)} \otimes \mathrm{id}} (4)$$

where

$$\begin{aligned}
(1) &= {}^X GY \otimes G^*(X \otimes Y) \otimes GX \otimes {}^{X \otimes Y} GZ \otimes G^*(X \otimes Y \otimes Z) \otimes G(X \otimes Y) \\
(2) &= {}^{X \otimes Y} GZ \otimes {}^X GY \otimes G^*(X \otimes Y \otimes Z) \otimes GX \\
(3) &= {}^X({}^Y GZ) \otimes {}^X G^*(Y \otimes Z) \otimes {}^X GY \otimes {}^X G(Y \otimes Z) \otimes G^*(X \otimes Y \otimes Z) \otimes GX \\
(4) &= {}^X({}^Y GZ \otimes G^*(Y \otimes Z) \otimes GY) \otimes {}^X G(Y \otimes Z) \otimes G^*(X \otimes Y \otimes Z) \otimes GX
\end{aligned}$$

and, on a morphism $\beta \colon G \Rightarrow G'$, *by*

$$\delta(\beta)_{X,Y} \colon\ {}^X GY \otimes G^*(X \otimes Y) \otimes GX \xrightarrow{{}^X \beta_Y \otimes \beta^*_{X \otimes Y} \otimes \beta_X} {}^X G'Y \otimes G'^*(X \otimes Y) \otimes G'X$$

Moreover, the kernel of $\delta_{\mathbb{G}}$ *is equivalent to* $\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}, \varphi)$.

The fact that the families $r_{\delta G}, l_{\delta G}$ and $a_{\delta G}$ constructed in Lemma 4.4 are canonical depends on the fact that $\mathbb{A}$ is symmetric, a braiding on $\mathbb{A}$ is not sufficient here.

Definition 4.5 The second cohomology categorical group $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ of $\mathbb{G}$ with coefficients in $\mathbb{A}$ is the cokernel of $\delta_{\mathbb{G}}$. Therefore, we have

$$\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}, \varphi) \xrightarrow{\mathrm{K}(\delta)} \mathbb{C}^1(\mathbb{G}, \mathbb{A}) \xrightarrow{\delta} \mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi) \xrightarrow{\mathrm{C}(\delta)} \mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$$

Example 4.6

1. If $\mathbb{G}$ is the discrete categorical group $[G]_0$ associated with a group G, then the set of objects of $\mathbb{Z}^2([G]_0, \mathbb{A})$ is identified with the set of Ulbrich's 3-cocycles of G with coefficients in $\mathbb{A}$, see [33]. In particular (see Example 3 in [12]), if $\mathbb{A}$ is the discrete symmetric $[G]_0$-categorical group $[A]_0$ associated with a G-module A, then $\mathbb{Z}^2([G]_0, [A]_0)$ is the discrete symmetric categorical group $[\mathrm{Z}^2(G, A)]_0$ associated with the abelian group of Eilenberg-Mac Lane 2-cocycles of G with coefficients in A.
2. If $\mathbb{G} = [G]_0$ and, moreover, $\mathbb{A} = [A]_1$ is the symmetric $[G]_0$-categorical group with only one object associated with a G-module A, then $\mathbb{Z}^2([G]_0, [A]_1)$ is the discrete symmetric categorical group associated with the abelian group of Eilenberg-Mac Lane 3-cocycles of G with coefficients in the G-module A.

This shows that the cohomology groups introduced by Eilenberg-Mac Lane [28] and Ulbrich [33] can be obtained as special instances of the symmetric categorical group $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ via the functor π_0.

5. The Hochschild-Serre sequences for categorical groups

In order to prepare the Hochschild-Serre 2-exact sequences, we generalise Lemma 2.1 and Lemma 2.2 to symmetric categorical groups.

Lemma 5.1 *Consider a diagram in* $\mathcal{SCG}$

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "D"] & \mathbb{K}\mathrm{er}(\delta) \arrow[r, "\mathrm{K}(\delta)"] \arrow[d, "\mathrm{C}(D)"'] & \mathbb{C} \arrow[r, "\delta"] \arrow[d, "\mathrm{C}(\mathrm{K}(\delta)\cdot D)"'] & \mathbb{Z} \arrow[r, "\mathrm{C}(\delta)"] \arrow[dr, "\mathrm{C}(\underline{\delta})"'] & \mathbb{C}\mathrm{oker}(\delta) \arrow[d, "S"] \\
 & \mathbb{C}\mathrm{oker}(D) \arrow[r, "\underline{\mathrm{K}}(\delta)"'] & \mathbb{C}\mathrm{oker}(\mathrm{K}(\delta)\cdot D) \arrow[ur, "\underline{\delta}"'] & & \mathbb{C}\mathrm{oker}(\underline{\delta})
\end{tikzcd}$$

where $\underline{\mathrm{K}}(\delta)$, $\underline{\delta}$ *and* S *are induced by the universal properties of* $\mathbb{C}\mathrm{oker}(D)$, $\mathbb{C}\mathrm{oker}(\mathrm{K}(\delta)\cdot D)$ *and* $\mathbb{C}\mathrm{oker}(\delta)$. *Then*

1. $\underline{\mathrm{K}}(\delta)\colon \mathbb{C}\mathrm{oker}(D) \to \mathbb{C}\mathrm{oker}(\mathrm{K}(\delta)\cdot D)$ *is the kernel of* $\underline{\delta}\colon \mathbb{C}\mathrm{oker}(\mathrm{K}(\delta)\cdot D) \to \mathbb{Z}$;
2. $S\colon \mathbb{C}\mathrm{oker}(\delta) \to \mathbb{C}\mathrm{oker}(\underline{\delta})$ *is full and essentially surjective.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathrm{K}(\delta)\cdot D"] \arrow[d] & \mathbb{C} \arrow[r, "\mathrm{C}(\mathrm{K}(\delta)\cdot D)"] \arrow[d, "\delta"'] & \mathrm{Coker}(\mathrm{K}(\delta)\cdot D) \arrow[d, "\underline{\delta}"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{Id}"'] & \mathbb{Z}
\end{tikzcd}$$

Apply the Kernel-Cokernel Lemma for symmetric categorical groups (Proposition 6.3 in [15]) and use Exercise 3.9. We get a 2-exact sequence

$$\mathbb{A} \xrightarrow{D} \mathbb{K}\mathrm{er}(\delta) \xrightarrow{T} \mathbb{K}\mathrm{er}(\underline{\delta}) \longrightarrow [\pi_0(\mathbb{A})]_1 \xrightarrow{R} \mathbb{C}\mathrm{oker}(\delta) \xrightarrow{S} \mathbb{C}\mathrm{oker}(\underline{\delta})$$

1. Consider the factorization through the cokernel

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "D"] & \mathbb{K}\mathrm{er}(\delta) \arrow[r, "T"] \arrow[d, "\mathrm{C}(D)"'] & \mathbb{K}\mathrm{er}(\underline{\delta}) \\
 & \mathbb{C}\mathrm{oker}(D) \arrow[ur, "T'"'] &
\end{tikzcd}$$

By 2-exactness in $\mathbb{K}\mathrm{er}(\delta)$, T' is full and faithful. Moreover, the sequence

$$\pi_0(\mathbb{K}\mathrm{er}(\delta)) \xrightarrow{\pi_0(T)} \pi_0(\mathbb{K}\mathrm{er}(\underline{\delta})) \longrightarrow \pi_0([\pi_0(\mathbb{A})]_1) = 0$$

is exact, so that $\pi_0(T)$ is surjective. This means that T is essentially surjective, and then T' also is essentially surjective.

2. Consider the factorization through the kernel

$$\begin{tikzcd}
{[\pi_0(\mathbb{A})]_1} \arrow[r, "R"] \arrow[dr, "R'"'] & \mathrm{Coker}(\delta) \arrow[r, "S"] & \mathrm{Coker}(\underline{\delta}) \\
& \mathbb{K}\mathrm{er}(S) \arrow[u, "\mathrm{K}(S)"'] &
\end{tikzcd}$$

By 2-exactness in $\mathbb{C}\mathrm{oker}(\delta)$, R' is (full and) essentially surjective. Therefore,

$$\pi_0(R') \colon \pi_0([\pi_0(\mathbb{A})]_1) = 0 \to \pi_0(\mathbb{K}\mathrm{er}(S))$$

is surjective, and then $\pi_0(\mathbb{K}\mathrm{er}(S)) = 0$. This means that S is full (see Proposition 2.1 in [34]). Finally, the fact that S is essentially surjective is obvious because $S \cdot \mathrm{C}(\delta) = \mathrm{C}(\underline{\delta})$ and $\mathrm{C}(\underline{\delta})$ is essentially surjective. ∎

Lemma 5.2 *Consider a diagram in $\mathcal{SCG}$*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "D"'] \arrow[dr, phantom, "\simeq"] & \mathbb{A}' \arrow[d, "D'"] & \\
\mathbb{C} \arrow[r, "G"'] \arrow[d, "\mathrm{C}(D)"'] & \mathbb{C}' \arrow[r, "\mathrm{C}(G)"] \arrow[d, "\mathrm{C}(D')"] & \mathbb{C}\mathrm{oker}(G) \arrow[d, "R"] \\
\mathbb{C}\mathrm{oker}(D) \arrow[r, "\underline{G}"'] & \mathbb{C}\mathrm{oker}(D') \arrow[r, "\mathrm{C}(\underline{G})"'] & \mathbb{C}\mathrm{oker}(\underline{G})
\end{tikzcd}$$

with F full and essentially surjective, and G faithful. Then

1. *the comparison $\underline{G} \colon \mathbb{C}\mathrm{oker}(D) \to \mathbb{C}\mathrm{oker}(D')$ is faithful;*
2. *the comparison $R \colon \mathbb{C}\mathrm{oker}(G) \to \mathbb{C}\mathrm{oker}(\underline{G})$ is an equivalence.*

Proof. Using the factorization $\underline{D}$ of D through the kernel of $\mathrm{C}(D)$, and the factorization $\underline{D}'$ of D' through the kernel of $\mathrm{C}(D')$, we split the upper part of the previous diagram as

$$\begin{tikzcd}
\mathbb{A} \arrow[rrr, "F"] \arrow[dr, "\underline{D}"] \arrow[dd, "D"'] & & & \mathbb{A}' \arrow[dl, "\underline{D}'"'] \arrow[dd, "D'"] \\
& \mathrm{Ker}(\mathrm{C}(D)) \arrow[r, "\underline{F}"'] \arrow[dl, "\mathrm{K}(\mathrm{C}(D))"] & \mathbb{K}\mathrm{er}(\mathrm{C}(D')) \arrow[dr, "\mathrm{K}(\mathrm{C}(D))"'] & \\
\mathbb{C} \arrow[rrr, "G"'] & & & \mathbb{C}'
\end{tikzcd}$$

Following Proposition 2.1 in [26], we have that $\underline{D}$ and $\underline{D}'$ are full and essentially surjective. Therefore, $\mathbb{C}\mathrm{oker}(D)$ is also a cokernel of $\mathrm{K}(\mathrm{C}(D))$, $\mathbb{C}\mathrm{oker}(D')$ is also a cokernel of

$\mathrm{K}(\mathrm{C}(D'))$, and $\underline{F}$ is full and essentially surjective. Apply now the Kernel-Cokernel Lemma to the diagram

$$\begin{tikzcd}
\mathbb{K}\mathrm{er}(\mathrm{C}(D)) \arrow[r, "\mathrm{K}(\mathrm{C}(D))"] \arrow[d, "\underline{F}"'] & \mathbb{C} \arrow[r, "\mathrm{C}(D)"] \arrow[d, "G"] & \mathbb{C}\mathrm{oker}(D) \arrow[d, "\underline{G}"] \\
\mathbb{K}\mathrm{er}(\mathrm{C}(D')) \arrow[r, "\mathrm{K}(\mathrm{C}(D'))"'] & \mathbb{C}' \arrow[r, "\mathrm{C}(D')"'] & \mathbb{C}\mathrm{oker}(D')
\end{tikzcd}$$

We get a 2-exact sequence

$$\mathbb{K}\mathrm{er}(\underline{F}) \longrightarrow \mathbb{K}\mathrm{er}(G) \longrightarrow \mathbb{K}\mathrm{er}(\underline{G}) \longrightarrow 0 \longrightarrow \mathbb{C}\mathrm{oker}(G) \xrightarrow{R} \mathbb{C}\mathrm{oker}(\underline{G})$$

where $\mathbb{C}\mathrm{oker}(\underline{F})$ is equivalent to 0 because $\underline{F}$ is full and essentially surjective (see Proposition 2.2 in [34]).
1. From the previous 2-exact sequence, we get an exact sequence of abelian groups

$$0 = \pi_1(\mathbb{K}\mathrm{er}(G)) \to \pi_1(\mathbb{K}\mathrm{er}(\underline{G})) \to 0$$

where $0 = \pi_1(\mathbb{K}\mathrm{er}(G))$ because G is faithful (see Proposition 2.1 in [34]). Therefore, $\pi_1(\mathbb{K}\mathrm{er}(\underline{G})) = 0$, and then $\underline{G}$ is faithful.
2. The 2-exactness in $\mathbb{C}\mathrm{oker}(G)$ of the previous sequence of symmetric categorical groups gives that R is full and faithful (use Exercise 3.9). Finally, R is also essentially surjective because $R \cdot \mathrm{C}(G) = \mathrm{C}(\underline{G}) \cdot \mathrm{C}(D')$. ■

5.3 Hochschild-Serre 2-exact sequences. In the rest of this section, we fix an essentially surjective homomorphism of categorical groups $P\colon \mathbb{E} \to \mathbb{G}$ and a symmetric $\mathbb{G}$-categorical group $\mathbb{A} = (\mathbb{A}, \varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A}))$. Clearly, P induces a symmetric action $\varphi \cdot P$ of $\mathbb{E}$ on $\mathbb{A}$ (by abuse of notation, we will write φ instead of $\varphi \cdot P$). Moreover, P induces by composition two faithful homomorphisms

$$P_{\mathbb{C}}\colon \mathbb{C}^1(\mathbb{G}, \mathbb{A}) \to \mathbb{C}^1(\mathbb{E}, \mathbb{A}) \;\text{ and }\; P_{\mathbb{Z}}\colon \mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi) \to \mathbb{Z}^2(\mathbb{E}, \mathbb{A}, \varphi)$$

We can now construct the following diagram in $\mathcal{SCG}$ (each part is either commutative or commutative up to a 2-arrow) :

