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On Algebraically Exact Categories and Essential Localizations of Varieties

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Algebraically exact categories have been introduced in J. Adámek, F. W. Lawvere, and J. Rosický (to appear), as an equational hull of the 2-category VAR of all varieties of finitary algebras. We will show that algebraically exact categories with a regular generator are precisely the essential localizations of varieties and that, in this case, algebraic exactness is equivalent to (1) exactness, (2) commutativity of filtered colimits with finite limits, (3) distributivity of filtered colimits over arbitrary products, and (4) product-stability of regular epimorphisms. This can be viewed as a nonadditive generalization of the classical Roos Theorem characterizing essential localizations of categories of modules. Analogously, precontinuous categories, introduced in J. Adámek, F. W. Lawrence, and J. Rosický (to appear) as an equational hull of the 2-category LFP (of locally finitely presentable categories), are characterized by the above properties (2) and (3). Essential localizations of locally finitely presentable categories and presheaf categories are fully described.
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1. INTRODUCTION

The category VAR of (finitary) varieties is not equational over CAT, the quasicategory of all categories, as shown in [ALR3]. There an equational hull of VAR with respect to "small operations" has been described, whereby it was shown that those small operations on VAR are just combinations of the following three kinds of operations:

(1) $\lim_{\mathscr{K}}$, the formation of limits of type $\mathscr{K}$ (a $\mathscr{K}$-ary operation for every small category $\mathscr{K}$),

(2) $\operatorname{colim}_{\mathscr{K}}$, the formation of filtered colimits of type $\mathscr{K}$ (a $\mathscr{K}$-ary operation for every small filtered category $\mathscr{K}$), and

(3) coeq, the formation of reflexive coequalizers, an operation of arity

$$\mathscr{K} = \bullet \overset{\overset{p}{\rightarrow}}{\underset{\underset{q}{\rightarrow}}{\overset{r}{\leftarrow}}} \bullet \qquad \text{where } pr = \text{id} = qr.$$

There are important equational rules concerning the operations (1)–(3) which hold in every variety and therefore hold in all the categories $\mathscr{A}$ lying in the equational hull of VAR:

(FLC) Finite limits commute with filtered colimits.

(PD) Products distribute over filtered colimits. That is, given a collection of filtered diagrams in $\mathscr{A}$

$$D_i : \mathscr{D}_i \to \mathscr{A} \qquad (i \in I),$$

and forming the "product diagram" $\prod_{i \in I} D_i$ by

$$\prod_{i \in I} \mathscr{D}_i \to \mathscr{A}, \qquad (d_i) \mapsto \prod (D_i d_i),$$

then the canonical morphism

$$\operatorname{colim} \prod_{i \in I} D_i \to \prod_{i \in I} \operatorname{colim} D_i$$

is an isomorphism.

(REP) Regular epimorphisms are product-stable. That is, $\prod_{i \in I} e_i : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$ is a regular epimorphism whenever each e_i is.

An open problem concerning VAR is whether the above three properties characterize the equational hull. In the present paper we characterize a large class of members of the equational hull of VAR, i.e., all essential localizations of varieties. Recall that, for a category $\mathscr{A}$, a *localization* is a full reflective subcategory $\mathscr{B}$ such that a reflector $R: \mathscr{A} \to \mathscr{B}$ preserves finite limits. And $\mathscr{B}$ is called a *complete localization* if R preserves limits and an *essential localization* if R is a right adjoint (recall that R is a left adjoint of the embedding $\mathscr{B} \hookrightarrow \mathscr{A}$). As observed in [ALR3], an equational hull of VAR is closed under complete localizations. Now for localizations of a variety, "essential" = "complete". Thus, essential localizations of varieties are important examples of categories "near" to VAR. One of the main results of our paper is the following.

THEOREM. *A category is an essential localization of a variety iff it is cocomplete, exact, has a regular generator, and satisfies* (FLC), (PD), *and* (REP).

We obtain a sharper result concerning the category LFP, of all locally finitely presentable categories. An equational hull with respect to all "small operations" has been described in [ALR2], and these operations are generated by the two kinds $\lim_{\mathscr{X}}$, formation of limits, and $\operatorname{colim}_{\mathscr{X}}$, formation of filtered colimits. It is clear that every LFP category has finite limits commuting with filtered colimits (FLC) and products distributing over filtered colimits (PD). In the present paper we prove that the equational rules (FLC) and (PD) generate all equations holding between small operations on LFP. And we again characterize all essential (= complete) localizations of LFP categories.

THEOREM. *A category is an essential localization of an* LFP *category iff it is cocomplete, has a regular generator, and satisfies* (FLC) *and* (PD).

To put the results of our paper into a historical perspective, let us recall some of the classical results on categories related to the module categories (more precisely, to the categories Mod-R of right modules over a unitary ring R).

(α) *A category is equivalent to a module category iff it*

- (i) *is abelian,*
- (ii) *satisfies* (AB3), *i.e., it is cocomplete, and*
- (iii) *has a finitely presentable, regularly projective regular generator.*

This has been proved by Gabriel [G] and Mitchell [M]. Let us stress here that the classical results, dealing with one-sorted algebras, always use "generator" as a single object. Below, we use "generator" to mean a set of objects; this is related to the fact that the algebras we consider are many-sorted, in general.

(β) *A category is equivalent to a localization of a module category iff it*

(i) *is abelian*,

(ii) *satisfies* (AB5), *i.e.*, *it is cocomplete and for any directed family of subobjects* $A_i \hookrightarrow B$ $(i \in I)$ and any subobject $A \hookrightarrow B$ we have $\bigcup_{i \in I}(A \cap A_i) = A \cap (\bigcup_{i \in I} A_i)$, and

(iii) *has a regular generator*.

See Popescu and Gabriel [PG]. Finally,

(γ) *A category is equivalent to an essential localization of a module category iff it*

(i) *is abelian*,

(ii) *has product-stable regular epimorphisms* (*this is the dual condition* (AB4)* *to* (AB4)), *and*

(iii) *satisfies* (AB6), *i.e.*, *it is cocomplete and for any family of directed families* $A_{ij} \hookrightarrow B$ $(j \in J_i)$ *of subobjects* $(i \in I)$ *we have*

$$\bigcup_{(j_i) \in \prod J_i} \bigcap_{i \in I} A_{ij_i} = \bigcap_{i \in I} \bigcup_{j \in J_i} A_{ij}$$

and

(iv) *has a regular generator*.

This is the classical Roos Theorem; see [R2]. Now, module categories are precisely the additive version of finitary varieties: recall from Lawvere [L1, L2] that varieties are given by algebraic (finite-product) theories. In categories with biproducts thus the models of algebraic theories are just models of the unary reducts, and so module categories are, in the abelian world, precisely what varieties are in general.

Thus, a nonadditive generalization of (α) above is, then, an abstract characterization of varieties of finitary algebras. This has been proved in [L1] for one-sorted algebras. We, however, want to consider many-sorted algebras in general. Lawvere's result immediately yields that case, too (as explicitly proved in [AR1]); one just has to understand a *regular generator* to mean a set $\mathscr{G}$ of objects such that for every object K the canonical morphism

$$\coprod_{G \in \mathscr{G}} \coprod_{\hom(G,K)} G \to K$$

is a regular epimorphism.

(α^*) A *category is equivalent to a variety iff it*

(i) *is exact* (*in Barr's sense* [B]) *and complete*, *and*

(ii) *has a regular generator consisting of finitely presentable regular projectives*.

A nonadditive generalization of (β) has been presented by the third author in [V2]:

(β^*) A *category is equivalent to a localization of a variety iff it*

(i) *is exact and cocomplete,*

(ii) *has filtered colimits which commute with finite limits*, *and*

(iii) *has a regular generator*.

The above theorem characterizing essential localizations of varieties is, then, a nonadditive extension (γ^*) of the Roos Theorem.

We also prove the same results for presheaf categories in place of varieties or locally finitely presentable categories. Here we obtain complete pretoposes and complete and cocomplete categories whose limits distribute over colimits in a sense analogous to the above situations. As a consequence, we obtain the result of Roos [R2] characterizing essential localizations of categories of presheaves. These are exactly complete pretoposes with a regular generator.

All our categories are supposed to be locally small.

2. PRECONTINUOUS CATEGORIES

Recall the 2-category LFP of locally finitely presentable categories of Gabriel and Ulmer. Its morphisms (1-cells) follow from the Gabriel–Ulmer duality—they are the right adjoints preserving filtered colimits (and 2-cells are the natural transformations). In [ALR2] it has been proved that LFP is not monadic over CAT, and an equational hull of LFP has been described. It is the following 2-category:

Objects are called *precontinuous categories*: They are the categories $\mathscr{K}$ with limits and filtered colimits which distribute in the sense made precise below:

morphisms (1-cells) are the functors preserving limits and filtered colimits; and

2-cells are the natural transformations.

However, little more has been told about precontinuous categories in [ALR2]. We now present a more straightforward characterization of these categories and exhibit one of the basic examples; essential localizations of locally finite presentable categories.

Recall that the completion

$$\eta_{\mathscr{K}}^{\mathrm{Ind}} : \mathscr{K} \to \mathrm{Ind}\, \mathscr{K}$$

of a category $\mathscr{K}$ under filtered colimits was described in [AGV] as follows: Ind $\mathscr{K}$ is the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ of all functors $\mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ which are (small) filtered colimits of representable functors. And η is the codomain restriction of the Yoneda embedding $Y_{\mathscr{K}} : \mathscr{K} \to \mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$. Recall that if $\mathscr{K}$ is complete then Ind $\mathscr{K}$ is closed under limits in $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ and therefore is also complete; moreover, filtered colimits distribute in Ind $\mathscr{K}$ over products (see [AGV]). If $\mathscr{K}$ has filtered colimits we denote by

$$C_{\mathscr{K}}^{\mathrm{Ind}} : \mathrm{Ind}\,\mathscr{K} \to \mathscr{K}$$

a functor computing filtered colimits in $\mathscr{K}$. This is the essentially unique functor preserving filtered colimits and satisfying

$$C_{\mathscr{K}}^{\mathrm{Ind}} \cdot \eta_{\mathscr{K}}^{\mathrm{Ind}} \cong \mathrm{Id}_{\mathscr{K}}. \tag{3}$$

Moreover,

$$\eta_{\mathscr{K}}^{\mathrm{Ind}} \vdash C_{\mathscr{K}}^{\mathrm{Ind}}.$$

If $\mathscr{K}$ has and $C_{\mathscr{K}}^{\mathrm{Ind}}$ preserves limits, we say that limits *distribute* over filtered colimits.

DEFINITION (see [ALR2]). A category is called *precontinuous* iff it is complete, has filtered colimits, and its limits distribute over filtered colimits.

In the following theorem we prove that distributivity of limits over filtered colimits is equivalent to two (less formal) conditions; the commutativity of finite limits and the distributivity of products as defined in the Introduction above.

THEOREM 2.1. *A category $\mathscr{K}$ is precontinuous iff it has limits and filtered colimits and*

(a) *filtered colimits commute with finite limits*, *and*

(b) *products distribute over filtered colimits*.

Proof. Let $\mathscr{K}$ be a category with limits and filtered colimits. We use the following description of Ind $\mathscr{K}$ (see e.g., [JJ]): objects are all filtered diagrams in $\mathscr{K}$. Morphisms from $D : \mathscr{D} \to \mathscr{K}$ to $D' : \mathscr{D}' \to \mathscr{K}$ are compatible families of equivalence classes $[f_d]$ ($d \in \mathrm{obj}\,\mathscr{D}$) of morphisms $f_d : Dd \to D'd'$ in $\mathscr{K}$ under smallest equivalence $\sim$ with $f_d \sim D'\delta \cdot f_d$ for every $\delta \in \mathrm{mor}\,\mathscr{D}'$ with domain d'; compatibility means $[f_d] = f_{\bar{d}} \cdot D\bar{\delta}]$ for all $\bar{\delta} : d \to \bar{d}$ in $\mathscr{D}$. The embedding $\eta_{\mathscr{K}}^{\mathrm{Ind}}$ sends an object X to the corresponding single-morphism diagram (X). The diagram

$$(d_i)_{i\in I} \mapsto \prod_{i\in I} D_i d_i$$

used in the above definition of distributivity of products over filtered colimits is obviously filtered, and it is a product of the diagrams D_i $(i \in I)$ as objects of Ind$\mathscr{K}$ w.r.t. the morphisms $[\pi_j]: \prod D_i d_i \to D_j d_j$, where π_j is the jth projection $(j \in I)$. Thus;

$$C_{\mathscr{K}}^{\mathrm{Ind}} \text{ preserves products} \Leftrightarrow \text{products distribute over filtered colimits.} \quad (*)$$

(A) Let $\mathscr{K}$ be precontinuous. In view of $(*)$ it remains to prove that filtered colimits commute with equalizers. In fact, they commute with finite products (since this is equivalent to distributing over them), hence they then commute with finite limits. Let $D, D' : \mathscr{D} \to \mathscr{K}$ be filtered diagrams, and let $f_d, g_d : Dd \to D'd$ $(d \in \mathscr{D})$ be natural transformations. Then $([f_d]_{d\in\mathscr{D}}([g_d])_{d\in\mathscr{D}} : D \to D'$ are morphisms in Ind$\mathscr{K}$. If $e_d : Ed \to Dd$ are (pointwise) equalizers in $\mathscr{K}$, then we obtain an obvious filtered diagram $E: \mathscr{D} \to \mathscr{K}$ with a morphism $([e_d])_{d\in\mathscr{D}} : E \to D$ which is easily seen to be an equalizer of $([f_d]), ([g_d])$. Since $C_{\mathscr{K}}^{\mathrm{Ind}}$ preserves equalizers, we conclude that

$$\operatorname*{colim}_{d\in\mathscr{D}} e_d \text{ is an equalizer of } \operatorname*{colim}_{d\in\mathscr{D}} f_d \text{ and } \operatorname*{colim}_{d\in\mathscr{D}} g_d,$$

which is precisely what we needed to prove.

(B) Let $\mathscr{K}$ have filtered colimits commuting with finite limits and distributing over products. In view of $(*)$, it remains to prove that $C_{\mathscr{K}}^{\mathrm{Ind}}$ preserves equalizers. Here we return to our description of Ind$\mathscr{K}$ as a full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ above. We abbreviate $\eta_{\mathscr{K}}^{\mathrm{Ind}}$ to η. Every object of Ind$\mathscr{K}$ is a colimit of $\eta(B)$ for some filtered diagram B in $\mathscr{K}$. Consider a pair of morphisms

$$f, g : \operatorname{colim} \eta(A) \to \operatorname{colim} \eta(B)$$

in Ind$\mathscr{K}$, where $A : \mathscr{I} \to \mathscr{K}$ and $B : \mathscr{J} \to \mathscr{K}$ are filtered diagrams.

I. At first assume that $\mathscr{I}$ has just one morphism, i.e., that

$$f, g : \eta(X) \to \operatorname{colim} \eta(B)$$

where $X \in \mathscr{K}$. Since $\mathscr{J}$ is filtered, there are $j_0 \in \mathscr{J}$ and $f_0, g_0 : X \to B_{j_0}$ in $\mathscr{K}$ such that $f = b_{j_0} \cdot \eta(f_0)$ and $g = b_{j_0} \cdot \eta(g_0)$; here $(b_j : \eta(B_j) \to \operatorname{colim} \eta(B))_{j\in\mathscr{J}}$ denotes a colimit cocone in Ind$\mathscr{K}$. Consider the comma category $\bar{\mathscr{J}} = j_0 \downarrow \mathscr{J}$. Then $\bar{\mathscr{J}}$ is filtered and we define a diagram $E : \bar{\mathscr{J}} \to \mathscr{K}$

as follows: given an object $\bar{j} : j_0 \to j$ of $\bar{\mathscr{J}}$, form an equalizer

$$E_{\bar{j}} \xrightarrow{e_{\bar{j}}} X \overset{B(\bar{j}) \cdot f_0}{\underset{B(\bar{j}) \cdot g_0}{\rightrightarrows}} B_j$$

in $\mathscr{K}$. Given a morphism $h : \bar{j} \to \bar{j}'$ in $\bar{\mathscr{J}}$, i.e., a commutative triangle

$$\begin{array}{ccc} & j_0 & \\ {}^{\bar{j}_0}\swarrow & & \searrow^{\bar{j}'} \\ j & \xrightarrow[h]{} & j' \end{array}$$

in $\mathscr{J}$, we have a unique morphism

$$E(h) : E_{\bar{j}} \to E_{\bar{j}'} \text{ with } e_{\bar{j}} = e_{\bar{j}'} \cdot E(h)$$

induced by the fact that $e_{\bar{j}}$ merges $B(\bar{j}') \cdot f_0$ and $B(\bar{j}') \cdot g_0$.