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, "\mathrm{Id}"] \arrow[d, "d_{\mathbb{G}}"'] & & \mathbb{A} \arrow[d, "d_{\mathbb{E}}"'] & & & \\
\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi) \arrow[rr] \arrow[dr, "\mathrm{C}(d_{\mathbb{G}})"] \arrow[dd, "\mathrm{K}(\delta_{\mathbb{G}})"'] & & \mathbb{D}\mathrm{er}(\mathbb{E},\mathbb{A},\varphi) \arrow[rr] \arrow[dr, "\mathrm{C}(d_{\mathbb{E}})"] \arrow[dd, "\mathrm{K}(\delta_{\mathbb{E}})"'] & & \mathbb{K}\mathrm{er}(\delta_P) \arrow[dr, "\pi_P"] \arrow[dd, "\mathrm{K}(\delta_P)"'] & \\
& \mathbb{H}^1(\mathbb{G},\mathbb{A},\varphi) \arrow[rr] \arrow[dd, "\underline{\mathrm{K}}(\delta_{\mathbb{G}})"'] & & \mathbb{H}^1(\mathbb{E},\mathbb{A},\varphi) \arrow[rr] \arrow[dd, "\underline{\mathrm{K}}(\delta_{\mathbb{E}})"'] & & \mathbb{K}\mathrm{er}(\underline{\delta}_P) \arrow[dd, "\mathrm{K}(\underline{\delta}_P)"] \\
\mathbb{C}^1(\mathbb{G},\mathbb{A}) \arrow[rr, "P_{\mathbb{C}}"] \arrow[dr, "\mathrm{C}(\mathrm{K}(\delta_{\mathbb{G}})\cdot d_{\mathbb{G}})"] \arrow[dd, "\delta_{\mathbb{G}}"'] & & \mathbb{C}^1(\mathbb{E},\mathbb{A}) \arrow[rr, "\mathrm{C}(P_{\mathbb{C}})"] \arrow[dr, "\mathrm{C}(\mathrm{K}(\delta_{\mathbb{E}})\cdot d_{\mathbb{E}})"] \arrow[dd, "\delta_{\mathbb{E}}"'] & & \mathbb{C}^1(P,\mathbb{A}) \arrow[dr, "\underline{\pi}_P"] \arrow[dd, "\delta_P"'] & \\
& \underline{\mathbb{C}}^1(\mathbb{G},\mathbb{A}) \arrow[rr, "\underline{P}_{\mathbb{C}}"] \arrow[dl, "\underline{\delta}_{\mathbb{G}}"'] & & \underline{\mathbb{C}}^1(\mathbb{E},\mathbb{A}) \arrow[rr, "\mathrm{C}(\underline{P}_{\mathbb{C}})"] \arrow[dl, "\underline{\delta}_{\mathbb{E}}"'] & & \underline{\mathbb{C}}^1(P,\mathbb{A}) \arrow[dl, "\underline{\delta}_P"] \\
\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[rr, "P_{\mathbb{Z}}"] \arrow[dr, "\mathrm{C}(\underline{\delta}_{\mathbb{G}})"] \arrow[dd, "\mathrm{C}(\delta_{\mathbb{G}})"'] & & \mathbb{Z}^2(\mathbb{E},\mathbb{A},\varphi) \arrow[rr, "\mathrm{C}(P_{\mathbb{Z}})"] \arrow[dr, "\mathrm{C}(\underline{\delta}_{\mathbb{E}})"] \arrow[dd, "\mathrm{C}(\delta_{\mathbb{E}})"'] & & \mathbb{Z}^2(P,\mathbb{A},\varphi) \arrow[dr, "\mathrm{C}(\underline{\delta}_P)"] \arrow[dd, "\mathrm{C}(\delta_P)"'] & \\
& \underline{\mathbb{H}}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[rr] & & \underline{\mathbb{H}}^2(\mathbb{E},\mathbb{A},\varphi) \arrow[rr] & & \mathbb{C}\mathrm{oker}(\underline{\delta}_P) \\
\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[rr] \arrow[ur, "s_{\mathbb{G}}"] & & \mathbb{H}^2(\mathbb{E},\mathbb{A},\varphi) \arrow[rr] \arrow[ur, "s_{\mathbb{E}}"] & & \mathbb{C}\mathrm{oker}(\delta_P) \arrow[ur, "s_P"'] &
\end{tikzcd}
$$

where

- $\underline{\mathbb{C}}^1(\mathbb{G},\mathbb{A})$ is the cokernel of $\mathrm{K}(\delta_{\mathbb{G}}) \cdot d_{\mathbb{G}}$ and $\underline{\mathbb{H}}^2(\mathbb{G},\mathbb{A},\varphi)$ is the cokernel of $\underline{\delta}_{\mathbb{G}}$;
- $\mathbb{C}^1(P,\mathbb{A})$ is the cokernel of $P_{\mathbb{C}}$, $\underline{\mathbb{C}}^1(P,\mathbb{A})$ is the cokernel of $\underline{P}_{\mathbb{C}}$, and $\mathbb{Z}^2(P,\mathbb{A},\varphi)$ is the cokernel of $P_{\mathbb{Z}}$.

Proposition 5.4 *Let $P\colon \mathbb{E} \to \mathbb{G}$ be an essentially surjective homomorphism of categorical groups and $\mathbb{A}$ a symmetric $\mathbb{G}$-categorical group. There exist 2-exact sequences of symmetric categorical groups*

$$
\begin{tikzcd}
\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi) \arrow[r] \arrow[d, "\mathrm{C}(d_{\mathbb{G}})"'] & \mathbb{D}\mathrm{er}(\mathbb{E},\mathbb{A},\varphi) \arrow[r] \arrow[d, "\mathrm{C}(d_{\mathbb{E}})"'] & \mathbb{K}\mathrm{er}(\delta_P) \arrow[r] \arrow[d, "\pi_P"'] & \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[r] \arrow[d, "s_{\mathbb{G}}"] & \mathbb{H}^2(\mathbb{E},\mathbb{A},\varphi) \arrow[r] \arrow[d, "s_{\mathbb{E}}"] & \mathbb{C}\mathrm{oker}(\delta_P) \arrow[d, "s_P"] \\
\mathbb{H}^1(\mathbb{G},\mathbb{A},\varphi) \arrow[r] & \mathbb{H}^1(\mathbb{E},\mathbb{A},\varphi) \arrow[r] & \mathbb{K}\mathrm{er}(\underline{\delta}_P) \arrow[r] & \underline{\mathbb{H}}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[r] & \underline{\mathbb{H}}^2(\mathbb{E},\mathbb{A},\varphi) \arrow[r] & \mathbb{C}\mathrm{oker}(\underline{\delta}_P)
\end{tikzcd}
$$

Moreover, for the connecting homomorphisms, we have :

- *$\mathrm{C}(d_{\mathbb{G}})$ and $\mathrm{C}(d_{\mathbb{E}})$ are essentially surjective ;*
- *$s_{\mathbb{G}}$ and $s_{\mathbb{E}}$ are full and essentially surjective ;*
- *π_P and s_P are equivalences.*

Proof. Apply the Kernel-Cokernel Lemma to the diagrams

$$\begin{tikzcd}
\mathbb{C}^1(\mathbb{G},\mathbb{A}) \arrow[r, "P_{\mathbb{C}}"] \arrow[d, "\delta_{\mathbb{G}}"'] & \mathbb{C}^1(\mathbb{E},\mathbb{A}) \arrow[d, "\delta_{\mathbb{E}}"] \\
\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[r, "P_{\mathbb{Z}}"'] & \mathbb{Z}^2(\mathbb{E},\mathbb{A},\varphi)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{C}}^1(\mathbb{G},\mathbb{A}) \arrow[r, "\underline{P}_{\mathbb{C}}"] \arrow[d, "\underline{\delta}_{\mathbb{G}}"'] & \underline{\mathbb{C}}^1(\mathbb{E},\mathbb{A}) \arrow[d, "\underline{\delta}_{\mathbb{E}}"] \\
\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[r, "P_{\mathbb{Z}}"'] & \mathbb{Z}^2(\mathbb{E},\mathbb{A},\varphi)
\end{tikzcd}$$

in order to construct the 2-exact sequeces. Then apply Lemma 5.1 to the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "d_{\mathbb{G}}"] & \mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi) \arrow[r, "\mathrm{K}(\delta_{\mathbb{G}})"] \arrow[d, "\mathrm{C}(d_{\mathbb{G}})"'] & \mathbb{C}^1(\mathbb{G},\mathbb{A}) \arrow[r, "\delta_{\mathbb{G}}"] \arrow[d, "\mathrm{C}(\mathrm{K}(\delta_{\mathbb{G}})\cdot d_{\mathbb{G}})"'] & \mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[r, "\mathrm{C}(\delta_{\mathbb{G}})"] \arrow[dr, "\mathrm{C}(\underline{\delta}_{\mathbb{G}})"'] & \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[d, "s_{\mathbb{G}}"] \\
& \mathbb{H}^1(\mathbb{G},\mathbb{A},\varphi) \arrow[r, "\underline{\mathrm{K}}(\delta_{\mathbb{G}})"'] & \underline{\mathbb{C}}^1(\mathbb{G},\mathbb{A}) \arrow[ur, "\underline{\delta}_{\mathbb{G}}"'] & & \underline{\mathbb{H}}^2(\mathbb{G},\mathbb{A},\varphi)
\end{tikzcd}$$

(and to the similar diagram with $\mathbb{G}$ replaced by $\mathbb{E}$) and Lemma 5.2 to the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{K}(\delta_{\mathbb{G}})\cdot d_{\mathbb{G}}"'] & \mathbb{A} \arrow[d, "\mathrm{K}(\delta_{\mathbb{E}})\cdot d_{\mathbb{E}}"] & \\
\mathbb{C}^1(\mathbb{G},\mathbb{A}) \arrow[r, "P_{\mathbb{C}}"] \arrow[d, "\mathrm{C}(\mathrm{K}(\delta_{\mathbb{G}})\cdot d_{\mathbb{G}})"'] & \mathbb{C}^1(\mathbb{E},\mathbb{A}) \arrow[r, "\mathrm{C}(P_{\mathbb{C}})"] \arrow[d, "\mathrm{C}(\mathrm{K}(\delta_{\mathbb{E}})\cdot d_{\mathbb{E}})"] & \mathbb{C}^1(P,\mathbb{A}) \arrow[d, "\underline{\pi}_P"] \\
\underline{\mathbb{C}}^1(\mathbb{G},\mathbb{A}) \arrow[r, "\underline{P}_{\mathbb{C}}"'] & \underline{\mathbb{C}}^1(\mathbb{E},\mathbb{A}) \arrow[r, "\mathrm{C}(\underline{P}_{\mathbb{C}})"'] & \underline{\mathbb{C}}^1(P,\mathbb{A})
\end{tikzcd}$$

The fact that $\underline{\pi}_P$ is an equivalence immediately implies that π_P and s_P also are equivalences. ∎

6. The middle point of the Hochschild-Serre 2-exact sequences

In this section we look for a description of $\mathbb{K}\mathrm{er}(\delta_P)$ similar to the one obtained with Proposition 2.4 in the case of groups.
We fix the following data

$$\mathbb{N} \xrightarrow{\mathrm{K}(P)} \mathbb{E} \xrightarrow{P} \mathbb{G} \xrightarrow{\varphi} \mathbb{E}\mathrm{q}_s(\mathbb{A})$$

where P is a homomorphism of categorical groups, $\mathbb{N}$ is the kernel of P, and $(\mathbb{A},\varphi)$ is a symmetric $\mathbb{G}$-categorical group. From Example 2.6.v in [13], recall that there is a canonical action $\bullet\colon \mathbb{E}\times\mathbb{N}\to\mathbb{N}$ with constraints $\bullet_0,\bullet_1,\bullet_2$. Explicitly, $E\bullet(N, n\colon PN\to I)$ is given by

$$(E\otimes N\otimes E^*,\ P(E\otimes N\otimes E^*) \xrightarrow{P_2} PE\otimes PN\otimes PE^* \xrightarrow{\mathrm{id}\otimes n\otimes \mathrm{id}} PE\otimes I\otimes PE^*\simeq I)$$

6.1 The equivariant hom. We denote by $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})$ the symmetric categorical group having equivariant homomorphisms as objects. This means that an object is a homomorphism $(H,H_2)\colon \mathbb{N}\to\mathbb{A}$ in $\mathcal{CG}$ together with a natural family of arrows

$$H_\varphi^{E,(N,n)}\colon H(E\bullet(N,n)) \to {}^{PE}H(N,n)$$

satisfying the following coherence conditions [2] :

(act7') $H_\varphi^{I,(N,n)}\cdot H(\bullet_0^{(N,n)}) = {}^{P_0}H(N,n)\cdot \varphi_0^{H(N,n)}$

(act8') ${}^{PE}H_2^{(N,n),(M,m)}\cdot \varphi_2^{PE,(N,n),(M,m)}\cdot (H_\varphi^{E,(N,n)}\otimes H_\varphi^{E,(M,m)}) =$
$= H_\varphi^{E,(N,n)\otimes(M,m)}\cdot H(\bullet_2^{E,(N,n),(M,m)})\cdot H_2^{E\bullet(N,n),E\bullet(M,m)}$

(act9') ${}^{P_2^{E,F}}H(N,n)\cdot\varphi_1^{PE,PF,(N,n)}\cdot {}^{PE}H_\varphi^{F,(N,n)}\cdot H_\varphi^{E,F\bullet(N,n)} = H_\varphi^{E\otimes F,(N,n)}\cdot H(\bullet_1^{E,F,(N,n)})$

A morphism

$$\alpha\colon (H,H_2,H_\varphi)\Rightarrow(K,K_2,K_\varphi)\colon \mathbb{N}\to\mathbb{A}$$

in $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})$ is a morphism $\alpha\colon H\Rightarrow K$ in $\mathcal{CG}$ such that

(act10') ${}^{PE}\alpha_{(N,n)}\cdot H_\varphi^{E,(N,n)} = K_\varphi^{E,(N,n)}\cdot\alpha_{E\bullet(N,n)}$

In order to prove that $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})$ and $\mathbb{K}\mathrm{er}(\delta_P)$ are equivalent symmetric categorical groups (Proposition 6.4), we need two lemmas.

Lemma 6.2 *Consider the group actions*

$$\pi_0(\mathbb{E})\times\pi_0(\mathbb{N})\xrightarrow{\pi_0(\bullet)}\pi_0(\mathbb{N}) \quad \textit{and} \quad \pi_0(\mathbb{E})\times\pi_1(\mathbb{A})\xrightarrow{\pi_0(P)\times\mathrm{id}}\pi_0(\mathbb{G})\times\pi_1(\mathbb{A})\xrightarrow{\underline{\varphi}}\pi_1(\mathbb{A})$$

(see Remark 3.6 for the action $\underline{\varphi}$*), and the abelian group* $\mathrm{Hom}_{\pi_0(\mathbb{E})}(\pi_0(\mathbb{N}),\pi_1(\mathbb{A}))$ *of equivariant group homomorphisms. Then*

$$\pi_1(\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A}))\simeq \mathrm{Hom}_{\pi_0(\mathbb{E})}(\pi_0(\mathbb{N}),\pi_1(\mathbb{A}))$$

Proof. As in Remark 3.3, a morphism $\alpha\colon 0\Rightarrow 0$ in $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})$ induces a group homomorphism

$$\pi_0(\mathbb{N})\to\pi_1(\mathbb{A}) \ , \ \ [N,n]\mapsto(\alpha_{(N,n)}\colon I\to I)$$

Moreover, by condition (act10'), such a homomorphism is equivariant. ■

2. We write these conditions just as equations, because they are special cases of conditions (act7) – (act10) defining homomorphisms and morphisms of actions, see Subsection 7.1.

Lemma 6.3 *Consider the following diagram in $\mathcal{CG}$*

$$\begin{tikzcd}
 & \mathbb{E} \arrow[rr, "0"] \arrow[dr, "P"] \arrow[d, phantom, "\Downarrow k(P)"] & & \mathbb{A} \\
\mathbb{N} \arrow[ur, "\mathrm{K}(P)"] \arrow[rr, "0"'] & {} & \mathbb{G} \arrow[ur, "U"'] \arrow[u, phantom, "\Uparrow\beta"] &
\end{tikzcd}$$

If P is essentially surjective and if β and $k(P)$ are compatible, that is, for every $(N,n) \in \mathbb{N}$

$$\begin{tikzcd}
UPN \arrow[rr, "U(n)"] \arrow[dr, "\beta_N"'] & & UI \\
 & I \arrow[ur, "U_0"'] &
\end{tikzcd}$$

commutes, then there exists a unique morphism $\overline{\beta}\colon U \Rightarrow 0$ in $\mathcal{CG}$ such that $\overline{\beta} \circ P = \beta$.

Proof. Using part 2 of Proposition 7.9, this Lemma can be seen as a special case of Lemma 7.5. For the reader convenience, we sketch here the direct argument. The uniqueness of $\overline{\beta}$ immediately follows from the fact that P is essentially surjective. To construct $\overline{\beta}$, consider an object $X \in \mathbb{G}$, fix $E \in \mathbb{E}$ and $x\colon X \to PE$ and put

$$\overline{\beta}_X\colon\ UX \xrightarrow{U(x)} UPE \xrightarrow{\beta_E} I$$

The naturality of $\overline{\beta}$ amounts to the equation $\beta_F \cdot U(f) = \beta_E$ for every arrow $f\colon PE \to PF$ in $\mathbb{G}$. To check this equation, consider the following object in $\mathbb{N}$

$$(F \otimes E^*, n_f\colon\ P(F \otimes E^*) \simeq PF \otimes PE^* \xrightarrow{f \otimes \mathrm{id}} PE \otimes PE^* \simeq I\ )$$

The compatibility between β and $k(P)$ applied to the object $(F \otimes E^*, n_f)$ gives the commutativity of the exterior of the following diagram

$$\begin{tikzcd}
U(PE \otimes PE^*) \arrow[ddd, "U(f \otimes \mathrm{id})"'] & & UI \arrow[ll, "U(\eta_{PE})"'] & I \arrow[l, "U_0"'] \arrow[dl, "\mathrm{can}"] \arrow[ddd, "\mathrm{id}"] \\
 & UPE \otimes UPE^* \arrow[ul, "U_2^{PE,PE^*}"] \arrow[r, "\beta_E \otimes \beta_{E^*}"] \arrow[d, "U(f) \otimes \mathrm{id}"'] \arrow[dr, phantom, "(*)"] & I \otimes I \arrow[d, "\mathrm{id}"] & \\
 & UPF \otimes UPE^* \arrow[r, "\beta_F \otimes \beta_{E^*}"'] \arrow[dl, "U_2^{PF,PE^*}"'] & I \otimes I & \\
U(PF \otimes PE^*) \arrow[rr, "U(P_2^{F,E^*})"'] & & UP(F \otimes E^*) \arrow[r, "\beta_{F \otimes E^*}"'] & I \arrow[ul, "\mathrm{can}"']
\end{tikzcd}$$

Since the four unlabelled regions commute by naturality and monoidality of β, the region labelled $(*)$ also commutes. Tensoring $(*)$ with UPE we get $\beta_F \cdot U(f) = \beta_E$, as requested. The rest of the proof is now easy. ∎

Consider now $\mathbb{N} = \mathbb{K}\mathrm{er}(P) \to \mathbb{E} \to \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ as at the beginning of this section, and $\mathbb{K}\mathrm{er}(\delta_P) \to \mathbb{C}^1(P, \mathbb{A}) \to \mathbb{Z}^2(P, \mathbb{A}, \varphi)$ as in Subsection 5.3. We have the following result.

Proposition 6.4 *There is a homomorphism of symmetric categorical groups*

$$\Sigma\colon \mathbb{K}\mathrm{er}(\delta_P) \to \underline{\mathbb{H}\mathrm{om}}(\mathbb{N}, \mathbb{A})$$

Moreover :

1. *If P is essentially surjective, then Σ is faithful.*
2. *If P has a functorial section, then Σ is an equivalence.*

Proof. Construction of the functor Σ.
Recall that an object in $\mathbb{K}\mathrm{er}(\delta_P)$ is given by $G\colon \mathbb{E} \to \mathbb{A}$ in $\mathbb{C}^1(\mathbb{E}, \mathbb{A})$ together with an equivalence class $[L, \alpha]$ with $L\colon \mathbb{G} \times \mathbb{G} \to \mathbb{A}$ in $\mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi)$ and $\alpha\colon \delta_{\mathbb{E}}(G) \Rightarrow P_{\mathbb{Z}}(L)$ in $\mathbb{Z}^2(\mathbb{E}, \mathbb{A}, \varphi)$. We define $\Sigma(G, [L, \alpha])$ to be the restriction of G to the kernel of P

$$\Sigma(G, [L, \alpha]) = \ \mathbb{N} \xrightarrow{\mathrm{K}(P)} \mathbb{E} \xrightarrow{G} \mathbb{A}$$

The monoidal structure $G(\mathrm{K}(P)(M, m)) \otimes G(\mathrm{K}(P)(N, n)) \to G(\mathrm{K}(P)((M, m) \otimes (N, n)))$ is given by

$$\begin{tikzcd}
GM \otimes GN \arrow[r, "c"] & GN \otimes GM \arrow[r, "\varphi_0^{GN} \otimes \mathrm{id}"] & {}^{I}GN \otimes GM \arrow[r, "m^{-1}GN \otimes \mathrm{id}"] & {}^{PM}GN \otimes GM \arrow[d, "\overline{\alpha}_{M,N}"] \\
G(M \otimes N) \arrow[r, "r_L^I = l_L^I"] & L(I, I) \otimes G(M \otimes N) & & L(PM, PN) \otimes G(M \otimes N) \arrow[ll, "L(m,n) \otimes \mathrm{id}"']
\end{tikzcd}$$

where $\overline{\alpha}_{M,N}$ is obtained in a canonical way from

$$\alpha_{M,N}\colon\ {}^{PM}GN \otimes G^*(M \otimes N) \otimes GM \to L(PM, PN)$$

The equivariant structure $G(\mathrm{K}(P)(E \bullet (N, n))) \to {}^{PE}G(\mathrm{K}(P)(N, n))$ is obtained in a similar way, using $\varphi_2, \alpha_{N,E}, \alpha_{E,N \otimes E^*}$ and α_{E,E^*}.
The functor Σ is well-defined on objects. Indeed, $[L, \alpha] = [L', \alpha']$ means that there exists $\lambda\colon L \Rightarrow L'$ in $\mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi)$ such that $P_{\mathbb{Z}}(\lambda) \cdot \alpha = \alpha'$. This implies the commutativity of the following diagram, which expresses the fact that the monoidal structure of $\Sigma(G, [L, \alpha])$ does not depend on the choice of L and α (and similarly for the equivariant structure)

$$\begin{tikzcd}
L(PM, PN) \otimes G(M \otimes N) \arrow[rr, "L(m,n) \otimes \mathrm{id}"] \arrow[dd, bend left, "\lambda_{PM,PN} \otimes \mathrm{id}"] & & L(I, I) \otimes G(M \otimes N) \arrow[dd, bend right, "\lambda_{I,I} \otimes \mathrm{id}"'] \\
{}^{PM}GN \otimes GM \arrow[u, "\overline{\alpha}_{M,N}"] \arrow[d, "\overline{\alpha'}_{M,N}"'] & & G(M \otimes N) \arrow[u, "r_L^I \otimes \mathrm{id}"'] \arrow[d, "r_{L'}^I \otimes \mathrm{id}"] \\
L'(PM, PN) \otimes G(M \otimes N) \arrow[rr, "L'(m,n) \otimes \mathrm{id}"'] & & L'(I, I) \otimes G(M \otimes N)
\end{tikzcd}$$

Recall now that an arrow $[U,\beta]\colon (G,[L,\alpha]) \Rightarrow (G',[L',\alpha'])$ in $\mathbb{K}\mathrm{er}(\delta_P)$ is a class of pairs with $U\colon \mathbb{G} \to \mathbb{A}$ in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ and $\beta\colon G \Rightarrow P_{\mathbb{C}}(U) \otimes G'$ in $\mathbb{C}^1(\mathbb{E},\mathbb{A})$ such that there exists $\lambda\colon L \Rightarrow \delta_{\mathbb{G}}(U) \otimes L'$ in $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ such that $P_{\mathbb{Z}}(\lambda)\cdot\alpha = (\delta_{\mathbb{E}}(P_{\mathbb{C}}(U))\otimes\alpha')\cdot\delta_{\mathbb{E}}(\beta)$. We define

$$\Sigma[U,\beta]\colon \Sigma(G,[L,\alpha]) = G\cdot \mathrm{K}(P) \Rightarrow G'\cdot \mathrm{K}(P) = \Sigma(G',[L',\alpha'])$$

$$\Sigma[U,\beta]_{(N,n)}\colon\ GN \xrightarrow{\beta_N} UPN \otimes G'N \xrightarrow{U(n)\otimes\mathrm{id}} UI \otimes G'N \simeq G'N$$

The functor Σ is well-defined on arrows. Indeed, $[U,\beta] = [U',\beta']$ means that there exists $\gamma\colon U \Rightarrow U'$ in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ such that $(P_{\mathbb{C}}(\gamma)\otimes G')\cdot\beta = \beta'$. This implies the commutativity of the following diagram, which expresses the fact that $\Sigma[U,\beta]$ does not depend on the choice of U and β

$$\begin{tikzcd}
UPN \otimes G'N \arrow[rr, "U(n)\otimes\mathrm{id}"] \arrow[dd, bend left, "\gamma_N\otimes\mathrm{id}"] & & UI \otimes G'N \arrow[dd, bend right, "\gamma_I\otimes\mathrm{id}"'] \\
GN \arrow[u, "\beta_N"] \arrow[d, "\beta'_N"'] & & I \otimes G'N \arrow[u, "U_0\otimes\mathrm{id}"'] \arrow[d, "U'_0\otimes\mathrm{id}"] \\
U'PN \otimes G'N \arrow[rr, "U'(n)\otimes\mathrm{id}"'] & & U'I \otimes G'N
\end{tikzcd}$$

Since the symmetric monoidal structure of $\mathbb{K}\mathrm{er}(\delta_P)$ and of $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})$ are both induced by that of $\mathbb{A}$, the functor Σ is a homomorphism in $\mathcal{SCG}$.

1. Σ is faithful. This is equivalent to prove that $\pi_1(\Sigma)\colon \pi_1(\mathbb{K}\mathrm{er}(\delta_P)) \to \pi_1(\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A}))$ is injective (Proposition 1.1 in [34]). Using Lemma 6.2, we get the following explicit description of $\pi_1(\Sigma)$: for an element $[U\colon \mathbb{G}\to\mathbb{A}, \beta\colon 0 \Rightarrow P_{\mathbb{C}}(U)]$ in $\pi_1(\mathbb{K}\mathrm{er}(\delta_P))$,

$$\pi_1(\Sigma)[U,\beta]\colon \pi_0(\mathbb{N}) \to \pi_0(\mathbb{A})\ ,\ \ [N,n] \mapsto\ I \xrightarrow{\beta_N} UPN \xrightarrow{U(n)} UI \xrightarrow{U_0^{-1}} I$$

Assume now that $[U,\beta]$ is in the kernel of $\pi_1(\Sigma)$. This means that, for any object (N,n) in the kernel of P, one has that

$$\begin{tikzcd}
I \arrow[r, "\beta_N"] \arrow[dr, "U_0"'] & UPN \arrow[d, "U(n)"] \\
 & UI
\end{tikzcd}$$

commutes. This is precisely the compatibility condition requested in Lemma 6.3. Since P is essentially surjective, from Lemma 6.3. we get a (unique) $\gamma\colon 0 \Rightarrow U$ in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ such that $\gamma \circ P = \beta$. Therefore, $[U,\beta] = 0$ in $\pi_1(\mathbb{K}\mathrm{er}(\delta_P))$ and $\pi_1(\Sigma)$ is injective. Observe that we can apply Lemma 6.3 even if U in general is not monoidal. Indeed, since $[U,\beta]$ is in the kernel of $\pi_1(\Sigma)$, there exists $\lambda\colon 0 \Rightarrow \delta_{\mathbb{G}}(U)$ in $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ such that $P_{\mathbb{Z}}(\lambda) = \delta_{\mathbb{E}}(\beta)$. Explicitly, this gives

$$\lambda_{X,Y}\colon I \to\ {}^{X}UY \otimes U^*(X\otimes Y)\otimes UX$$

so that U is a derivation and β is a morphism of derivations, and derivations correspond to homomorphisms into the semi-direct product (see [19] or [25]).
2. Assume now that P has a functorial section, that is, there exists a functor $S\colon \mathbb{G} \to \mathbb{E}$ and a natural transformation $s\colon \mathrm{Id} \Rightarrow P \cdot S$. With no lost of generality, we can assume that there exists an arrow $S_0\colon I \to SI$ such that $P(S_0) \cdot P_0 = s_I$.
The functor Σ is essentially surjective. Let

$$(H, H_2)\colon \mathbb{N} \to \mathbb{A}\ , \quad H_\varphi^{E,(N,n)}\colon H(E \bullet (N,n)) \to {}^{PE}H(N,n)$$

be an object in $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N}, \mathbb{A})$. We look for an object $(G_S, [L_S, \alpha_S])$ in $\mathbb{K}\mathrm{er}(\delta_P)$ such that $\Sigma(G_S, [L_S, \alpha_S]) \simeq (H, H_2, H_\varphi)$. From an object $E \in \mathbb{E}$, we construct an arrow

$$e_s\colon\ P(E \otimes (SPE)^*) \simeq PE \otimes (PSPE)^* \xrightarrow{\mathrm{id} \otimes s_{PE}^*} PE \otimes (PE)^* \simeq I$$

and we define

$$G_S\colon \mathbb{E} \to \mathbb{A}\ , \quad G_S(E) = H(E \otimes (SPE)^*, e_s)$$

The functor G_S is defined in a similar way on arrows. Using S_0 we get that G_S is in $\mathbb{C}^1(\mathbb{E}, \mathbb{A})$. From objects $X, Y \in \mathbb{G}$, we construct now an arrow

$$(x, y)_s\colon P(S(X \otimes Y) \otimes (SY)^* \otimes (SX)^*) \to I$$

Up to the constraint P_2, $(x, y)_s$ is given by

$$PS(X \otimes Y) \otimes (PSY)^* \otimes (PSX)^* \xrightarrow{s_{X \otimes Y} \otimes s_Y^* \otimes s_X^*} X \otimes Y \otimes Y^* \otimes X^* \simeq I$$

We define

$$L_S\colon \mathbb{G} \times \mathbb{G} \to \mathbb{A}\ , \quad L_S(X, Y) = H(S(X \otimes Y) \otimes (SY)^* \otimes (SX)^*, (x, y)_s)$$

L_S is defined in a similar way on arrows. To have a factor set, we still need the constraints a_L, r_L, l_L as in Definition 4.2. We will go back to this point in few lines, let us now construct $\alpha_S\colon \delta_{\mathbb{E}}(G_S) \Rightarrow P_{\mathbb{Z}}(L_S)$. The idea is to go back to the proof of Proposition 2.4, check that $\delta_E(g_s) = p_2(f_s)$ and then follow step-by-step such a proof. The output is the following natural family of arrows (without writing the arrow-part of the objects of $\mathbb{N}$)

$$(1) \xrightarrow{H_\varphi^{-1} \otimes \mathrm{can} \otimes \mathrm{id}} (2) \xrightarrow{c \otimes \mathrm{id}} (3) \xrightarrow{H_2} (4) \xrightarrow{\mathrm{can}} (5)$$

where

$$\begin{aligned}
(1) &= {}^{PX}H(Y \otimes (SPY)^*) \otimes H(X \otimes Y \otimes (SP(X \otimes Y))^*)^* \otimes H(X \otimes (SPX)^*) \\
(2) &= H(X \otimes Y \otimes (SPY)^* \otimes X^*) \otimes H(SP(X \otimes Y) \otimes Y^* \otimes X^*) \otimes H(X \otimes (SPX)^*) \\
(3) &= H(SP(X \otimes Y) \otimes Y^* \otimes X^*) \otimes H(X \otimes Y \otimes (SPY)^* \otimes X^*) \otimes H(X \otimes (SPX)^*) \\
(4) &= H(SP(X \otimes Y) \otimes Y^* \otimes X^* \otimes X \otimes Y \otimes (SPY)^* \otimes X^* \otimes X \otimes (SPX)^*)
\end{aligned}$$

(5) $= H(SP(X \otimes Y) \otimes (SPY)^* \otimes (SPX)^*)$

In the same way, to construct the constraints a_L, r_L, l_L one can make explicit the fact that the map $f_s\colon G \times G \to A$ considered in the proof of Proposition 2.4 is a 2-cocycle, and then follow step-by-step such a proof. We omit this part, as well as the straightforward construction of an isomorphism $\Sigma(G_S, [L_S, \alpha_S]) \simeq (H, H_2, H_\varphi)$.