Denote by $U : \bar{\mathscr{J}} \to \mathscr{J}$ the usual forgetful functor. Composed with B, this yields a filtered diagram $B \cdot U : \bar{\mathscr{J}} \to \mathscr{K}$ and two natural transformations

$$\bar{f}, \bar{g} : \Delta_X \to B \cdot U$$

where $\Delta_X : \bar{\mathscr{J}} \to \mathscr{K}$ is the constant functor with value X and $\bar{f}_{\bar{j}} = B(\bar{j}) \cdot f_0$, $\bar{g}_{\bar{j}} = B(\bar{j}) \cdot g_0$. The above pointwise equalizers $e_{\bar{j}}$ define a natural transformation

$$e : E \to \Delta_X$$

which is an equalizer of $\bar{f}$ and $\bar{g}$ in $\mathscr{K}^{\bar{\mathscr{J}}}$. Since filtered colimits commute with equalizers in $\mathscr{K}$, we conclude that

$$\operatorname{colim} E \xrightarrow{\operatorname{colim} e} X \overset{\operatorname{colim} \bar{f}}{\underset{\operatorname{colim} \bar{g}}{\rightrightarrows}} \operatorname{colim} B \cdot U$$

is an equalizer in $\mathscr{K}$. Since B is a filtered diagram, we have $\operatorname{colim} B \cdot U \cong \operatorname{colim} B$, and we can write

$$\operatorname{colim} \bar{f} = C_{\mathscr{K}}^{\mathrm{Ind}}(f) \qquad \text{and} \qquad \operatorname{colim} \bar{g} = C_{\mathscr{K}}^{\mathrm{Ind}}(g).$$

Analogously, since

$$\eta(e) \cdot \eta(E) \to \eta(\Delta X)$$

is an equalizer of $\eta(\bar{f})$ and $\eta(\bar{g})$ in $(\mathrm{Ind}\,\mathscr{K})^{\bar{\mathscr{J}}}$ and equalizers commute with filtered colimits in $\mathrm{Ind}\,\mathscr{K}$ (because $\mathrm{Ind}\,\mathscr{K}$ is closed in $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ under limits and filtered colimits), we get that

$$\operatorname{colim} \eta(E) \xrightarrow{\operatorname{colim} \eta(e)} \eta(X) \underset{\operatorname{colim} \eta(\bar{g})}{\overset{\operatorname{colim} \eta(\bar{f})}{\rightrightarrows}} \operatorname{colim} \eta(B \cdot U)$$

is an equalizer in $\operatorname{Ind} \mathscr{K}$. Colimits commute with colimits, thus we have

$$C_{\mathscr{K}}^{\operatorname{Ind}}(\operatorname{colim} \eta(e)) = \operatorname{colim} e.$$

We have proved that $C_{\mathscr{K}}^{\operatorname{Ind}}$ preserves the above equalizer.

II. Consider a general case and denote by $(a_i : \eta(A_i) \to \operatorname{colim} \eta(A))_{i \in \mathscr{I}}$ a colimit cocone in $\operatorname{Ind} \mathscr{K}$. We form equalizers

$$E_i \xrightarrow{e_i} \eta(A_i) \underset{g \cdot a_i}{\overset{f \cdot a_i}{\rightrightarrows}} \operatorname{colim} \eta(B)$$

in $\operatorname{Ind} \mathscr{K}$. This defines a filtered diagram $E : \mathscr{I} \to \operatorname{Ind} \mathscr{K}$ whose connecting morphism $E(h)$ for $h : i \to i'$ in $\mathscr{I}$ is given by the fact that $A(h) \cdot e_i$ merges $f \cdot a'_i$ and $g \cdot a'_i$. We get a morphism

$$e = \operatorname{colim} e_i : \operatorname{colim} E \to \operatorname{colim} \eta(A)$$

which is an equalizer of

$$\operatorname{colim} \eta(A) \underset{g}{\overset{f}{\rightrightarrows}} \operatorname{colim} \eta(B)$$

because filtered colimits commute with equalizers in $\operatorname{Ind} \mathscr{K}$. Following **Condition** I, we know that

$$C_{\mathscr{K}}^{\operatorname{Ind}}(E(i)) \xrightarrow{C_{\mathscr{K}}^{\operatorname{Ind}}(e_i)} A_i \underset{C_{\mathscr{K}}^{\operatorname{Ind}}(g \cdot a_i)}{\overset{C_{\mathscr{K}}^{\operatorname{Ind}}(f \cdot a_i)}{\rightrightarrows}} \operatorname{colim} B$$

is an equalizer in $\mathscr{K}$. Since filtered colimits commute with equalizers in $\mathscr{K}$,

$$C_{\mathscr{K}}^{\operatorname{Ind}}(\operatorname{colim} E) \xrightarrow{C_{\mathscr{K}}^{\operatorname{Ind}}(f)} \operatorname{colim} A \underset{C_{\mathscr{K}}^{\operatorname{Ind}}(g)}{\overset{C_{\mathscr{K}}^{\operatorname{Ind}}(f)}{\rightrightarrows}} \operatorname{colim} B$$

is an equalizer in $\mathscr{K}$. Hence $C_{\mathscr{K}}^{\operatorname{Ind}}$ preserves equalizers. ∎

Remark 2.2. Recall that a *complete localization* of a category $\mathscr{L}$ is a full reflective subcategory $\mathscr{K}$ of $\mathscr{L}$ whose reflector R preserves limits. If $\mathscr{L}$ is locally presentable then this is the same concept as essential localization: by SAFT, R is then a right adjoint.

EXAMPLE 2.3. (1) $\mathscr{K} = \operatorname{Ind}\mathscr{A}$ is precontinuous (for any category $\mathscr{A}$) iff $\mathscr{K}$ is complete; see [ALR2]. This is the case whenever $\mathscr{A}$ is complete, or whenever $\mathscr{A}$ is small and finitely cocomplete. The latter case precisely characterizes the locally finitely presentable categories.

(2) Complete localizations of precontinuous categories are precontinuous. In fact, let $E : \mathscr{K} \to \mathscr{L}$ be a complete localization of $\mathscr{L}$ with a (limit-preserving) reflector $R : \mathscr{L} \to \mathscr{K}$. Then $C_{\mathscr{K}}^{\mathrm{Ind}} = R \cdot C_{\mathscr{L}}^{\mathrm{Ind}} \cdot \operatorname{Ind} E : \operatorname{Ind}\mathscr{K} \to \mathscr{K}$, and since E preserves limits so does $\operatorname{Ind} E$ (see [ALR2]). Thus, whenever $C_{\mathscr{L}}^{\mathrm{Ind}}$ preserves limits, so does $C_{\mathscr{K}}^{\mathrm{Ind}}$.

Remark 2.4. The examples (1) and (2) above are generic: every precontinuous category $\mathscr{K}$ is a complete localization of $\operatorname{Ind}\mathscr{A}$, with $\mathscr{A}$ complete. In fact, put $\mathscr{A} = \mathscr{K}$, then $\eta : \mathscr{A} \to \operatorname{Ind}\mathscr{K}$ is a full embedding whose left adjoint (reflector) is $C_{\mathscr{K}}^{\mathrm{Ind}}$.

We thus concentrate on complete localizations of the basic examples of precontinuous categories, viz, locally finitely presentable (LFP) categories.

Remark 2.5. (i) Categories which are (noncomplete) localizations of LFP categories have been fully characterized in [DS] as cocomplete categories which have

(a) finite limits commuting with filtered colimits, and

(b) a regular generator.

(ii) Every localization of an LFP category is locally λ-presentable for some infinite regular cardinal λ. In fact, let $E : \mathscr{K} \to \mathscr{L}$ be a localization with $\mathscr{L}$ an LFP category. As proved in [BK], E preserves λ-filtered colimits for some λ; thus, $\mathscr{K}$ is equivalent to a full reflective subcategory of a locally λ-presentable category $\mathscr{L}$ closed under λ-filtered colimits. It follows that $\mathscr{K}$ is locally λ-presentable; see Theorem 1.20 in [AR]. However, $\mathscr{k}$ is not LFP in general.

Remark 2.6. Every continuous lattice L which is not algebraic (e.g., a closed real interval) is a complete localization of $\operatorname{Ind} L$ which, being an algebraic lattice, is LFP. But L itself is not LFP.