The functor Σ is full. This is equivalent to prove that $\pi_0(\mathbb{K}\mathrm{er}(\Sigma)) = 0$ (Proposition 2.1 in [34]). An object in $\mathbb{K}\mathrm{er}(\Sigma)$ is an object $(G, [L, \alpha])$ in $\mathbb{K}\mathrm{er}(\delta_P)$ together with a morphism $\mu\colon 0 \Rightarrow \Sigma(G, [L, \alpha])$ in $\underline{\mathbb{H}\mathrm{om}}(\mathbb{N}, \mathbb{A})$, that is, $\mu\colon 0 \Rightarrow G \cdot \mathrm{K}(P)$ is a monoidal and equivariant natural transformation. To prove that $[G, [L, \alpha]] = 0$, we need $U_S\colon \mathbb{G} \to \mathbb{A}$ in $\mathbb{C}^1(\mathbb{G}, \mathbb{A})$, $\beta_S\colon G \Rightarrow P_{\mathbb{C}}(U_S)$ in $\mathbb{C}^1(\mathbb{E}, \mathbb{A})$ and $\lambda_S\colon L \Rightarrow \delta_{\mathbb{G}}(U_S)$ in $\mathbb{Z}^2(\mathbb{G}, \mathbb{A}, \varphi)$. We define

$$U_S\colon\ \mathbb{G} \xrightarrow{S} \mathbb{E} \xrightarrow{G} \mathbb{A}$$

We construct β_S in two steps. Consider first the following natural family of arrows depending on the object $X \in \mathbb{G}$

$$\begin{tikzcd}
I \arrow[r, "G_0"] & GI \arrow[r, "G(\eta_{SX})"] & G(SX \otimes (SX)^*) \arrow[r, "\overline{\alpha}"] & GSX \otimes {}^{PSX}G(S^*X) \otimes L^*(PSX, (PSX)^*) \arrow[d, "\mathrm{id} \otimes {}^{s_X}G(S^*X) \otimes L^*(s_X, s_X^*)"'] \\
& & & GSX \otimes {}^{X}G(S^*X) \otimes L^*(X, X^*)
\end{tikzcd}$$

with $\overline{\alpha}$ is obtained in a canonical way from α. Tensoring with G^*SX, we get

$$b_S(X)\colon G^*SX \to {}^{X}G(S^*X) \otimes L^*(X, X^*)$$

Fix now an object $E \in \mathbb{E}$ and consider the object $(E \otimes S^*PE, e_s\colon P(E \otimes S^*PE) \to I)$ of $\mathbb{N}$ already used to prove that Σ is essentially surjective. We can construct the following natural family of arrows depending on $E \in \mathbb{E}$

$$\begin{tikzcd}
I \arrow[r, "\mu_{(E \otimes S^*PE, e_s)}"] & G(E \otimes S^*PE) \arrow[r, "\widetilde{\alpha}"] & GE \otimes {}^{PE}G(S^*PE) \otimes L^*(PE, (PSPE)^*) \arrow[d, "\mathrm{id} \otimes L^*(\mathrm{id}, s_{PE}^*)"'] \\
& GE \otimes G^*SPE & GE \otimes {}^{PE}G(S^*PE) \otimes L^*(PE, (PE)^*) \arrow[l, "\mathrm{id} \otimes b_S(PE)^{-1}"']
\end{tikzcd}$$

with $\widetilde{\alpha}$ obtained once again from α in a canonical way. Tensoring with $GSPE$ we get

$$\beta_S(E)\colon U_SPE = GSPE \to GE$$

as required. It remains to construct

$$\lambda_S(X, Y)\colon L(X, Y) \to {}^{X}U_SX \otimes U_S^*(X \otimes Y) \otimes U_SX$$

The argument is similar to the one used to prove that Σ is faithful, and we only sketch it. Using β_S and α, we construct

$$\begin{tikzcd} \tau_{E,F}\colon\ L(PE,PF) \arrow[r, "\alpha^{-1}_{E,F}"] & {}^{PE}GF\otimes G^*(E\otimes F)\otimes GE \arrow[d, "{}^{PE}\beta_S(F)\otimes\beta^*_S(E\otimes F)\otimes\beta_S(E)"] \\ & {}^{PE}U_SPF\otimes U^*_SP(E\otimes F)\otimes U_SPE \end{tikzcd}$$

Now we have to prove that τ is compatible with $k(P)$. This means that for every $(N,n),(M,m)\in\mathbb{N}$, the following diagram must be commutative

$$\begin{tikzcd} L(PN,PM) \arrow[rr, "\tau_{N,M}"] \arrow[d, "L(n,m)"'] & & {}^{PN}U_SPM\otimes U^*_S(PN\otimes PM)\otimes U_SPN \arrow[d, "{}^{n}U_S(m)\otimes U^*_S(n\otimes m)\otimes U_S(n)"] \\ L(I,I) & I \arrow[l, "\mathrm{can}"] \arrow[r, "\mathrm{can}"'] & {}^{I}U_SI\otimes U^*_S(I\otimes I)\otimes U_SI \end{tikzcd}$$

If this is the case, then there exists a unique $\lambda_S\colon L\Rightarrow\delta_{\mathbb{G}}(U)$ such that $\lambda_S\circ(P\times P)=\mu$, and we have done. Through a (quite intricate) diagram chasing, the commutativity of the above diagram reduces to the fact that if $E=\mathrm{K}(P)(N,n)$, then the arrow $e_s\colon P(E\otimes S^*PE)\to I$ is equal to

$$P(N\otimes S^*PN)\xrightarrow{P(\mathrm{id}\otimes S^*(n))}P(N\otimes I)\simeq PN\xrightarrow{n}I$$

and this is easy to prove using the naturality of s and Lemma 3.2 applied to the arrow $n\colon PN\to I$. ∎

Putting together Proposition 5.4 and Proposition 6.4, we get the following result.

Corollary 6.5 *Let $P\colon\mathbb{E}\to\mathbb{G}$ be an essentially surjective homomorphism of categorical groups and $\mathbb{A}$ a symmetric $\mathbb{G}$-categorical group. There exist 2-exact sequences of symmetric categorical groups*

$$\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{D}\mathrm{er}(\mathbb{E},\mathbb{A},\varphi)\to\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})\to\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{H}^2(\mathbb{E},\mathbb{A},\varphi)\to\mathbb{C}\mathrm{oker}(\delta_P)$$

$$\mathbb{H}^1(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{H}^1(\mathbb{E},\mathbb{A},\varphi)\to\underline{\mathbb{H}\mathrm{om}}(\mathbb{N},\mathbb{A})\to\underline{\mathbb{H}}^2(\mathbb{G},\mathbb{A},\varphi)\to\underline{\mathbb{H}}^2(\mathbb{E},\mathbb{A},\varphi)\to\mathbb{C}\mathrm{oker}(\underline{\delta}_P)$$

7. Categorical crossed modules

Categorical precrossed modules and categorical crossed modules have been introduced in [13], and used in [10] as algebraic models for connected homotopy 3-types. In this section we complete the definitions of [13] so to organize categorical (pre)crossed modules in a 2-category. We also add some results on the quotient categorical group associated with a categorical crossed module.

7.1 The 2-category of actions. Consider two actions

$$(\varphi\colon \mathbb{G}\times\mathbb{A}\to\mathbb{A},\varphi_0,\varphi_1,\varphi_2) \text{ and } (\varphi'\colon \mathbb{G}'\times\mathbb{A}'\to\mathbb{A}',\varphi'_0,\varphi'_1,\varphi'_2)$$

in the sense of Subsection 3.5. A homomorphism of actions is a triple

$$(R,F,\lambda)\colon (\mathbb{G},\mathbb{A},\varphi)\to(\mathbb{G}',\mathbb{A}',\varphi')$$

with $R\colon \mathbb{G}\to\mathbb{G}'$ and $F\colon \mathbb{A}\to\mathbb{A}'$ two homomorphisms in $\mathcal{CG}$ (with F in $\mathcal{SCG}$ if the actions φ and φ' are symmetric), and

$$\lambda^{X,A}\colon F({}^{X}A)\to {}^{RX}FA$$

a natural family of arrows satisfying the following coherence conditions :

(act7)
$$\begin{tikzcd} FA \arrow[r, "F(\varphi_0^A)"] \arrow[d, "{\varphi'}_0^{FA}"'] & F({}^{I}A) \arrow[d, "\lambda^{I,A}"] \\ {}^{I}FA \arrow[r, "R_0FA"'] & {}^{RI}FA \end{tikzcd}$$

(act8)
$$\begin{tikzcd} F({}^{X}A)\otimes F({}^{X}B) \arrow[r, "F_2^{{}^XA,{}^XB}"] \arrow[d, "\lambda^{X,A}\otimes\lambda^{X,B}"'] & F({}^{X}A\otimes{}^{X}B) \arrow[r, "F(\varphi_2^{X,A,B})"] & F({}^{X}(A\otimes B)) \arrow[d, "\lambda^{X,A\otimes B}"] \\ {}^{RX}FA\otimes{}^{RX}FB \arrow[r, "{\varphi'}_2^{RX,FA,FB}"'] & {}^{RX}(FA\otimes FB) \arrow[r, "{}^{RX}F_2^{A,B}"'] & {}^{RX}F(A\otimes B) \end{tikzcd}$$

(act9)
$$\begin{tikzcd} {}^{RX}F({}^{Y}A) \arrow[d, "{}^{RX}\lambda^{Y,A}"'] & F({}^{X}({}^{Y}A)) \arrow[l, "\lambda^{X,{}^YA}"'] \arrow[r, "F(\varphi_1^{X,Y,A})"] & F({}^{X\otimes Y}A) \arrow[d, "\lambda^{X\otimes Y,A}"] \\ {}^{RX}({}^{RY}FA) \arrow[r, "{\varphi'}_1^{RX,RY,FA}"'] & {}^{RX\otimes RY}FA \arrow[r, "{}^{R_2^{X,Y}}FA"'] & {}^{R(X\otimes Y)}FA \end{tikzcd}$$

It follows that $\lambda^{X,I}\cdot F(\varphi_\bullet^X)\cdot F_0 = {}^{RX}F_0\cdot{\varphi'}_\bullet^{RX}$.
Given two homomorphisms of actions

$$(R,F,\lambda),(R',F',\lambda')\colon (\mathbb{G},\mathbb{A},\varphi)\to(\mathbb{G}',\mathbb{A}',\varphi')$$

a morphisms of actions is a pair of morphisms

$$\beta\colon R\Rightarrow R' \qquad \alpha\colon F\Rightarrow F'$$

in $\mathcal{CG}$ such that the following diagram commutes

(act10)
$$\begin{tikzcd} F\cdot\varphi \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\alpha\circ\varphi"'] & \varphi'\cdot(R\times F) \arrow[d, Rightarrow, "\varphi'\circ(\beta\times\alpha)"] \\ F'\cdot\varphi \arrow[r, Rightarrow, "\lambda'"'] & \varphi'\cdot(R'\times F') \end{tikzcd}$$

Actions with their homomorphisms and morphisms form a 2-category denoted $\mathcal{ACT}$ (or $\mathcal{SACT}$ if we restrict to symmetric actions).
Observe that, if $(R,F,\lambda)\colon (\mathbb{G},\mathbb{A},\varphi)\to(\mathbb{G}',\mathbb{A}',\varphi')$ is a homomorphism of actions with $\mathbb{A}=\mathbb{A}'$ and $F=\mathrm{Id}_{\mathbb{A}}$, then λ becomes a morphism in $\mathcal{CG}$:

$$\begin{tikzcd} & \mathbb{G}' \arrow[dr, "\varphi'"] & \\ \mathbb{G} \arrow[ur, "R"] \arrow[rr, "\varphi"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow\lambda"] & & \mathbb{E}\mathrm{q}(\mathbb{A}) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & \mathbb{G}' \arrow[dr, "\varphi'"] & \\ \mathbb{G} \arrow[ur, "R"] \arrow[rr, "\varphi"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow\lambda"] & & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \end{tikzcd}$$

7.2 The 2-category of categorical precrossed modules. A categorical precrossed module is a 5-tuple

$$(\mathbb{G},\mathbb{A},\varphi\colon \mathbb{G}\to\mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A}\to\mathbb{G},\nu)$$

with $(\mathbb{G},\mathbb{A},\varphi)$ an action, $T\colon\mathbb{A}\to\mathbb{G}$ a homomorphism in $\mathcal{CG}$, and

$$\nu_{X,A}\colon T({}^{X}A)\otimes X\to X\otimes TA$$

a natural family of arrows satisfying the following coherence conditions (see Definition 2.2 in [13]) :

(pcm1) $\nu_{X\otimes Y,A}\cdot(T(\varphi_1^{X,Y,A})\otimes\mathrm{id})=(\mathrm{id}\otimes\nu_{Y,A}\otimes\mathrm{id})\cdot(\nu_{X,{}^{Y}A}\otimes\mathrm{id})$

(pcm2) $(\mathrm{id}\otimes T_2^{A,B})\cdot(\nu_{X,A}\otimes\mathrm{id})\cdot(\mathrm{id}\otimes\nu_{X,B})=\nu_{X,A\otimes B}\cdot(T(\varphi_2^{X,A,B})\otimes\mathrm{id})\cdot(T_2^{{}^{X}A,{}^{X}B}\otimes\mathrm{id})$