THEOREM 2.7. *A category is a complete* (= *essential*) *localization of a locally finitely presentable category iff it*

(i) *is cocomplete*,

(ii) *has finite limits commuting with filtered colimits*,

(iii) *has products distributing over filtered colimits*, *and*

(iv) *has a regular generator*.

COROLLARY. *Complete localizations of LFP categories are precisely the cocomplete precontinuous categories which have a regular generator*.

Proof. The necessity of (i)–(iv) is obvious. To prove sufficiency, let $\mathscr{K}$ fulfill (i)–(iv) and be locally λ-presentable (see Remark 2.5.). Let $\mathscr{C}$ be a small full subcategory representing all λ-presentable objects; $J : \mathscr{C} \to \mathscr{K}$ denotes the full embedding. Since $\mathscr{C}$ is closed under finite colimits in $\mathscr{K}$, it follows that $\operatorname{Ind}\mathscr{C}$ is a locally finitely presentable category. The functor

$$E : \mathscr{K} \to \operatorname{Ind}\mathscr{C}, \qquad K \mapsto \mathscr{K}(J-, K)$$

is fully faithful and preserves limits and λ-filtered colimits (since objects of $\mathscr{C}$ are λ-presentable). It follows that E has a left adjoint (reflector)

$$H : \operatorname{Ind}\mathscr{C} \to \mathscr{K},$$

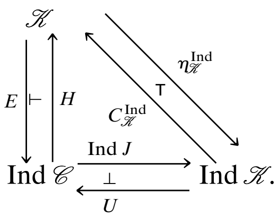
see [AR, 1.66]. It is sufficient to prove that H preserves limits, then $\mathscr{K}$ is (equivalent to) a complete localization of $\operatorname{Ind}\mathscr{C}$. Observe that

$$H \cdot E \cong \mathrm{Id}_{\mathscr{K}}. \tag{4}$$

The functor $\operatorname{Ind} J : \operatorname{Ind}\mathscr{C} \to \operatorname{Ind}\mathscr{K}$ has a right adjoint, viz, the functor

$$U : \operatorname{Ind}\mathscr{K} \to \operatorname{Ind}\mathscr{C}$$

of restriction of a presheaf from $\mathscr{K}^{\mathrm{op}}$ to $\mathscr{C}^{\mathrm{op}}$ (recall that $\mathscr{C}$ is closed in $\mathscr{K}$ under finite colimits, in fact, under λ-small colimits). Consider the diagram



We will prove that H preserves limits. Since

$$E = U \cdot \eta^{\mathrm{Ind}}_{\mathscr{K}}, \tag{5}$$

we have, for the corresponding left adjoints,

$$H \cong C^{\mathrm{Ind}}_{\mathscr{K}} \cdot \operatorname{Ind} J. \tag{6}$$

Moreover, clearly

$$U \cdot \operatorname{Ind} J \cong \mathrm{Id}_{\operatorname{Ind}\mathscr{C}}. \tag{7}$$

The above (5), (4), and (3) imply

$$H \cdot U \cdot \eta^{\mathrm{Ind}}_{\mathscr{K}} \cong H \cdot E \cong \mathrm{Id}_{\mathscr{K}} \cong C^{\mathrm{Ind}}_{\mathscr{K}} \cdot \eta^{\mathrm{Ind}}_{\mathscr{K}}. \tag{8}$$

Both of the functors $H \cdot U$ and $C_{\mathscr{K}}^{\mathrm{Ind}}$ preserve filtered colimits (U preserves them because they are calculated pointwise in $\mathrm{Ind}\,\mathscr{K}$); thus from (8) we derive

$$H \cdot U \cong C_{\mathscr{K}}^{\mathrm{Ind}}. \tag{9}$$

We are ready to prove that H preserves limits, using the fact that $C_{\mathscr{K}}^{\mathrm{Ind}}$ preserves them (since $\mathscr{K}$ is precontinuous), and so does, obviously, U. We thus have, for every diagram D in $\mathrm{Ind}\,\mathscr{C}$, canonical isomorphisms as follows:

$$\begin{aligned} H(\lim D) &\cong H(\lim U \cdot \mathrm{Ind}\, J \cdot D) && \text{by } (7) \\ &\cong HU(\lim \mathrm{Ind}\, J \cdot D) && \text{continuity of } U \\ &\cong C_{\mathscr{K}}^{\mathrm{Ind}}(\lim \mathrm{Ind}\, J \cdot D) && \text{by } (9) \\ &\cong \lim\big(C_{\mathscr{K}}^{\mathrm{Ind}} \cdot \mathrm{Ind}\, J \cdot D\big) && \text{precontinuity of } \mathscr{K} \\ &\cong \lim H \cdot D && \text{by } (7), (9). \end{aligned}$$

∎

3. ALGEBRAICALLY EXACT CATEGORIES

In [ALR1] a duality between varieties and algebraic theories has been introduced which leads naturally to the 2-category VAR of finitary varieties (as objects): its morphisms (1-cells), called *algebraically exact functors*, are precisely the right adjoint functors preserving filtered colimits and regular epimorphisms. And 2-cells are the natural transformations. Now algebraically exact functors are precisely those preserving limits and *sifted colimits*. Recall that a small category $\mathscr{D}$ is called *sifted* if $\mathscr{D}$-colimits commute in **Set** with finite products (see [ALR1]). Filtered categories are sifted. Also, the "reflexive pair" category

$$A_1 \underset{f_2}{\overset{f_1}{\underset{\rightarrow}{\overset{\rightarrow}{\overset{d}{\leftarrow}}}}} A_0$$

(with $df_1 = df_2 = \mathrm{id}_{A_1}$) is sifted. Following [La], a nonempty category $\mathscr{D}$ is sifted iff for every pair A, B of objects the category of all cospans

$$A \to X \leftarrow B$$

is connected; a full proof of this result is also presented in [AR2]. By *sifted colimits* we mean colimits with sifted domains.

Let

$$\eta_{\mathscr{K}}^{\text{Sind}} : \mathscr{K} \to \text{Sind}\,\mathscr{K}$$

be a free completion of $\mathscr{K}$ under sifted colimits (see [AR2]). Sind $\mathscr{K}$ can be described as the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\text{op}}}$ consisting of all (small) sifted colimits of representable functors; $\eta_{\mathscr{K}}^{\text{Sind}}$ is the codomain restriction of the Yoneda embedding. For any category $\mathscr{K}$ with sifted colimits, we denote by

$$C_{\mathscr{K}}^{\text{Sind}} : \text{Sind}\,\mathscr{K} \to \mathscr{K}$$

a functor computing sifted colimits in $\mathscr{K}$. This is the essentially unique functor preserving sifted colimits and satisfying

$$C_{\mathscr{K}}^{\text{Sind}} \cdot \eta_{\mathscr{K}}^{\text{Sind}} \cong \text{Id}_{\mathscr{K}}. \tag{10}$$

Again,

$$\eta_{\mathscr{K}}^{\text{Sind}} \vdash C_{\mathscr{K}}^{\text{Sind}}.$$

If $\mathscr{K}$ is complete then Sind $\mathscr{K}$ is complete as well (see [ALR3]).

DEFINITION (see [ALR3]). A category $\mathscr{K}$ is called *algebraically exact* if it has limits and sifted colimits such that $C_{\mathscr{K}}^{\text{Sind}} : \text{Sind}\,\mathscr{K} \to \mathscr{K}$ preserves limits.

An equational hull of VAR has been presented in [ALR3] as the 2-category of all

algebraically exact categories (0-cells),

functors preserving limits and sifted colimits (1-cells), and

natural transformations (2-cells).

Any algebraically exact category $\mathscr{K}$ is exact and precontinuous and has product-stable regular epimorphisms; see [ALR3].

Problem 3.1. Let $\mathscr{K}$ be a complete category with sifted colimits, which is exact, precontinuous and has product-stable regular epimorphisms. Is $\mathscr{K}$ then algebraically exact?

We will show that the answer is affirmative for cocomplete categories having a regular generator. In this case, the resulting categories are precisely the essential localizations of varieties. We will start with a characterization of complete localizations of monadic categories over many-sorted sets (i.e., categories of T-algebras for monads T over $\mathbf{Set}^S$, S a set). If T is finitary, these monadic categories are precisely the (many-sorted) varieties; the general case includes infinitary varieties and is outside the scope of algebraically exact categories. We start with this result as a preparation for the characterization of algebraically exact categories with a regular generator and because we find it interesting per se.

THEOREM 3.2. *Complete localizations of monadic categories over many sorted sets are precisely the cocomplete exact categories with a regular generator whose regular epimorphisms are product-stable.*

Proof. Necessity is evident. To prove sufficiency, let $\mathscr{C} = \{C_s;\ s \in S\}$ be a regular generator of $\mathscr{K}$ and consider $U : \mathscr{K} \to \mathbf{Set}^S$ given by

$$U(K)(s) = \mathscr{K}(C_s, K).$$

Then U has a left adjoint H given by

$$H(X) = \coprod_{s \in S} X_s \cdot C_s.$$

Let $T = UH$ be the induced monad on $\mathbf{Set}^S$ and let $G : \mathscr{K} \to \mathrm{Alg}(T)$ be the comparison functor. Since $\mathscr{C}$ is a regular generator, G is full and faithful. Following [V1], G is a localization; denote by $F \dashv G$ a left adjoint of G.

Let Q be the regular epireflective hull of $G(\mathscr{K})$ in $\mathrm{Alg}(T)$, i.e., the full subcategory of all subalgebras of the algebras $G(K)$, $K \in \mathscr{K}$. Following [PR, Theorem 1.5], $G : \mathscr{K} \to Q$ is a complete localization. Hence $F : \mathrm{Alg}(T) \to \mathscr{K}$ preserves finite limits and products of objects from Q. We will prove that F preserves all products, which proves that $\mathscr{K}$ is a complete localization of $\mathrm{Alg}(T)$.

Given T-algebras X_i $(i \in I)$, consider

$$Y_i \underset{v_i}{\overset{u_i}{\rightrightarrows}} GK_i \xrightarrow{e_i} X_i,$$

where e_i represents X_i as a quotient of a free T-algebra and u_i, v_i is a kernel pair of e_i in $\mathrm{Alg}(T)$. Then $Y_i \in Q$. In

$$FY_i \underset{Fv_i}{\overset{Fu_i}{\rightrightarrows}} FGK_i \xrightarrow{Fe_i} FX_i,$$

Fe_i is regular epi and Fu_i, Fv_i is a kernel pair of Fu_i, Fv_i. Hence, in

$$\prod FY_i \cong F\prod Y_i \underset{\prod Fv_i = F\prod v_i}{\overset{\prod Fu_i = F\prod u_i}{\rightrightarrows}} \prod K_i \xrightarrow{\prod Fe_i} \prod FX_i, \tag{11}$$

$\prod Fe_i$ is regular epi and $\prod Fu_i, \prod Fv_i$ is a kernel pair of $\prod Fe_i$. The isomorphism $\prod FY_i \cong F\prod Y_i$ follows from $Y_i \in Q$. Hence (11) is a coequalizer diagram.

Analogously,

$$\prod Y_i \underset{\prod v_i}{\overset{\prod u_i}{\rightrightarrows}} \prod GK_i \cong G \prod K_i \xrightarrow{\prod e_i} \prod X_i$$

is a kernel pair coequalizer diagram. Therefore

$$F \prod Y_i \underset{F \prod F v_i}{\overset{F \prod F u_i}{\rightrightarrows}} FG \prod K_i = \prod K_i \xrightarrow{F \prod e_i} F \prod X_i$$

is a coequalizer diagram. Hence $F \prod X_i \cong \prod FX_i$. ▮

THEOREM 3.3. *A category is a complete* $(=$ *essential*$)$ *localization of a variety iff it*

(i) *is cocomplete,*

(ii) *is exact,*

(iii) *has a regular generator,*

(iv) *has filtered colimits which commute with finite limits,*

(v) *has products which distribute over filtered colimits, and*

(vi) *has regular epimorphisms which are product-stable.*

COROLLARY. *A category is an essential localization of a variety iff it is cocomplete and algebraically exact and has a regular generator.*

Proof. Necessity follows from the fact that essential localizations of algebraically exact categories are algebraically exact (see [ALR3]).

Sufficiency. Assume that $\mathscr{K}$ satisfies the conditions (i)–(v). Consider $G : \mathscr{K} \to \mathrm{Alg}(T)$ from the proof of Theorem 3.2. Let T_0 be the finitary core of T. Let $P : \mathrm{Alg}(T) \to \mathrm{Alg}(T_0)$ be the induced functor and put

$$G_0 : \mathscr{K} \overset{G}{\to} \mathrm{Alg}(T) \overset{P}{\to} \mathrm{Alg}(T_0).$$

Following [V2], G_0 is a localization; we denote by $F_0 \dashv G_0$ a left adjoint of G_0. We will prove that F_0 preserves products. Then $\mathscr{K}$ is a complete localization of $\mathrm{Alg}(T_0)$. Since $\mathrm{Alg}(T_0)$ is locally finitely presentable, $\mathscr{K}$ is locally presentable by [BK, 6.7]. Hence, G_0 is an essential localization.

(a) F_0 preserves products of finitely generated free T_0-algebras (i.e., the algebras (X, μ_X) where the set $\coprod_{s \in S} X_s$ is finite). This follows from the fact that they coincide with freely finitely generated T-algebras: recall that G is a complete localization (by Theorem 3.2).

(b) F_0 preserves products of free T_0-algebras. In fact, every free T_0-algebra is a filtered colimit of finitely generated free T_0-algebras. Let

$A_i = \operatorname{colim}_{j \in J_i} A_{ij}$, $i \in I$, be free T_0-algebras expressed as filtered colimits of free finitely generated T_0-algebras A_{ij}. By condition (v), $\prod A_i$ is a canonical filtered colimit of products $\prod_{i \in I} A_{if(i)}$ where f ranges through $\prod J_i$. Then $F(\prod A_i) \cong \prod FA_i$ follows from (a) since F preserves filtered colimits.

(c) F_0 preserves all products. In fact, given algebras $A_i \in \mathrm{Alg}(T_0)$, $i \in I$, consider

$$D_i \xrightarrow{g_i} C_i \underset{v_i}{\overset{u_i}{\rightrightarrows}} B_i \xrightarrow{e_i} A_i$$

where e_i represents A_i as a quotient of a free T_0-algebra B_i; u_i, v_i is a kernel pair of e_i; and g_i represents C_i as a quotient of a free T_0-algebra D_i. In

$$\prod D_i \xrightarrow{\prod g_i} \prod C_i \underset{\prod v_i}{\overset{\prod u_i}{\rightrightarrows}} \prod B_i \xrightarrow{\prod e_i} \prod A_i$$

$\prod e_i$ and $\prod g_i$ are regular epimorphisms and $\prod u_i$ and $\prod v_i$ is a kernel pair of $\prod e_i$. Hence

$$\prod D_i \underset{\prod v_i g_i}{\overset{\prod u_i g_i}{\rightrightarrows}} \prod B_i \xrightarrow{\prod e_i} \prod A_i$$

is a coequalizer. Therefore

$$\prod F_0 D_i \cong F_0 \prod D_i \underset{\prod F_0(v_i g_i)}{\overset{\prod F_0(u_i g_i)}{\rightrightarrows}} \prod F_0 B_i \cong F_0 \prod B_i \xrightarrow{F_0 \prod e_i} F_0 \prod A_i$$

is a coequalizer. Moreover, in

$$F_0 D_i \xrightarrow{F_0 g_i} F_0 C_i \underset{F_0 v_i}{\overset{F_0 u_i}{\rightrightarrows}} F_0 B_i \xrightarrow{F_0 e_i} F_0 A_i,$$

$F_0 e_i$ and $F_0 g_i$ are regular epimorphisms and $F_0 u_i, F_0 v_i$ is a kernel pair of $F_0 e_i$. Hence, in

$$\prod F_0 D_i \xrightarrow{\prod F_0 g_i} \prod F_0 C_i \underset{\prod F_0 v_i}{\overset{\prod F_0 u_i}{\rightrightarrows}} \prod F_0 B_i \xrightarrow{\prod F_0 e_i} \prod F_0 A_i,$$

$\prod F_0 g_i$ and $\prod F_0 e_i$ are regular epimorphisms and $\prod F_0 u_i, \prod F_0 v_i$ is a kernel pair of $\prod F_0 e_i$. Therefore

$$\prod F_0 D_i \underset{\prod F_0(v_i g_i)}{\overset{\prod F_0(u_i g_i)}{\rightrightarrows}} \prod F_0 B_i \xrightarrow{\prod F_0 e_i} \prod F_0 A_i$$

is a coequalizer. We conclude $F_0 \prod A_i \cong \prod F_0 A_i$. ∎

Remark 3.4. As a consequence of Theorem 3.3 we get that an abelian category $\mathscr{K}$ is equivalent to an essential localization of a category of modules iff its satisfies (AB5), (AB4*) and if filtered colimits distribute over products.