(pcm3) $\nu_{I,A}\cdot T(\varphi_0^{A}\otimes\mathrm{id})\cdot T(l_A)=l_{TA}$

It follows that $\nu_{X,I}\cdot T(\varphi_\bullet^{X}\otimes\mathrm{id})\cdot(T_0\otimes\mathrm{id})\cdot l_X=(\mathrm{id}\otimes T_0)\cdot r_X$.
Given two categorical precrossed modules

$$(\mathbb{G},\mathbb{A},\varphi\colon\mathbb{G}\to\mathbb{E}\mathrm{q}(\mathbb{A}),T\colon\mathbb{A}\to\mathbb{G},\nu)\ \text{ and }\ (\mathbb{G}',\mathbb{A}',\varphi'\colon\mathbb{G}'\to\mathbb{E}\mathrm{q}(\mathbb{A}'),T'\colon\mathbb{A}'\to\mathbb{G}',\nu')$$

a homomorphism of categorical precrossed modules is a 4-tuple

$$(R,F,\lambda,\tau)\colon(\mathbb{G},\mathbb{A},\varphi,T,\nu)\to(\mathbb{G}',\mathbb{A}',\varphi',T',\nu')$$

with $(R,F,\lambda)\colon(\mathbb{G},\mathbb{A},\varphi)\to(\mathbb{G}',\mathbb{A}',\varphi')$ a homomorphism in $\mathcal{ACT}$, and $\tau\colon R\cdot T\Rightarrow T'\cdot F$ a morphism in $\mathcal{CG}$ such that the following diagram commutes

(pcm4)
$$\begin{tikzcd} R(T({}^{X}A)\otimes X) \arrow[rr, "R(\nu_{X,A})"] & & R(X\otimes TA) \arrow[d, "(R_2^{X,TA})^{-1}"] \\ RT({}^{X}A)\otimes RX \arrow[u, "R_2^{T({}^{X}A),X}"] \arrow[d, "\tau_{{}^{X}A}\otimes\mathrm{id}"'] & & RX\otimes RTA \arrow[d, "\mathrm{id}\otimes\tau_A"] \\ T'F({}^{X}A)\otimes RX \arrow[dr, "T'(\lambda^{X,A})\otimes\mathrm{id}"'] & & RX\otimes T'(FA) \arrow[dl, "\nu'_{RX,FA}"] \\ & T'({}^{RX}FA)\otimes RX & \end{tikzcd}$$

Given two homomorphisms of categorical precrossed modules

$$(R, F, \lambda, \tau), (R', F', \lambda', \tau')\colon (\mathbb{G}, \mathbb{A}, \varphi, T, \nu) \to (\mathbb{G}', \mathbb{A}', \varphi', T', \nu')$$

a morphism of categorical precrossed modules is a morphism

$$(\beta, \alpha)\colon (R, F, \lambda) \Rightarrow (R', F', \lambda')$$

in $\mathcal{ACT}$ such that the following diagram commutes

(pcm5)
$$\begin{tikzcd} R \cdot T \arrow[r, Rightarrow, "\tau"] \arrow[d, Rightarrow, "\beta \circ T"'] & T' \cdot F \arrow[d, Rightarrow, "T' \circ \alpha"] \\ R' \cdot T \arrow[r, Rightarrow, "\tau'"'] & T' \cdot F' \end{tikzcd}$$

Categorical precrossed modules with their homomorphisms and morphisms form a 2-category denoted $\mathcal{CPCM}$ (or $\mathcal{SCPCM}$ if we restrict to symmetric actions).

7.3 The 2-category of categorical crossed modules. A categorical crossed module is a 6-tuple

$$(\mathbb{G}, \mathbb{A}, \varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi)$$

with $(\mathbb{G}, \mathbb{A}, \varphi, T, \nu)$ a categorical precrossed module, and

$$\chi_{A,B}\colon {}^{TA}B \otimes A \to A \otimes B$$

a natural family of arrows satisfying the following coherence conditions (see Definition 2.4 in [13]) :

(cm1) $(\mathrm{id} \otimes \chi_{B,C}) \cdot (\chi_{A,{}^{TB}C} \otimes \mathrm{id}) \cdot ((\varphi_1^{TA,TB,C})^{-1} \otimes \mathrm{id}) \cdot ({}^{T_2^{A,B}}C \otimes \mathrm{id}) = \chi_{A \otimes B,C}$

(cm2) $(\chi_{A,B} \otimes \mathrm{id}) \cdot (\mathrm{id} \otimes \chi_{A,C}) = \chi_{A,B \otimes C} \cdot (\varphi_2^{TA,B,C} \otimes \mathrm{id})$

(cm3) ${}^{X}\chi_{A,B} \cdot \varphi_2^{X,{}^{TA}B,A} \cdot (\varphi_1^{X,TA,B} \otimes \mathrm{id})^{-1} \cdot ({}^{\nu_{X,A}}B \otimes \mathrm{id}) \cdot (\varphi_1^{T({}^{X}A),X,B} \otimes \mathrm{id}) = \varphi_2^{X,A,B} \cdot \chi_{{}^{X}A,{}^{X}B}$

(cm4) $T_2^{A,B} \cdot \nu_{TA,B} = T(\chi_{A,B}) \cdot T_2^{{}^{TA}B,A}$

(cm5) $\chi_{I,B} \cdot r_{{}^{TI}B} \cdot {}^{T_0}B \cdot \varphi_0^{B} = l_B$

It follows that $\chi_{A,I} \cdot (\varphi_\bullet^{TA} \otimes \mathrm{id}) \cdot l_A = r_A$.
Given two categorical crossed modules

$$(\mathbb{G}, \mathbb{A}, \varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi) \ \text{ and } \ (\mathbb{G}', \mathbb{A}', \varphi'\colon \mathbb{G}' \to \mathbb{E}\mathrm{q}(\mathbb{A}'), T'\colon \mathbb{A}' \to \mathbb{G}', \nu', \chi')$$

a homomorphism of categorical crossed modules is a homomorphism

$$(R, F, \lambda, \tau)\colon (\mathbb{G}, \mathbb{A}, \varphi, T, \nu) \to (\mathbb{G}', \mathbb{A}', \varphi', T', \nu')$$

in $\mathcal{CPCM}$ such that the following diagram commutes

(cm6)
$$\begin{tikzcd}
F({}^{TA}B \otimes A) \arrow[r, "F(\chi_{A,B})"] & F(A \otimes B) \\
F({}^{TA}B) \otimes FA \arrow[u, "F_2^{{}^{TA}B,A}"] & FA \otimes FB \arrow[u, "F_2^{A,B}"'] \\
{}^{RTA}FB \otimes FA \arrow[u, "\lambda^{TA,B}\otimes\mathrm{id}"] \arrow[r, "{\tau_A}_{FB}\otimes\mathrm{id}"'] & {}^{T'EA}FB \otimes FA \arrow[u, "\chi'_{FA,FB}"']
\end{tikzcd}$$

Finally, morphisms of categorical crossed modules are the same as morphisms of categorical precrossed modules.
Categorical crossed modules with their homomorphisms and morphisms form a 2-category denoted $\mathcal{CCM}$ (or $\mathcal{SCCM}$ if we restrict to symmetric actions).

7.4 The quotient categorical group. Let us recall from [13] that, given a categorical crossed module $(\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi)$, we can construct the quotient categorical group

$$\begin{tikzcd}
 & \mathbb{G} \arrow[dr, "\mathrm{C}(T)"] \arrow[d, phantom, "\Downarrow c(T)"] & \\
\mathbb{A} \arrow[ur, "T"] \arrow[rr, "0"'] & & \mathbb{G}/T
\end{tikzcd}$$

Its construction is the same as for the cokernel of a homomorphism in $\mathcal{SCG}$, see Subsection 3.8, the only difference is the tensor product of arrows : given two arrows $[A,f]\colon\ X \multimap Y$ and $[B,g]\colon\ H \multimap K$, their tensor product is given by the class of the prearrow with object part $A \otimes {}^{Y}B$ and arrow part

$$X \otimes H \xrightarrow{f\otimes g} TA \otimes Y \otimes TB \otimes K \xrightarrow{\mathrm{id}\otimes\nu_{Y,B}^{-1}\otimes\mathrm{id}} TA \otimes T({}^{Y}B) \otimes Y \otimes K \simeq T(A \otimes {}^{Y}B) \otimes Y \otimes K$$

Here is the universal property (in the bilimit style) of the quotient categorical group : given a diagram in $\mathcal{CG}$

$$\begin{tikzcd}
 & \mathbb{G} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\delta"] & \\
\mathbb{A} \arrow[ur, "T"] \arrow[rr, "0"'] & & \mathbb{H}
\end{tikzcd}$$

if the following diagram commutes

(qcg1)
$$\begin{tikzcd}
GX \otimes GTA \arrow[r, "G_2^{X,TA}"] \arrow[d, "\mathrm{id}\otimes\delta_A"'] & G(X \otimes TA) & G(T({}^{X}A) \otimes X) \arrow[l, "G(\nu_{X,A})"'] \\
GX \otimes I \arrow[r, "\mathrm{can}"'] & I \otimes GX & GT({}^{X}A) \otimes GX \arrow[l, "\delta_{{}^{X}A}\otimes\mathrm{id}"] \arrow[u, "G_2^{T({}^{X}A),X}"']
\end{tikzcd}$$

then there exists

$$\begin{tikzcd} & \mathbb{G}/T \arrow[dr, "G'"] \arrow[d, phantom, "\Downarrow\delta'"] & \\ \mathbb{G} \arrow[ur, "\mathrm{C}(T)"] \arrow[rr, "G"'] & {} & \mathbb{H} \end{tikzcd}$$

in $\mathcal{CG}$ such that the following diagram commutes

$$\text{(qcg2)} \quad \begin{tikzcd} G' \cdot \mathrm{C}(T) \cdot T \arrow[r, Rightarrow, "\delta' \circ T"] \arrow[d, Rightarrow, "G' \circ c(T)"'] & G \cdot T \arrow[d, Rightarrow, "\delta"] \\ G' \cdot 0 & 0 \arrow[l, Rightarrow, "\mathrm{can}"] \end{tikzcd}$$

Moreover, the pair (G', δ') is essentially unique.
Conversely, given a pair (G', δ') as above, condition (qcg2) defines a morphism $\delta\colon G \cdot T \Rightarrow 0$ in $\mathcal{CG}$ satisfying condition (qcg1).

As a consequence of its universal property, the quotient categorical group satisfies the following cancellation property.

Lemma 7.5 *Let* $(\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi)$ *be in* $\mathcal{CCM}$, *and consider a diagram in* $\mathcal{CG}$

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\mathrm{C}(T)"] \arrow[d, "\mathrm{C}(T)"'] \arrow[dr, phantom, "\Downarrow\beta"] & \mathbb{G}/T \arrow[d, "H"] \\ \mathbb{G}/T \arrow[r, "K"'] & \mathbb{H} \end{tikzcd}$$

such that the following diagram commutes

$$\textit{(qcg3)} \quad \begin{tikzcd} H \cdot \mathrm{C}(T) \cdot P \arrow[r, Rightarrow, "H \circ c(T)"] \arrow[d, Rightarrow, "\beta \circ T"'] & H \cdot 0 \arrow[d, Leftrightarrow, "\mathrm{can}"] \\ K \cdot \mathrm{C}(T) \cdot T \arrow[r, Rightarrow, "K \circ c(T)"'] & K \cdot 0 \end{tikzcd}$$

There exists a unique morphism $\alpha\colon H \Rightarrow K$ *such that* $\alpha \circ \mathrm{C}(T) = \beta$.

Here is how homomorphisms of categorical crossed modules extend to the quotient categorical groups.

Lemma 7.6 *Consider a homomorphism in* $\mathcal{CCM}$

$$(R, F, \lambda, \tau)\colon (\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi) \to (\varphi'\colon \mathbb{G} \to \mathbb{E}\mathrm{q}(\mathbb{A}'), T'\colon \mathbb{A}' \to \mathbb{G}', \nu', \chi')$$

There exists a diagram in $\mathcal{CG}$

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\mathrm{C}(T)"] \arrow[d, "R"'] \arrow[dr, phantom, "\Downarrow\widetilde{\tau}"] & \mathbb{G}/T \arrow[d, "\widetilde{R}"] \\ \mathbb{G}' \arrow[r, "\mathrm{C}(T')"'] & \mathbb{G}'/T' \end{tikzcd}$$

such that the following diagram commutes

$$
\text{(qcg4)}\quad
\begin{tikzcd}
\widetilde{R}\cdot \mathrm{C}(T)\cdot T \arrow[r, Rightarrow, "\widetilde{\tau}\circ T"] \arrow[d, Rightarrow, "\widetilde{R}\circ c(T)"'] & \mathrm{C}(T')\cdot R\cdot T \arrow[r, Rightarrow, "\mathrm{C}(T')\circ\tau"] & \mathrm{C}(T')\cdot T'\cdot F \arrow[d, Rightarrow, "c(T')\circ F"] \\
\widetilde{R}\cdot 0 \arrow[rr, Leftrightarrow, "\mathrm{can}"'] & & 0\cdot F
\end{tikzcd}
$$

Proof. Define

$$\delta\colon\ \mathrm{C}(T')\cdot R\cdot T \xRightarrow{\mathrm{C}(T')\circ\tau} \mathrm{C}(T')\cdot T'\cdot F \xRightarrow{c(T')\circ F} 0\cdot F \xRightarrow{\mathrm{can}} 0$$

Check condition (qcg1) on δ using condition (pcm4) on τ, and apply the universal property of $\mathbb{G}/T$. Condition (qcg4) is now a special case of condition (qcg2). ■

Remark 7.7 To complete Lemma 7.6, consider a morphism in $\mathcal{CCM}$

$$(\beta,\alpha)\colon (R,F,\lambda,\tau)\Rightarrow(R',F',\lambda',\tau')\colon(\mathbb{G},\mathbb{A},\varphi,T,\nu,\chi)\to(\mathbb{G}',\mathbb{A}',\varphi',T',\nu',\chi')$$

There exists a unique morphism $\widetilde{\beta}\colon \widetilde{R}\Rightarrow\widetilde{R}'$ such that the following diagram commutes

$$
\text{(qcg5)}\quad
\begin{tikzcd}
\widetilde{R}\cdot \mathrm{C}(T) \arrow[r, Rightarrow, "\widetilde{\tau}"] \arrow[d, Rightarrow, "\widetilde{\beta}\circ \mathrm{C}(T)"'] & \mathrm{C}(T')\cdot R \arrow[d, Rightarrow, "\mathrm{C}(T')\circ\beta"] \\
\widetilde{R}' \arrow[r, Rightarrow, "\widetilde{\tau}'"'] & \mathrm{C}(T')\cdot R'
\end{tikzcd}
$$

We adopt the following terminology, introduced in [13].

Definition 7.8 Let $\mathbb{G}$ be a categorical group. A normal sub-categorical group of $\mathbb{G}$ is a categorical crossed module $(\mathbb{G},\mathbb{A},\varphi\colon\mathbb{G}\to\mathbb{E}\mathrm{q}(\mathbb{A}),T\colon\mathbb{A}\to\mathbb{G},\nu,\chi)$ with T faithful.

From Example 2.6.v in [13], we know that the kernel of a homomorphism $F\colon\mathbb{G}\to\mathbb{H}$ in $\mathcal{CG}$ has a canonical structure of normal sub-categorical group. Moreover, condition (qcg1) is verified if we take $\delta=k(F)$. Here is the relation between kernels and quotients (Propositions 3.6 and 3.8 in [13]).