Following the Roos Theorem mentioned in the Introduction, in abelian categories satisfying (AB4*) the condition (AB6) is equivalent to the fact that filtered colimits commute with finite limits and distribute over products, i.e., is equivalent to precontinuity. In general abelian categories, we do not know whether precontinuity is equivalent to (AB6).

A localization $G : \mathscr{K} \to \mathscr{L}$ is called *finitary* if G preserves filtered colimits.

COROLLARY 3.5. *Let $\mathscr{K}$ be a category. Then the following conditions are equivalent*:

(i) *$\mathscr{K}$ is equivalent to a finitary essential localization of a variety*,

(ii) *$\mathscr{K}$ is locally finitely presentable and algebraically exact*,

(iii) *$\mathscr{K}$ is locally finite presentable and exact and regular epimorphisms are product stable*.

Proof. (i) ⇒ (ii) ⇒ (iii) is evident.

(iii) ⇒ (i). Let $\mathscr{C}$ be a set of representatives of finitely presentable objects of $\mathscr{K}$. Then the monad T from the proof of Theorem 3.2 is finitary and $G : \mathscr{K} \to \mathrm{Alg}(T)$ is finitary as well. Hence (i) follows from Theorem 3.2. ∎

4. ESSENTIAL LOCALIZATIONS OF PRESHEAF CATEGORIES

4.1. We will denote by

$$\eta_{\mathscr{K}}^{\mathrm{Colim}} : \mathscr{K} \to \mathrm{Colim}\,\mathscr{K}$$

a free completion of a category $\mathscr{K}$ under all colimits. $\mathrm{Colim}\,\mathscr{K}$ can be described as the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ consisting of all (small) colimits of representable functors and $\eta_{\mathscr{K}}^{\mathrm{Colim}}$ is the codomain restriction of the Yoneda embedding. If $\mathscr{K}$ is cocomplete we denote by

$$C_{\mathscr{K}}^{\mathrm{Colim}} : \mathrm{Colim}\,\mathscr{K} \to \mathscr{K}$$

a functor computing colimits in $\mathscr{K}$, i.e., the essentially unique functor preserving colimits and such that

$$C_{\mathscr{K}}^{\mathrm{Colim}} \cdot \eta_{\mathscr{K}}^{\mathrm{Colim}} \cong \mathrm{Id}_{\mathscr{K}}. \tag{12}$$

We have

$$\eta_{\mathscr{K}}^{\mathrm{Colim}} \vdash C_{\mathscr{K}}^{\mathrm{Colim}}.$$

If $\mathscr{K}$ is complete then $\operatorname{Colim}\mathscr{K}$ is complete, too, even closed under limits in $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ (see [F] and [Ro]).

DEFINITION 4.2. A category $\mathscr{K}$ is called a *complete pretopos* if it is complete and cocomplete and $C_{\mathscr{K}}^{\mathrm{Colim}}$ preserves limits.

LEMMA 4.3. *Every complete pretopos is algebraically exact.*

Proof. A complete pretopos $\mathscr{K}$ is, by definition, a complete localization of $\operatorname{Colim}\mathscr{K}$. Since complete localizations of algebraically exact categories are algebraically exact (see [ALR3]), it suffices to prove that $\operatorname{Colim}\mathscr{K}$ is algebraically exact. $\operatorname{Colim}\mathscr{K}$ is complete and $\operatorname{Sind}\operatorname{Colim}\mathscr{K}$ is closed under limits in $\operatorname{Colim}\operatorname{Colim}\mathscr{K}$ (see [ALR3]). Thus, $C_{\operatorname{Colim}\mathscr{K}}^{\mathrm{Sind}}$ is the domain restriction of

$$C_{\operatorname{Colim}\mathscr{K}}^{\mathrm{Colim}} : \operatorname{Colim}\operatorname{Colim}\mathscr{K} \to \operatorname{Colim}\mathscr{K},$$

and the latter functor preserves limits because Colim is a KZ-doctrine (see [K, Ma]).

Remark 4.4. (1) A free completion of a category $\mathscr{K}$ under coproducts is denoted by

$$\eta_{\mathscr{K}}^{\mathrm{Fam}} : \mathscr{K} \to \operatorname{Fam}\mathscr{K}.$$

It can be described as the codomain restriction of the Yoneda embedding into the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ consisting of all coproducts of representable functors.

(2) An exact completion of a finitely complete category $\mathscr{K}$ is denoted by

$$\eta_{\mathscr{K}}^{\mathrm{ex}} : \mathscr{K} \to \mathscr{K}_{\mathrm{ex}}$$

(see [C, CV]). It is defined by the following universal property: $\mathscr{K}_{\mathrm{ex}}$ is an exact category and $\eta_{\mathscr{K}}^{\mathrm{ex}}$ preserves finite limits, and for any finite-limits preserving functor $H : \mathscr{K} \to \mathscr{L}$ into an exact category $\mathscr{L}$ there is, up to an isomorphism, a unique exact extension $\hat{H} : \mathscr{K}_{\mathrm{ex}} \to \mathscr{L}$ satisfying $\hat{H} \cdot \eta_{\mathscr{K}}^{\mathrm{ex}} \cong H$. (Exactness of $\hat{H}$ means preservation of finite limits and regular epimorphisms.)

Following [CV], $\mathscr{K}_{\mathrm{ex}}$ can be described as the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ consisting of all functors $X : \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ such that there exists a regular epimorphism $e : Y(K) \to X$ whose kernel pair

$$Z \underset{v}{\overset{u}{\rightrightarrows}} Y(K) \xrightarrow{\ e\ } X$$

has the property that Z is a regular quotient of a representable functor. Thus, $\mathscr{K}_{\text{ex}}$ consists of coequalizers of pseudoequivalences of representable functors. And $\eta_{\mathscr{K}}^{\text{ex}}$ is the codomain restriction of the Yoneda embedding $Y : \mathscr{K} \to \mathbf{Set}^{\mathscr{K}^{\text{op}}}$.

Recall that $\mathscr{K}_{\text{ex}}$ is complete whenever $\mathscr{K}$ is complete (see [CV]).

LEMMA 4.5. *Let $\mathscr{K}$ be a complete category and $\mathscr{L}$ a complete exact category having product-stable regular epimorphisms. For every functor $H : \mathscr{K} \to \mathscr{L}$ preserving limits, the exact extension $\hat{H} : \mathscr{K}_{\text{ex}} \to \mathscr{L}$ preserves limits too.*

Proof. We abbreviate $\eta_{\mathscr{K}}^{\text{ex}}$ to η. We are to prove that $\hat{H}$ preserves products of objects $X_i \in \mathscr{K}_{\text{ex}}$, $i \in I$. We have representations

$$\eta(L_i) \xrightarrow{g_i} Z_i \underset{v_i}{\overset{u_i}{\rightrightarrows}} \eta(K_i) \xrightarrow{e_i} X_i$$

from Remark 4.4, with e_i and g_i regular epimorphisms. Further, form a product

$$\eta\left(\prod L_i\right) \cong \prod \eta(L_i) \xrightarrow{\prod g_i} \prod Z_i \underset{\prod v_i}{\overset{\prod u_i}{\rightrightarrows}} \prod \eta(K_i) \xrightarrow{\prod e_i} \prod X_i.$$

Since $\prod e_i$ and $\prod g_i$ are regular epimorphisms in $\mathbf{Set}^{\mathscr{K}^{\text{op}}}$ and $(\prod u_i, \prod v_i)$ is a kernel pair of $\prod e_i$, the diagram

$$\eta\left(\prod L_i\right) \underset{\prod v_i g_i}{\overset{\prod v_i g_i}{\rightrightarrows}} \eta\left(\prod K_i\right) \xrightarrow{\prod e_i} \prod X_i$$

is a coequalizer (of a pseudoequivalence) in $\mathscr{K}_{\text{ex}}$. Hence

$$H\eta\left(\prod L_i\right) \underset{H(\prod v_i g_i)}{\overset{H(\prod u_i g_i)}{\rightrightarrows}} H\eta\left(\prod K_i\right) \xrightarrow{\hat{H} \prod e_i} \hat{H} \prod X_i$$

is a coequalizer in $\mathscr{L}$. At the same time,

$$H\eta(L_i) \underset{H(v_i g_i)}{\overset{H(u_i g_i)}{\rightrightarrows}} H\eta(K_i) \xrightarrow{\hat{H} e_i} \hat{H} X_i$$

is a coequalizer in $\mathscr{L}$ for any $i \in I$. Since regular epimorphisms are product stable in $\mathscr{L}$,

$$H\eta\left(\prod L_i\right) \underset{H(\prod v_i g_i)}{\overset{H(\prod u_i g_i)}{\rightrightarrows}} H\eta\left(\prod K_i\right) \xrightarrow{\prod \hat{H} e_i} \prod \hat{H} X_i$$

is a coequalizer in $\mathscr{L}$ as well. Therefore $\hat{H}$ preserves products and thus all limits. ▮

4.6. Recall that a category with finite limits is called ∞-*extensive* iff it has universal and disjoint coproducts (see [CLW]). We say that *coproducts distribute over products* if we have a canonical isomorphism

$$\prod_{i\in I}\left(\coprod_{j\in J_i} K_{ij}\right) \cong \coprod_{f\in\prod J_i}\left(\prod_{i\in I} K_{if(i)}\right).$$

THEOREM. *Let $\mathscr{K}$ be a complete and cocomplete category. Then $\mathscr{K}$ is a complete pretopos iff $\mathscr{K}$ is exact and ∞-extensive, regular epimorphisms are product-stable, and products distribute over coproducts.*

Proof. Necessity is evident because $\mathscr{K}$ is a complete localization in $\operatorname{Colim}\mathscr{K}$ and $\operatorname{Colim}\mathscr{K}$ is closed under limits and colimits in $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$. Assume that $\mathscr{K}$ satisfies the conditions listed above. Following [C, (4.1)] and [Ro, Lemma 3], we have

$$\operatorname{Colim}\mathscr{K} \cong (\operatorname{Fam}\mathscr{K})_{\mathrm{ex}}.$$

Let

$$C_{\mathscr{K}}^{\mathrm{Fam}} : \operatorname{Fam}\mathscr{K} \to \mathscr{K}$$

be the essentially unique functor preserving coproducts with

$$C_{\mathscr{K}}^{\mathrm{Fam}} \cdot \eta_{\mathscr{K}}^{\mathrm{Fam}} \cong \mathrm{Id}_{\mathscr{K}}. \tag{13}$$

Since $\mathscr{K}$ is ∞-extensive, $C_{\mathscr{K}}^{\mathrm{Fam}}$ preserves finite limits (see [HT1]). We will prove that it preserves all limits. Let $\coprod_{j\in J_i}\eta K_{ij}$, $i \in I$, belong to $\operatorname{Fam}\mathscr{K}$. Abbreviate $\eta_{\mathscr{K}}^{\mathrm{Fam}}$ to η and $C_{\mathscr{K}}^{\mathrm{Fam}}$ to C. We have

$$\begin{aligned}
&C\left(\prod_{i\in I}\coprod_{j\in J_i}\eta(K_{ij})\right)\\
&\qquad\cong C\left(\coprod_{(j_i)\in\prod J_i}\prod_{i\in I}\eta(K_{ij_i})\right) && \text{by distributivity}\\
&\qquad\cong \coprod_{(j_i)\in\prod J_i} C\eta\left(\prod_{i\in I}K_{ij_i}\right) && \text{by preservations of coproducts by } C\\
&\qquad\cong \coprod_{(j_i)\in\prod J_i}\prod_{i\in I}K_{ij_i} && \text{by (13)}
\end{aligned}$$

$$\cong \prod_{i\in I} \coprod_{j\in J_i} K_{ij} \qquad \text{by distributivity}$$

$$\cong \prod_{i\in I} \coprod_{j\in J_i} C\eta(K_{ij}) \qquad \text{by (13)}$$

$$\cong \prod_{i\in I} C\Bigl(\coprod_{j\in J_i} \eta(K_{ij})\Bigr) \qquad \text{by preservation of coproducts by } C.$$

Hence C preserves products and therefore all limits.

Following Lemma 4.2, the induced functor

$$\hat{C} : (\operatorname{Fam} \mathscr{K})_{\text{ex}} \to \mathscr{K}$$

preserves limits.

We will prove that $\hat{C}$ is naturally isomorphic to the composite

$$(\operatorname{Fam} \mathscr{K})_{\text{ex}} \xrightarrow{C_{\text{ex}}} \mathscr{K}_{\text{ex}} \xrightarrow{C^{\text{ex}}_{\mathscr{K}}} \mathscr{K}$$

where C_{ex} is an exact functor essentially given by the commutativity of

$$\begin{array}{ccc} (\operatorname{Fam} \mathscr{K})_{\text{ex}} & \xrightarrow{C_{\text{ex}}} & \mathscr{K}_{\text{ex}} \\ {\scriptstyle \eta^{\text{ex}}_{\operatorname{Fam}\mathscr{K}}}\uparrow & & \uparrow{\scriptstyle \eta^{\text{ex}}_{\mathscr{K}}} \\ \operatorname{Fam} \mathscr{K} & \xrightarrow[C]{} & \mathscr{K} \end{array} \quad .$$

Since both $\hat{C}$ and $C^{\text{ex}}_{\mathscr{K}} \cdot C_{\text{ex}}$ are exact and

$$C^{\text{ex}}_{\mathscr{K}} \cdot C_{\text{ex}} \cdot \eta^{\text{ex}}_{\operatorname{Fam}\mathscr{K}} \cong C^{\text{ex}}_{\mathscr{K}} \cdot \eta^{\text{ex}}_{\mathscr{K}} \cdot C \cong C \cong \hat{C} \cdot \eta^{\text{ex}}_{\operatorname{Fam}\mathscr{K}}.$$

We conclude that $C^{\text{ex}}_{\mathscr{K}} \cdot C_{\text{ex}} \cong \hat{C}$.

Finally, since $\eta^{\text{Colim}}_{\mathscr{K}}$ is the composite

$$\mathscr{K} \xrightarrow{\eta} \operatorname{Fam} \mathscr{K} \xrightarrow{\eta^{\text{ex}}_{\operatorname{Fam}\mathscr{K}}} (\operatorname{Fam} \mathscr{K})_{\text{ex}}$$

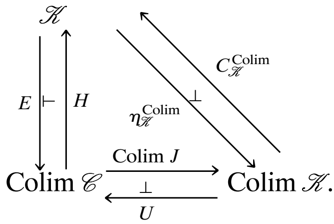
and

$$\begin{array}{ccc} \operatorname{Fam} \mathscr{K} & \xrightarrow{\eta^{\text{ex}}_{\operatorname{Fam}}} & (\operatorname{Fam} \mathscr{K})_{\text{ex}} \\ {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \eta_{\text{ex}}} \\ \mathscr{K} & \xrightarrow[\eta^{\text{ex}}_{\mathscr{K}}]{} & \mathscr{K}_{\text{ex}} \end{array}$$

commutes, a left adjoint $C^{\text{Colim}}_{\mathscr{K}}$ to $\eta^{\text{Colim}}_{\mathscr{K}}$ is equal to the composite $C^{\text{ex}}_{\mathscr{K}} \cdot C_{\text{ex}}$. In fact, $C_{\text{ex}} \dashv \eta_{\text{ex}}$ (because $C \dashv \eta$) and $C^{\text{ex}}_{\mathscr{K}} \dashv \eta^{\text{ex}}_{\mathscr{K}}$. Hence $\hat{C} \cong C^{\text{Colim}}_{\mathscr{K}}$. Therefore $C^{\text{Colim}}_{\mathscr{K}}$ preserves limits. ∎

COROLLARY 4.7. *Essential localizations of presheaf categories are precisely the complete pretoposes with a regular generator.*

Proof. Necessity is obvious. Let $\mathscr{K}$ be a complete pretopos with a regular generator. Following Lemma 4.3, $\mathscr{K}$ is precontinuous. Now, we can follow the proof of Theorem 2.7; we only have to replace the diagram there with the following one:



5. MORE ON ALGEBRAICALLY EXACT CATEGORIES

A free completion of a category $\mathscr{K}$ under coequalizers of reflexive pairs will be denoted by

$$\eta_{\mathscr{K}}^{\mathrm{Rec}} : \mathscr{K} \to \mathrm{Rec}\,\mathscr{K}.$$