Proposition 7.9

1. *Let $(\varphi\colon\mathbb{G}\to\mathbb{E}\mathrm{q}(\mathbb{A}),T\colon\mathbb{A}\to\mathbb{G},\nu,\chi)$ be in $\mathcal{CCM}$ and consider the factorization through the kernel as in the following diagram*

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "T"] \arrow[dr, "T'"'] & \mathbb{G} \arrow[r, "\mathrm{C}(T)"] & \mathbb{G}/T \\
& \mathbb{K}\mathrm{er}(\mathrm{C}(T)) \arrow[u, "\mathrm{K}(\mathrm{C}(T))"'] &
\end{tikzcd}
$$

(a) *T' is a full and essentially surjective homomorphism of $\mathbb{G}$-categorical crossed modules ;*

(b) *if T is faithful, then $\mathbb{A}$ and $\mathrm{Ker}(\mathrm{C}(T))$ are equivalent normal sub-categorical groups of $\mathbb{G}$.*

2. *Let $F\colon \mathbb{G} \to \mathbb{H}$ be in $\mathcal{CG}$ and consider the factorization through the quotient as in the following diagram*

$$\begin{tikzcd} \mathbb{K}\mathrm{er}(F) \arrow[r, "\mathrm{K}(F)"] & \mathbb{G} \arrow[r, "F"] \arrow[d, "\mathrm{C}(\mathrm{K}(F))"'] & \mathbb{H} \\ & \mathbb{G}/\mathrm{K}(F) \arrow[ur, "F'"'] & \end{tikzcd}$$

(a) *F' is full and faithful ;*

(b) *in particular, F is a quotient (i.e., F' is an equivalence) iff F is essentially surjective.*

To end this section, we point out a simple but relevant situation (see Remark 8.5) where the universal property of the quotient categorical group holds.

Lemma 7.10 *Let $(\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{G}, \nu, \chi)$ be in $\mathcal{SCCM}$. The morphism in $\mathcal{CG}$*

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "\varphi"] \arrow[d, phantom, "\Downarrow\delta"] & \\ \mathbb{A} \arrow[ur, "T"] \arrow[rr, "0"'] & {} & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \end{tikzcd}$$

with $\delta_{A,B}\colon {}^{TA}B \to B$ defined by

$${}^{TA}B \simeq {}^{TA}B \otimes A \otimes A^* \xrightarrow{\chi_{A,B}\otimes \mathrm{id}} A \otimes B \otimes A^* \xrightarrow{c_{A,B}\otimes \mathrm{id}} B \otimes A \otimes A^* \simeq B$$

satisfies condition (qcg1).

Proof. This follows from conditions (act6) in Subsection 3.5 and (cm3) in Subsection 7.3. ∎

8. Extensions with symmetric kernel

Definition 8.1 Let $\mathbb{A}$ and $\mathbb{G}$ be categorical groups, with $\mathbb{A}$ symmetric. An extension of $\mathbb{G}$ by $\mathbb{A}$ is a 6-tuple

$$(\mathbb{B}, \psi\colon \mathbb{B} \to \mathbb{E}\mathrm{q}_s(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{B}, \nu, \chi, S)$$

where $(\mathbb{B}, \mathbb{A}, \psi, T, \nu, \chi)$ is an object in $\mathcal{SCCM}$ with T faithful (that is, $\mathbb{A}$ is a symmetric normal sub-categorical group of $\mathbb{B}$), and $S\colon \mathbb{B}/T \to \mathbb{G}$ is an equivalence in $\mathcal{CG}$.
When ψ, ν and χ are understood, we denote an extension of $\mathbb{G}$ by $\mathbb{A}$ by

$$\mathbb{A} \xrightarrow{T} \mathbb{B} \xrightarrow{\mathrm{C}(T)} \mathbb{B}/T \xrightarrow{S} \mathbb{G}$$

A homomorphism of extensions is a 4-tuple

$$(R,\lambda,\tau,m)\colon (\mathbb{B},\mathbb{A},\psi,T,\nu,\chi,S)\to(\mathbb{B}',\mathbb{A},\psi',T',\nu',\chi',S')$$

where $m\colon S\Rightarrow S'\cdot\widetilde{R}$ is a morphism in $\mathcal{CG}$ (with $\widetilde{R}$ induced by λ and τ as in Lemma 7.6), and $(R,\mathrm{Id}_{\mathbb{A}},\lambda,\tau)\colon(\mathbb{B},\mathbb{A},\psi,T,\nu,\chi)\to(\mathbb{B}',\mathbb{A},\psi',T',\nu',\chi',)$ is a homomorphism in $\mathcal{SCCM}$

$$\begin{tikzcd}
 & \mathbb{B} \arrow[rr, "\mathrm{C}(T)"] \arrow[dr, "\psi"] \arrow[dd, "R\,\Downarrow\lambda"] & & \mathbb{B}/T \arrow[dd, "\widetilde{R}"'] \arrow[dr, "S"] & \\
\mathbb{A} \arrow[ur, "T"] \arrow[dr, "T'"'] \arrow[r, phantom, "\Downarrow\tau"] & & \mathbb{E}\mathrm{q}_s(\mathbb{A}) & & \mathbb{G} \arrow[l, phantom, "\Downarrow m"] \\
 & \mathbb{B}' \arrow[ur, "\psi'"'] \arrow[rr, "\mathrm{C}(T')"'] & & \mathbb{B}'/T' \arrow[ur, "S'"'] &
\end{tikzcd}$$

A morphism of extensions

$$\beta\colon(R,\lambda,\tau,m)\Rightarrow(R',\lambda',\tau',m')\colon(\mathbb{B},\mathbb{A},\psi,T,\nu,\chi,S)\to(\mathbb{B}',\mathbb{A},\psi',T',\nu',\chi',S')$$

is a morphism $(\beta,\mathrm{id})\colon(R,\mathrm{Id}_{\mathbb{A}},\lambda,\tau)\Rightarrow(R',\mathrm{Id}_{\mathbb{A}},\lambda',\tau')$ in $\mathcal{SCCM}$ such that the following diagram commutes

$$\text{(ext1)}\quad\begin{tikzcd}
S'\cdot\widetilde{R} \arrow[rr, Rightarrow, "S'\circ\widetilde{\beta}"] & & S'\cdot\widetilde{R'} \\
 & S \arrow[ul, Rightarrow, "m"] \arrow[ur, Rightarrow, "m'"'] &
\end{tikzcd}$$

where $\widetilde{\beta}$ is obtained by β as in Remark 7.7.

Extensions of $\mathbb{G}$ by $\mathbb{A}$ with their homomorphisms and morphisms form a 2-category denoted $\mathbb{E}\mathrm{xt}(\mathbb{G},\mathbb{A})$.

Remark 8.2 Because of Proposition 7.9, in any extension of $\mathbb{G}$ by $\mathbb{A}$ the symmetric categorical group $\mathbb{A}$ is equivalent, as a normal sub-categorical group, to the kernel of $S\cdot\mathrm{C}(T)\colon\mathbb{B}\to\mathbb{B}/T\to\mathbb{G}$.

Remark 8.3 Using Lemma 7.5, in the definition of homomorphism of extensions we can replace the morphism $m\colon S\Rightarrow S'\cdot\widetilde{R}$ by a morphism

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "\mathrm{C}(T)"] \arrow[dd, "R"'] & \mathbb{B}/T \arrow[dr, "S"] & \\
 & \Downarrow\mu & \mathbb{G} \\
\mathbb{B}' \arrow[r, "\mathrm{C}(T')"'] & \mathbb{B}'/T' \arrow[ur, "S'"'] &
\end{tikzcd}$$

such that the following diagram commutes

$$\text{(ext2)}\quad \begin{tikzcd} S\cdot \mathrm{C}(T)\cdot T \arrow[r, Rightarrow, "\mu\circ T"] \arrow[d, Rightarrow, "S\circ c(T)"'] & S'\cdot \mathrm{C}(T')\cdot R\cdot T \arrow[r, Rightarrow, "S'\cdot \mathrm{C}(T')\circ\tau"] & S'\cdot \mathrm{C}(T')\cdot T' \arrow[d, Rightarrow, "S'\circ c(T)"] \\ S\cdot 0 \arrow[rr, Leftrightarrow, "\mathrm{can}"'] & & S'\cdot 0 \end{tikzcd}$$

In this case, condition (ext1) for a morphism of extensions is to be written as

$$\text{(ext1')}\quad \begin{tikzcd} S'\cdot \mathrm{C}(T')\cdot R \arrow[rr, Rightarrow, "S'\cdot \mathrm{C}(T')\circ\beta"] & & S'\cdot \mathrm{C}(T')\cdot R' \\ & S\cdot \mathrm{C}(T) \arrow[ul, Rightarrow, "\mu"] \arrow[ur, Rightarrow, "\mu'"'] & \end{tikzcd}$$

Remark 8.4 In fact, the 2-category $\mathbb{E}\mathrm{xt}(\mathbb{G},\mathbb{A})$ is a 2-groupoid, that is, any morhism is invertible and any homomorphism is an equivalence. The proof of this fact follows the same lines of the proof of Proposition 2.8 in [6], just replace the cokernel of a homomorphism in $\mathcal{SCG}$ used in [6] by the quotient categorical group (see also Proposition 10.3 in [30]).

Remark 8.5 Consider an extension of $\mathbb{G}$ by $\mathbb{A}$

$$\mathbb{A} \xrightarrow{T} \mathbb{B} \xrightarrow{\mathrm{C}(T)} \mathbb{B}/T \xrightarrow{S} \mathbb{G}$$

Since $\mathbb{A}$ is symmetric, we get a symmetric action of $\mathbb{G}$ on $\mathbb{A}$ as follows

$$\mathbb{G} \xrightarrow{S^{-1}} \mathbb{B}/T \xrightarrow{\widetilde{\psi}} \mathbb{E}\mathrm{q}_s(\mathbb{A})$$

where $(\widetilde{\psi},\widetilde{\delta})$ is the factorization through the quotient

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[dr, "0"'] & \mathbb{B} \arrow[r, "\mathrm{C}(T)"] \arrow[d, "\psi"] \arrow[dl, phantom, "\Downarrow\delta"] \arrow[dr, phantom, "\Downarrow\widetilde{\delta}"] & \mathbb{B}/T \arrow[dl, "\widetilde{\psi}"] \\ & \mathbb{E}\mathrm{q}_s(\mathbb{A}) & \end{tikzcd}$$

the morphism $\delta\colon \psi\cdot T \Rightarrow 0$ is as in Lemma 7.10, and S^{-1} is the essentially unique adjoint quasi-inverse of S. We recall $\widetilde{\psi}\cdot S^{-1} : \mathbb{G}\to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ as the action of $\mathbb{G}$ on $\mathbb{A}$ induced by a given extension of $\mathbb{G}$ by $\mathbb{A}$.

The next proposition allows us to use the action induced by an extension as a parameter to classify the extensions with symmetric kernel.

Proposition 8.6 *Consider a homomorphism of extensions as in Definition 8.1*

$$\begin{tikzcd} & \mathbb{B} \arrow[rr, "\mathrm{C}(T)"] \arrow[dd, "R"] \arrow[dr, "\psi"] & & \mathbb{B}/T \arrow[dd, "\widetilde{R}"'] \arrow[dr, "S"] & \\ \mathbb{A} \arrow[ur, "T"] \arrow[dr, "T'"'] \arrow[r, phantom, "\Downarrow\tau"] & {} \arrow[r, phantom, "\Downarrow\lambda"] & \mathbb{E}\mathrm{q}_s(\mathbb{A}) & {} \arrow[r, phantom, "\Downarrow m"] & \mathbb{G} \\ & \mathbb{B}' \arrow[rr, "\mathrm{C}(T')"'] \arrow[ur, "\psi'"] & & \mathbb{B}'/T' \arrow[ur, "S'"'] & \end{tikzcd}$$

The actions of $\mathbb{G}$ on $\mathbb{A}$ induced by the two extensions are equivalent.

This statement means that there exists an equivalence of the form

$$(\mathrm{Id}, \mathrm{Id}, \simeq)\colon (\mathbb{G}, \mathbb{A}, \widetilde{\psi} \cdot S^{-1}) \to (\mathbb{G}, \mathbb{A}, \widetilde{\psi'} \cdot S'^{-1})$$

in the 2-category $\mathcal{SACT}$ of symmetric actions described in Subsection 7.1.
To help reading the proof, let us resume the whole situation in the following diagram

$$\begin{tikzcd}
 & \mathbb{E}\mathrm{q}_s(\mathbb{A}) & & \\
 & \mathbb{B} \arrow[u, "\psi"] \arrow[r, "\mathrm{C}(T)"'] \arrow[d, "R"] \arrow[ur, phantom, "\Leftarrow\widetilde{\delta}"] & \mathbb{B}/T \arrow[ul, bend right, "\widetilde{\psi}"'] \arrow[dr, "S"] \arrow[d, "\widetilde{R}"'] & \\
\mathbb{A} \arrow[uur, bend left, "0"] \arrow[ur, "T"] \arrow[dr, "T'"'] \arrow[ddr, bend right, "0"'] \arrow[r, phantom, "\Downarrow\tau"] & {} \arrow[r, phantom, "\Downarrow\widetilde{\tau}"] & {} \arrow[r, phantom, "\Downarrow m"] & \mathbb{G} \\
 & \mathbb{B}' \arrow[r, "\mathrm{C}(T')"] \arrow[d, "\psi'"'] \arrow[dr, phantom, "\Leftarrow\widetilde{\delta'}"] & \mathbb{B}'/T' \arrow[ur, "S'"'] \arrow[dl, bend left, "\widetilde{\psi'}"] & \\
 & \mathbb{E}\mathrm{q}_s(\mathbb{A}) & &
\end{tikzcd}$$

$$\Leftarrow\delta \qquad \Leftarrow\delta'$$

Proof. The proof essentially consists in finding a morphism $\widetilde{\lambda}$ to fill-in the following diagram in $\mathcal{CG}$

$$\begin{tikzcd}
 & \mathbb{G} & \\
\mathbb{B}/T \arrow[ur, "S"] \arrow[rr, "\widetilde{R}"] \arrow[dr, "\widetilde{\psi}"'] \arrow[rr, phantom, shift left=4, "\Rightarrow m"] \arrow[rr, phantom, shift right=4, "\Rightarrow\widetilde{\lambda}"] & & \mathbb{B}'/T' \arrow[ul, "S'"'] \arrow[dl, "\widetilde{\psi'}"] \\
 & \mathbb{E}\mathrm{q}_s(\mathbb{A}) &
\end{tikzcd}$$

We start constructing the following morphism

$$\widetilde{\psi'} \cdot \widetilde{R} \cdot \mathrm{C}(T) \xRightarrow{\widetilde{\psi'} \circ \widetilde{\tau}} \widetilde{\psi'} \cdot \mathrm{C}(T') \cdot R \xRightarrow{\widetilde{\delta'} \circ R} \psi' \cdot R \xLeftarrow{\lambda} \psi \xLeftarrow{\widetilde{\delta}} \widetilde{\psi} \cdot \mathrm{C}(T)$$

In order to use the cancellation property of the quotient (Lemma 7.5), we have to check condition (qcg3). Using condition (qcg2) on $\widetilde{\delta}$ and on $\widetilde{\delta'}$ and condition (qcg4) on $\widetilde{\tau}$, condition (qcg3) reduces to the commutativity of

$$\begin{tikzcd}
\psi' \cdot T' \arrow[d, Rightarrow, "\delta'"'] & \psi' \cdot R \cdot T \arrow[l, Rightarrow, "\psi' \circ \tau"'] \arrow[d, Rightarrow, "\lambda \circ T"] \\
0 & \psi \cdot T \arrow[l, Rightarrow, "\delta"]
\end{tikzcd}$$

Finally, using the description of δ and δ' given in Lemma 7.10, the commutativity of the previous diagram amounts to condition (cm6). ■

In the next definition we specialize extensions (Definition 8.1) to get φ-extensions for φ a fixed symmetric action.

Definition 8.7 Let $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ be a symmetric action. A φ-extension of $\mathbb{G}$ by $\mathbb{A}$ is a 7-tuple

$$(\mathbb{B}, \psi\colon \mathbb{B} \to \mathbb{E}\mathrm{q}_s(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{B}, \nu, \chi, S, s)$$

where

$$\mathbb{A} \xrightarrow{T} \mathbb{B} \xrightarrow{\mathrm{C}(T)} \mathbb{B}/T \xrightarrow{S} \mathbb{G}$$

is an extension of $\mathbb{G}$ by $\mathbb{A}$, and

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "\varphi"] \arrow[d, phantom, "\Downarrow s"] & \\ \mathbb{B}/T \arrow[ur, "S"] \arrow[rr, "\widetilde{\psi}"'] & {} & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \end{tikzcd}$$

is a morphism in $\mathcal{CG}$ ($\widetilde{\psi}$ is as in Remark 8.5).
A homomorphism of φ-extensions

$$(R, \lambda, \tau, m)\colon (\mathbb{B}, \mathbb{A}, \psi, T, \nu, \chi, S, s) \to (\mathbb{B}', \mathbb{A}, \psi', T', \nu', \chi', S', s')$$

is a homomorphism of extensions (R, λ, τ, m) such that the following diagram commutes

$$(\varphi\text{ext1}) \quad \begin{tikzcd} \widetilde{\psi} \cdot S \arrow[r, Rightarrow, "\widetilde{\lambda} \circ S"] \arrow[d, Rightarrow, "s"'] & \widetilde{\psi}' \cdot R \cdot S \arrow[d, Rightarrow, "\widetilde{\psi}' \circ m"] \\ \varphi & \widetilde{\psi}' \cdot S' \arrow[l, Rightarrow, "s'"] \end{tikzcd}$$

A morphism of φ-extensions

$$\beta\colon (R, \lambda, \tau, m) \Rightarrow (R', \lambda', \tau', m')\colon (\mathbb{B}, \mathbb{A}, \psi, T, \nu, \chi, S, s) \to (\mathbb{B}', \mathbb{A}, \psi', T', \nu', \chi', S', s')$$

is just a morphism of extensions.
φ-Extensions of $\mathbb{G}$ by $\mathbb{A}$ with their homomorphisms and morphisms form a 2-groupoid denoted $\mathbb{O}\mathrm{pext}(\varphi, \mathbb{G}, \mathbb{A})$.

Remark 8.8 Using Lemma 7.5 once again, we can reformulate Definition 8.7 as follows : a φ-extension of $\mathbb{G}$ by $\mathbb{A}$ is a 7-tuple

$$(\mathbb{B}, \psi\colon \mathbb{B} \to \mathbb{E}\mathrm{q}_s(\mathbb{A}), T\colon \mathbb{A} \to \mathbb{B}, \nu, \chi, S, \sigma)$$

where

$$\mathbb{A} \xrightarrow{T} \mathbb{B} \xrightarrow{\mathrm{C}(T)} \mathbb{B}/T \xrightarrow{S} \mathbb{G}$$

is an extension of $\mathbb{G}$ by $\mathbb{A}$, and

$$\begin{tikzcd} \mathbb{E}q_s(\mathbb{A}) \arrow[dr, phantom, "\Leftarrow\sigma"] & \mathbb{G} \arrow[l, "\varphi"'] \\ \mathbb{B} \arrow[u, "\psi"] \arrow[r, "\mathrm{C}(T)"'] & B/T \arrow[u, "S"'] \end{tikzcd}$$

is a morphism in $\mathcal{CG}$ such that the following diagram commutes

$$(\varphi\text{ext2}) \quad \begin{tikzcd} \varphi \cdot S \cdot \mathrm{C}(T) \cdot T \arrow[r, Rightarrow, "\sigma\circ T"] \arrow[d, Rightarrow, "\varphi\cdot S\circ c(T)"'] & \psi \cdot T \arrow[d, Rightarrow, "\delta"] \\ \varphi \cdot S \cdot 0 & 0 \arrow[l, Rightarrow, "\mathrm{can}"] \end{tikzcd}$$

A homomorphism of φ-extensions

$$(R, \lambda, \tau, \mu) \colon (\mathbb{B}, \mathbb{A}, \psi, T, \nu, \chi, S, \sigma) \to (\mathbb{B}', \mathbb{A}, \psi', T', \nu', \chi', S', \sigma')$$

is a homomorphism of extensions (R, λ, τ, μ) as in Remark 8.3, such that the following diagram commutes

$$(\varphi\text{ext1'}) \quad \begin{tikzcd} \varphi \cdot S \cdot \mathrm{C}(T) \arrow[r, Rightarrow, "\varphi\circ\mu"] \arrow[d, Rightarrow, "\sigma"'] & \varphi \cdot S' \cdot \mathrm{C}(T') \cdot R \arrow[d, Rightarrow, "\sigma'\circ R"] \\ \psi \arrow[r, Rightarrow, "\lambda"'] & \psi' \cdot R \end{tikzcd}$$

Finally, a morphism of φ-extensions

$$\beta \colon (R, \lambda, \tau, \mu) \Rightarrow (R', \lambda', \tau', \mu') \colon (\mathbb{B}, \mathbb{A}, \psi, T, \nu, \chi, S, \sigma) \to (\mathbb{B}', \mathbb{A}, \psi', T', \nu', \chi', S', \sigma')$$

is just a morphism of extensions as in Remark 8.3.

9. Classification of extensions with symmetric kernel

In Subsection 3.8 we have described the cokernel $\mathbb{C}\text{oker}(T)$ of a morphism $T \colon \mathbb{G} \to \mathbb{H}$ in $\mathcal{SCG}$ as a categorical group. In fact, $\mathbb{C}\text{oker}(T)$ is obtained from a monoidal bicategory (that, by abuse of notation, we still denote $\mathbb{C}\text{oker}(T)$) by taking 2-isomorphism classes of 1-arrows as arrows. Explicitly, the objects of $\mathbb{C}\text{oker}(T)$ are those of $\mathbb{H}$, a 1-arrow from A to B is a pair $(X \in \mathbb{G}, f \colon A \to TX \otimes B)$, and a 2-arrow from $(X, f \colon A \to TX \otimes B)$ to $(X', f' \colon A \to TX' \otimes B)$ is an arrow $x \colon X \to X'$ in $\mathbb{G}$ such that $(T(x) \otimes \mathrm{id}) \cdot f = f'$. (The fact that $\mathbb{C}\text{oker}(T)$ is a monoidal bicategory is mentioned in the introduction of [34], and the whole proof has been done in [24]. The same argument can be developed for the quotient categorical group associated with a categorical crossed module.)
In order to compare $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ with $\mathbb{O}\text{pext}(\varphi, \mathbb{G}, \mathbb{A})$, in the next proposition we look at $\mathbb{H}^2(\mathbb{G}, \mathbb{A}, \varphi)$ as a monoidal bicategory.

Proposition 9.1 *Let $\mathbb{G}$ be a categorical group and $\varphi\colon \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ a symmetric action. There exists a homomorphism of bicategories*

$$\overline{\mathcal{E}}\colon \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi) \to \mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$$

which is locally an equivalence.

Proof. We split the proof in four steps.
Step 1. We construct a 2-functor

$$\mathcal{E}\colon \mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \to \mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$$

where $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ is seen as a 2-category with only identity 2-arrows.
Let $F\colon \mathbb{G}\times\mathbb{G} \to \mathbb{A}$ be a factor set and consider the categorical group $\mathbb{A}\times_F\mathbb{G}$ described in Remark 4.3, together with the canonical injection and the canonical projection

$$\mathbb{A} \xrightarrow{i_{\mathbb{A}}} \mathbb{A}\times_F\mathbb{G} \xrightarrow{p_{\mathbb{G}}} \mathbb{G}$$

Since $i_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}\times_F\mathbb{G}$ is equivalent to the kernel of $p_{\mathbb{G}}$, we get a canonical structure of normal sub-categorical group on $i_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}\times_F\mathbb{G}$. In particular, the symmetric action $\psi\colon \mathbb{A}\times_F\mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ is given by

$$^{(A,X)}B = p_{\mathbb{A}}((A,X)\otimes_F(B,I)\otimes_F(A,X)^*)$$

Moreover, since $p_{\mathbb{G}}$ is surjective on objects, by Proposition 7.9 we get a monoidal equivalence $S\colon (\mathbb{A}\times_F\mathbb{G})/i_{\mathbb{A}} \to \mathbb{G}$ such that $S\cdot \mathrm{C}(i_{\mathbb{A}}) = p_{\mathbb{G}}$. In this way, we get an extension $\mathcal{E}(F)$ of $\mathbb{G}$ by $\mathbb{A}$

$$\mathbb{A} \xrightarrow{i_{\mathbb{A}}} \mathbb{A}\times_F\mathbb{G} \xrightarrow{\mathrm{C}(i_{\mathbb{A}})} (\mathbb{A}\times_F\mathbb{G})/i_{\mathbb{A}} \xrightarrow{S} \mathbb{G}$$

which in fact is a φ-extension because there is an obvious morphism $\sigma\colon \varphi\cdot p_{\mathbb{G}} \Rightarrow \psi$ in $\mathcal{CG}$ obtained using φ_2, φ_1 and the symmetry of $\mathbb{A}$.
Let now $\alpha\colon F \Rightarrow F'\colon \mathbb{G}\times\mathbb{G} \to \mathbb{A}$ be a morphism of factor sets. We get a homomorphism of extensions $\mathcal{E}(\alpha)\colon \mathcal{E}(F) \Rightarrow \mathcal{E}(F')$ as follows

$$\begin{tikzcd}
 & \mathbb{A}\times_F\mathbb{G} \arrow[dd, "\mathrm{Id}"] \arrow[dr, "p_{\mathbb{G}}"] & \\
\mathbb{A} \arrow[ur, "i_{\mathbb{A}}"] \arrow[dr, "i_{\mathbb{A}}"'] & & \mathbb{G} \\
 & \mathbb{A}\times_{F'}\mathbb{G} \arrow[ur, "p_{\mathbb{G}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}\times_F\mathbb{G} \arrow[dd, "\mathrm{Id}"'] \arrow[dr, "\psi"] & \\
\arrow[r, phantom, "\Downarrow\lambda"] & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \\
\mathbb{A}\times_{F'}\mathbb{G} \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

where λ is given by $\lambda_{(A,X),B} = \mathrm{id}\otimes {}^{X}(\mathrm{id}\otimes {}^{X^*}\alpha^*_{X,X^*})\otimes\alpha_{X,X^*}$ and $\mathrm{Id}\colon \mathbb{A}\times_F\mathbb{G} \to \mathbb{A}\times_{F'}\mathbb{G}$ is the identity functor with monoidal structure given by

$$(A,X)\otimes_F(B,Y) \xrightarrow{(\mathrm{id}\otimes\alpha_{X,Y},\mathrm{id})} (A,X)\otimes_{F'}(B,Y)$$

In order to check that $\mathcal{E}(\alpha)$ is a homomorphism in $\mathbb{O}\mathrm{pext}(\varphi, \mathbb{G}, \mathbb{A})$, one has to apply Lemma 3.2 to the arrow $\alpha_{X,X^*}\colon F(X,X^*) \to F'(X,X^*)$.

Step 2. We construct now a 2-natural transformation

$$\begin{tikzcd} & \mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \arrow[dr, "\mathcal{E}"] \arrow[d, phantom, "\Downarrow\Phi"] & \\ \mathbb{C}^1(\mathbb{G},\mathbb{A}) \arrow[ur, "\delta"] \arrow[rr, "0"'] & {} & \mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A}) \end{tikzcd}$$

Let $G\colon \mathbb{G} \to \mathbb{A}$ be a cobord. We get a homomorphism of extensions as follows

$$\begin{tikzcd} & \mathbb{A} \times_{\delta G} \mathbb{G} \arrow[dr, "p_{\mathbb{G}}"] \arrow[dd, "\Phi_G"] & \\ \mathbb{A} \arrow[ur, "i_{\mathbb{A}}"] \arrow[dr, "i_{\mathbb{A}}"'] \arrow[r, phantom, "\Downarrow\tau"] & {} & \mathbb{G} \\ & \mathbb{A} \times_{\varphi} \mathbb{G} \arrow[ur, "p_{\mathbb{G}}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \times_{\delta G} \mathbb{G} \arrow[dr, "\psi"] \arrow[dd, "\Phi_G"'] & \\ {} \arrow[r, phantom, "\Downarrow\lambda"] & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \\ \mathbb{A} \times_{\varphi} \mathbb{G} \arrow[ur, "\psi'"'] & \end{tikzcd}$$

where $\mathbb{A} \times_{\varphi} \mathbb{G}$ is constructed by taking $F = 0$ in Remark 4.3, and the symmetric action $\psi'\colon \mathbb{A} \times_{\varphi} \mathbb{G} \to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ is constructed as the action ψ in Step 1, but taking once again $F = 0$; explicitly :

$$^{(A,X)}B = A \otimes {}^{X}B \otimes A^* \xrightarrow{\mathrm{can}} {}^{X}B$$

The functor Φ_G is defined on arrows by

$$(f,h)\colon (A,X) \to (B,Y) \;\mapsto\; (f \otimes G(h), h)\colon (A \otimes GX, X) \to (B \otimes GY, Y)$$

and its monoidal structure is obtained using φ_2, φ_1 and the symmetry of $\mathbb{A}$. Finally, the morphism $\tau\colon \Phi_G \cdot i_{\mathbb{A}} \Rightarrow i_{\mathbb{A}}$ is defined by

$$\Phi_G(i_{\mathbb{A}}(A)) \xrightarrow{(\mathrm{id} \otimes G_0^{-1}, \mathrm{id})} (A \otimes I, I) \simeq i_{\mathbb{A}}(A)$$

and the morphism $\lambda\colon \psi \Rightarrow \psi' \cdot \Phi_G$ is the opposite of the morphism σ used in Step 1.

Step 3. Now we extend the 2-functor $\mathcal{E}\colon \mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi) \to \mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$, constructed in Step 1, to a homomorphism of bicategories

$$\overline{\mathcal{E}}\colon \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi) \to \mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$$

(we limit ourselves to define $\overline{\mathcal{E}}$ on objects, 1-arrows and 2-arrows ; to check that $\overline{\mathcal{E}}$ is indeed a homomorphism of bicategories is long but essentially straightforward). On objects, $\overline{\mathcal{E}}$ is defined as $\mathcal{E}$. Consider now a 1-arrow in $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$, that is, a pair (G,α) with G in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ and $\alpha\colon F \otimes \delta G \Rightarrow F'$ in $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$; the homomorphism $\overline{\mathcal{E}}(G,\alpha)\colon \mathcal{E}(F) \Rightarrow \mathcal{E}(F')$ factors through $\mathcal{E}(F \otimes \delta G)$ and is described in the following diagram, where Φ_G and τ are as in Step 2, and Id is the identity functor with monoidal structure determined by α as in Step 1.

$$\begin{tikzcd}
& \mathbb{A}\times_F\mathbb{G} \arrow[dr, bend left, "p_{\mathbb{G}}"] & \\
\mathbb{A} \arrow[ur, bend left, "i_{\mathbb{A}}"] \arrow[ur, phantom, "\Uparrow\tau"] \arrow[r, "i_{\mathbb{A}}"] \arrow[dr, bend right, "i_{\mathbb{A}}"'] & \mathbb{A}\times_{F\otimes\delta G}\mathbb{G} \arrow[u, "\Phi_G"'] \arrow[r, "p_{\mathbb{G}}"] \arrow[d, "\mathrm{Id}"] & \mathbb{G} \\
& \mathbb{A}\times_{F'}\mathbb{G} \arrow[ur, bend right, "p_{\mathbb{G}}"'] &
\end{tikzcd}$$

Consider now a 2-arrow $\beta\colon (G,\alpha)\Rightarrow(G',\alpha')$ in $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$, that is, a morphism $\beta\colon G\Rightarrow G'$ in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$ such that

$$\begin{tikzcd}
F\otimes\delta G \arrow[rr, Rightarrow, "F\otimes\delta(\beta)"] \arrow[dr, Rightarrow, "\alpha"'] & & F\otimes\delta G' \arrow[dl, Rightarrow, "\alpha'"] \\
& F' &
\end{tikzcd}$$

commutes; the morphism $\overline{\mathcal{E}}(\beta)$ is described in the diagram

$$\begin{tikzcd}
& \mathbb{A}\times_F\mathbb{G} & \\
\mathbb{A}\times_{F\otimes\delta G}\mathbb{G} \arrow[ur, "\Phi_G"] \arrow[rr, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"'] \arrow[rr, phantom, shift left=6, "\overline{\beta}\Rightarrow"] & & \mathbb{A}\times_{F\otimes\delta G'}\mathbb{G} \arrow[ul, "\Phi_{G'}"'] \\
& \mathbb{A}\times_{F'}\mathbb{G} \arrow[ur, "\mathrm{Id}"'] &
\end{tikzcd}$$

where $\overline{\beta}_{A,X}=(\mathrm{id}\otimes\beta_X,\mathrm{id})\colon (A\otimes GX,X)\to(A\otimes G'X,X)$, and the monoidal structure on the identity functor Id is determined by $F\otimes\delta(\beta)$ as in Step 1. Finally, the fact that the identity natural transformation in the bottom triangle is monoidal is precisely condition $\alpha'\cdot(F\otimes\delta(\beta))=\alpha$ above, and the fact that the natural transformation $\overline{\beta}$ is monoidal follows from Lemma 3.2 applied to the arrow $\beta_{X\otimes Y}\colon G(X\otimes Y)\to G'(X\otimes Y)$.

Step 4. Finally, we prove that the homomorphism $\overline{\mathcal{E}}\colon \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$, constructed in Step 3, is locally an equivalence.

$\overline{\mathcal{E}}$ is locally faithful : if $\beta,\beta'\colon G\Rightarrow G'$ are such that $\overline{\mathcal{E}}(\beta)=\overline{\mathcal{E}}(\beta')$, then for all $A\in\mathbb{A}$ and $X\in\mathbb{G}$ we have

$$\mathrm{id}\otimes\beta_X=\mathrm{id}\otimes\beta'_X\colon A\otimes GX\to A\otimes G'X$$

and taking $A=I$ we get $\beta_X=\beta'_X$.

$\overline{\mathcal{E}}$ is locally full : let $(G,\alpha),(G',\alpha')\colon F\to F'$ be 1-arrows in $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$ and consider a 2-arrow $\overline{\beta}\colon\overline{\mathcal{E}}(G,\alpha)\Rightarrow\overline{\mathcal{E}}(G',\alpha')$ in $\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$. Explicitly, $\overline{\beta}$ is a natural transformation

$$\overline{\beta}_{A,X}\colon \Phi_G(A,X)=(A\otimes GX,X)\to(A\otimes G'X,X)=\Phi_{G'}(A,X)$$

and so it has two components

$$\overline{\beta}_1(A,X)\colon A\otimes GX\to A\otimes G'X\ ,\ \ \overline{\beta}_2(A,X)\colon X\to X$$

We define $\beta_X\colon GX\to G'X$ as follows :

$$GX\simeq I\otimes GX\xrightarrow{\overline{\beta}_1(I,X)} I\otimes G'X\simeq G'X$$

In this way, condition (pcm5) on $\overline{\beta}$ gives condition (cob1) on β, that is, $\beta\colon G\Rightarrow G'$ is a morphism in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$, and the monoidal character of $\overline{\beta}$ gives condition $\alpha'\cdot(F\otimes\delta(\beta))=\alpha$ on β, so that $\beta\colon(G,\alpha)\Rightarrow(G',\alpha')$ is a 2-arrow in $\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)$. Moreover, condition (act10) on $\overline{\beta}$ gives that

$$\overline{\beta}_1(A,X)=\mathrm{id}\otimes\beta_X\colon A\otimes GX\to A\otimes G'X$$

and condition (ext1') on $\overline{\beta}$ gives that

$$\overline{\beta}_2(A,X)=\mathrm{id}\colon X\to X$$

so that $\overline{\mathcal{E}}(\beta)=\overline{\beta}$.
$\overline{\mathcal{E}}$ is locally essentially surjective : consider two cocycles $F,F'\colon\mathbb{G}\times\mathbb{G}\to\mathbb{A}$ and a homomorphism $(R,\tau,\mu)\colon\overline{\mathcal{E}}(F')\to\overline{\mathcal{E}}(F)$ in $\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$

$$\begin{tikzcd}
& \mathbb{A}\times_F\mathbb{G} \arrow[dr, "p_{\mathbb{G}}"] & \\
\mathbb{A} \arrow[ur, "i_{\mathbb{A}}"] \arrow[dr, "i_{\mathbb{A}}"'] \arrow[r, phantom, "\Downarrow\tau"] & {} \arrow[r, phantom, "\Uparrow\mu"] & \mathbb{G} \\
& \mathbb{A}\times_{F'}\mathbb{G} \arrow[uu, "R"] \arrow[ur, "p_{\mathbb{G}}"'] &
\end{tikzcd}$$

We are going to construct G in $\mathbb{C}^1(\mathbb{G},\mathbb{A})$, $\alpha\colon F\otimes\delta G\Rightarrow F'$ in $\mathbb{Z}^2(\mathbb{G},\mathbb{A},\varphi)$ and $\beta\colon\Phi_G\Rightarrow R$ in $\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$. Let us write

$$R(A,X)=(R^1(A,X),R^2(A,X))$$

for the two components of R, and

$$\tau_A=(\tau^1(A),\tau^2(A))\colon(A,I)\to(R^1(A,I),R^2(A,I))$$

for the two components of τ, and observe that

$$\mu_{A,I}=\tau^2(A)\colon I\to R^2(A,I)$$

because of condition (ext2). Now we put

$$G=R^1(I,-)\colon\mathbb{G}\to\mathbb{A}$$

As far as the natural transformation β is concerned, it is of the form

$$\beta_{A,X} = (\beta^1(A,X), \beta^2(A,X))\colon \Phi_G(A,X) = (A \otimes GX, X) \to (R^1(A,X), R^2(A,X))$$

and we put

$$\beta^2(A,X) = \mu_{A,X}\colon X \to R^2(A,X)$$

To construct the first component $\beta^1(A,X)$, observe that for any cocycle F one has $(A,I) \otimes_F (I,X) \simeq (A,X)$, so that the monoidal structure of R provides a natural family of arrows $R_2^{A,X}$ with two components of the form

$$(\rho^1(A,X), \rho^2(A,X))\colon (R^1(A,I) \otimes R^1(I,X), R^2(A,I) \otimes R^2(I,X)) \to (R^1(A,X), R^2(A,X))$$

and we put

$$\beta^1(A,X)\colon\ A \otimes R^1(I,X) \xrightarrow{\tau^1(A)\otimes \mathrm{id}} R^1(A,I) \otimes R^1(I,X) \xrightarrow{\rho^1(A,X)} R^1(A,X)$$

To construct the natural transformation α, consider the following arrow, where we use $\mu_{I,X}$, $\mu_{I,Y}$ and the definition of $\otimes_F$ in the first step, the monoidal structure of R in the second step, and $\varphi_\bullet^X$ and the definition of $\otimes_{F'}$ in the third step

$$\begin{array}{c}
(R^1(I,X) \otimes {}^X R^1(I,Y) \otimes F(X,Y), R^2(I,X) \otimes R^2(I,Y)) \\
\downarrow \\
(R^1(I,X), R^2(I,X)) \otimes_F (R^1(I,Y), R^2(I,Y)) \\
\downarrow \\
R((I,X) \otimes_F (I,Y)) \\
\downarrow \\
(R^1(F'(X,Y), X \otimes Y), R^2(F'(X,Y), X \otimes Y))
\end{array}$$

Composing the first component of such arrow with

$$\beta^1(F'(X,Y), X \otimes Y)^{-1}\colon R^1(F'(X,Y), X \otimes Y) \to F'(X,Y) \otimes R^1(I, X \otimes Y)$$

we get an arrow

$$GX \otimes {}^X GY \otimes F(X,Y) \to F'(X,Y) \otimes G(X \otimes Y)$$

and then, using symmetry and inverses in $\mathbb{A}$, the needed arrow

$$\alpha_{X,Y}\colon F(X,Y) \otimes {}^X GY \otimes G^*(X \otimes Y) \otimes GX \to F'(X,Y)$$

This concludes the proof of Proposition 9.1. ■

Remark 9.2 For any cocycle $F\colon \mathbb{G}\times\mathbb{G}\to\mathbb{A}$, the φ-extension

$$\mathcal{E}(F)\colon\ \mathbb{A}\xrightarrow{i_{\mathbb{A}}}\mathbb{A}\times_F\mathbb{G}\xrightarrow{p_{\mathbb{G}}}\mathbb{G}$$

described in Step 1 of the proof of Proposition 9.1, has a functorial section given by the canonical injection $i_{\mathbb{G}}\colon \mathbb{G}\to\mathbb{A}\times_F\mathbb{G}$ (note that $i_{\mathbb{G}}$ is not monoidal, unless $F=0$).
We will say that an extension

$$\mathbb{A}\xrightarrow{T}\mathbb{B}\xrightarrow{\mathrm{C}(T)}\mathbb{B}/T\xrightarrow{S}\mathbb{G}$$

has a functorial section if there exists a (non necessarily monoidal) functor $U\colon\mathbb{G}\to\mathbb{B}$ and a natural transformation

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "S\cdot\mathrm{C}(T)"] \arrow[d, phantom, "\Downarrow u"] & \\ \mathbb{G} \arrow[ur, "U"] \arrow[rr, "\mathrm{Id}"'] & {} & \mathbb{G} \end{tikzcd}$$

and we will denote by $\mathbb{O}\mathrm{pext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$ the full and 2-full sub-2-category of $\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$ of those φ-extensions which have a functorial section.

Theorem 9.3 *Let $\mathbb{G}$ be a categorical group and $\varphi\colon\mathbb{G}\to\mathbb{E}\mathrm{q}_s(\mathbb{A})$ a symmetric action. The corestriction*

$$\overline{\mathcal{E}}\colon \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{O}\mathrm{pext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$$

of the homomorphism of bicategories $\overline{\mathcal{E}}\colon\mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)\to\mathbb{O}\mathrm{pext}(\varphi,\mathbb{G},\mathbb{A})$ of Proposition 9.1, is a biequivalence.

Proof. We have to prove that $\overline{\mathcal{E}}$ is essentially surjective on objects, that is, surjective up to equivalence. Let

$$\begin{tikzcd} & \mathbb{E}\mathrm{q}_s(\mathbb{A}) \arrow[r, phantom, "\Leftarrow\sigma"] & \mathbb{G} \arrow[l, "\varphi"'] \\ \mathbb{A} \arrow[r, "T"'] & \mathbb{B} \arrow[u, "\psi"] \arrow[r, "\mathrm{C}(T)"'] & \mathbb{B}/T \arrow[u, "S"'] \end{tikzcd}$$

be a φ-extension with functorial section. Up to the equivalences $\mathbb{G}\simeq\mathbb{B}/T$ and $\mathbb{A}\simeq\mathbb{K}\mathrm{er}(\mathrm{C}(T))$, we can describe a cocycle $F\colon\mathbb{B}/T\times\mathbb{B}/T\to\mathbb{K}\mathrm{er}(\mathrm{C}(T))$ using a section

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "\mathrm{C}(T)"] \arrow[d, phantom, "\Downarrow u"] & \\ \mathbb{B}/T \arrow[ur, "U"] \arrow[rr, "\mathrm{Id}"'] & {} & \mathbb{B}/T \end{tikzcd}$$

The functor F is defined by

$$F(X,Y)=(UX\otimes UY\otimes U^*(X\otimes Y),u_X\otimes u_Y\otimes u^*_{X\otimes Y})$$

Let us look for example to its associator

$$a_F^{X,Y,Z}\colon F(X,Y)\otimes F(X\otimes Y,Z)\to {}^{X}F(Y,Z)\otimes F(X,Y\otimes Z)$$

It is given by the following arrow, where the first step is canonical (use the explicit definition of F) and the second step comes from the fact that the equivalence $\mathbb{A}\simeq \mathbb{K}\mathrm{er}(\mathrm{C}(T))$ is equivariant (first part of Proposition 7.9)

$$\begin{tikzcd}
F(X,Y)\otimes F(X\otimes Y,Z) \arrow[d] \\
UX\otimes F(Y,Z)\otimes U^{*}X\otimes F(X,Y\otimes Z) \arrow[d] \\
{}^{X}F(Y,Z)\otimes F(X,Y\otimes Z) \arrow[d, "({}^{\sigma(X)}F(Y,Z))^{-1}\otimes \mathrm{id}"] \\
{}^{X}F(Y,Z)\otimes F(X,Y\otimes Z)
\end{tikzcd}$$

Finally, the needed arrow in $\mathbb{O}\mathrm{pext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$ is given by

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "T"] \arrow[d, "\simeq"'] \arrow[drr, phantom, "\simeq \mathrm{can}"] & & \mathbb{B} \arrow[r, "\mathrm{C}(T)"] \arrow[d, "R"] & \mathbb{B}/T \\
\mathbb{K}\mathrm{er}(\mathrm{C}(T)) \arrow[rr, "i_{\mathbb{K}\mathrm{er}(\mathrm{C}(T))}"'] & & \mathbb{K}\mathrm{er}(\mathrm{C}(T))\times_F \mathbb{B}/T \arrow[ur, "p_{\mathbb{B}/T}"'] &
\end{tikzcd}$$

with

$$RX=((X\otimes U^{*}X,\ X\otimes U^{*}X \xrightarrow{\mathrm{id}\otimes u_X^{*}} X\otimes X^{*}\simeq I\ ),X)$$

∎

The previous theorem allows us to define the Baer sum of extensions with symmetric kernel and functorial section. More precisely, if we denote by $\mathrm{Opext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$ the groupoid obtained from $\mathbb{O}\mathrm{pext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$ taking as arrows 2-equivalence classes of homomorphisms, we get the following corollary.

Corollary 9.4 *Let $\mathbb{G}$ be a categorical group and $\varphi\colon \mathbb{G}\to \mathbb{E}\mathrm{q}_s(\mathbb{A})$ a symmetric action. The biequivalence*

$$\overline{\mathcal{E}}\colon \mathbb{H}^2(\mathbb{G},\mathbb{A},\varphi)\to \mathbb{O}\mathrm{pext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$$

induces a structure of symmetric categorical group on the groupoid $\mathrm{Opext}_{FS}(\varphi,\mathbb{G},\mathbb{A})$.

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