$\mathrm{Rec}\,\mathscr{K}$ can be described as the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ of all functors $\mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ which belong to the iterated closure of representable functors under reflexive coequalizers, and $\eta_{\mathscr{K}}^{\mathrm{Rec}}$ is the codomain restriction of the Yoneda embedding $Y_{\mathscr{K}} : \mathscr{K} \to \mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$. If $\mathscr{K}$ has reflexive coequalizers we denote by

$$C_{\mathscr{K}}^{\mathrm{Rec}} : \mathrm{Rec}\,\mathscr{K} \to \mathscr{K}$$

a functor computing reflexive coequalizers in $\mathscr{K}$. This is the essentially unique functor preserving coequalizers and satisfying

$$C_{\mathscr{K}}^{\mathrm{Rec}} \cdot \eta_{\mathscr{K}}^{\mathrm{Rec}} \cong \mathrm{Id}_{\mathscr{K}}. \tag{14}$$

If $\mathscr{K}$ has finite coproducts, this completion has been introduced previously by Pitts (see [P] and [BC]). In this case, $\mathrm{Rec}\,\mathscr{K}$ consists of reflexive coequalizers of representable functors and

$$\mathrm{Sind}\,\mathscr{K} \cong \mathrm{Ind}\,\mathrm{Rec}\,\mathscr{K} \tag{15}$$

(see [AR2]). We will prove that this description of $\mathrm{Sind}\,\mathscr{K}$ also holds for any complete category $\mathscr{K}$.

For any full subcategory $\mathscr{X}$ of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$, let $E(\mathscr{X})$ denote the full subcategory of $\mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$ consisting of all coequalizers of pseudoequivalences in $\mathscr{X}$. In this notation,

$$\mathscr{K}_{\mathrm{ex}} = E\big(Y_{\mathscr{K}}(\mathscr{K})\big).$$

PROPOSITION 5.1. *Let $\mathscr{K}$ be a complete category. Then*

$$\operatorname{Sind}\mathscr{K} \cong \operatorname{Ind}\operatorname{Rec}\mathscr{K} \cong E(\operatorname{Ind}\mathscr{K}).$$

More precisely, there exist isomorphisms of categories forming commutative triangles such as

$$\begin{array}{ccccc} & & \mathscr{K} & & \\ & \eta_K^{\mathrm{Sind}} \swarrow & & \searrow \eta_{\mathscr{K}}^{\mathrm{Rec}} & \\ & & & & \operatorname{Rec}\mathscr{K} \\ & & & & \downarrow \eta_{\operatorname{Rec}\mathscr{K}}^{\mathrm{Ind}} \\ \operatorname{Sind} K & & \xrightarrow[\cong]{} & & \operatorname{Ind}\operatorname{Rec}\mathscr{K}, \end{array} \qquad \begin{array}{ccc} & \mathscr{K} & \\ \eta_{\mathscr{K}}^{\mathrm{Sind}} \swarrow & & \searrow Y'_{\mathscr{K}} \\ \operatorname{Sind}\mathscr{K} & \xrightarrow[\cong]{} & E(\operatorname{Ind}\mathscr{K}), \end{array}$$

where $Y'_{\mathscr{K}}$ is the codomain restriction of $Y_{\mathscr{K}}$.

Proof. We are going to prove the isomorphism for various types of categories $\mathscr{K}$ first.

I. Let $\mathscr{K}$ be a small category having finite limits and finite coproducts. Then, following [AR2],

$$\operatorname{Sind}\mathscr{K} \cong \operatorname{Ind}\operatorname{Rec}\mathscr{K}.$$

Moreover, $\operatorname{Sind}\mathscr{K}$ is a variety. Consequently,

$$\operatorname{Sind}\mathscr{K} = \mathscr{F}_{\mathrm{ex}},$$

where $\mathscr{F}$ is the full subcategory of $\operatorname{Sind}\mathscr{K}$ consisting of free algebras (see [V1]). Since any free algebra is a filtered colimit of finitely generated free algebras, and the latter belong to $\mathscr{K}$ (more precisely, to $\eta_{\mathscr{K}}^{\mathrm{Sind}}(\mathscr{K})$), we have $\mathscr{F} \subseteq \operatorname{Ind}\mathscr{K}$. Hence

$$\operatorname{Sind}\mathscr{K} = E(\operatorname{Ind}\mathscr{K}).$$

II. If $\mathscr{K}$ is a large category having finite limits and finite coproducts, then $\mathscr{K}$ can be expressed as directed union $\mathscr{K} = \bigcup_{i \in I}\mathscr{K}_i$ of small full subcategories $\mathscr{K}_i \subseteq \mathscr{K}$, $i \in I$, closed in $\mathscr{K}$ under finite limits and finite coproducts. The inclusions $U_{ij}: \mathscr{K}_i \to \mathscr{K}_j$ induce finite-limit-preserving functors $\operatorname{Ind} U_{ij}: \operatorname{Ind}\mathscr{K}_i \to \operatorname{Ind}\mathscr{K}_j$ (see [AGV, 8.9.8]). Hence $\operatorname{Ind} U_{ij}$ preserves pseudoequivalences and therefore induces the functor

$$E(\operatorname{Ind} U_{ij}): E(\operatorname{Ind}\mathscr{K}_i) \to E(\operatorname{Ind}\mathscr{K}_j).$$

Thus

$$E(\operatorname{Ind}\mathscr{K}) \cong \operatorname*{colim}_{i \in I} E(\operatorname{Ind}\mathscr{K}_i) \cong \operatorname*{colim}_{i \in I} \operatorname{Sind}\mathscr{K}_i \cong \operatorname{Sind}\mathscr{K}.$$

III. Let $\mathscr{K} = \operatorname{Lim}\mathscr{A}$ be a free limit completion of a category $\mathscr{A}$ with finite limits. If $\mathscr{A}$ is small then $\mathscr{K} = (\mathbf{Set}^{\mathscr{A}})^{\mathrm{op}}$ and the result follows from II. If $\mathscr{A}$ is large, we again express $\mathscr{A}$ as a directed union $\mathscr{A} = \bigcup_{i \in I}\mathscr{A}_i$ of small full subcategories closed under finite limits and get the result for $\mathscr{A}$.

IV. Let $\mathscr{K}$ be an arbitrary complete category. The inclusion $\operatorname{Rec}\mathscr{K} \subseteq \operatorname{Sind}\mathscr{K}$ induces a filtered colimits preserving functor

$$J_{\mathscr{K}} : \operatorname{Ind}\operatorname{Rec}\mathscr{K} \to \operatorname{Sind}\mathscr{K}.$$

Denote by $\eta_{\mathscr{K}}^{\mathrm{Lim}} : \mathscr{K} \to \operatorname{Lim}\mathscr{K}$ a free completion of $\mathscr{K}$ under limits and by

$$L_{\mathscr{K}} : \operatorname{Lim}\mathscr{K} \to \mathscr{K}$$

the essentially unique limit-preserving functor with

$$L_{\mathscr{K}} \cdot \eta_{\mathscr{K}}^{\mathrm{Lim}} \cong \mathrm{Id}_{\mathscr{K}}.$$

Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Ind}\operatorname{Rec}\mathscr{K} & \xrightarrow{\quad J_{\mathscr{K}} \quad} & \operatorname{Sind}\mathscr{K} \\ {\scriptstyle \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}}}\big\uparrow\big\downarrow{\scriptstyle \operatorname{Ind}\operatorname{Rec}\eta_{\mathscr{K}}^{\mathrm{Lim}}} & & {\scriptstyle \operatorname{Sind} L_{\mathscr{K}}}\big\uparrow\big\downarrow{\scriptstyle \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}}} \\ \operatorname{Ind}\operatorname{Rec}\operatorname{Lim}\mathscr{K} & \xrightarrow[\quad J_{\operatorname{Lim}\mathscr{K}} \quad]{} & \operatorname{Sind}\operatorname{Lim}\mathscr{K}. \end{array}$$

Following Part III, $J_{\operatorname{Lim}\mathscr{K}}$ is an isomorphism. Consider

$$M = \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}}^{-1} \cdot \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}} : \operatorname{Sind}\mathscr{K} \to \operatorname{Ind}\operatorname{Rec}\mathscr{K}.$$

Then

$$\begin{aligned} J_{\mathscr{K}} \cdot M &= J_{\mathscr{K}} \cdot \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}}^{-1} \cdot \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}} \\ &= \operatorname{Sind} L_{\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}}^{-1} \cdot \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}} \\ &= \operatorname{Sind} L_{\mathscr{K}} \cdot \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}} \\ &= \mathrm{Id} \end{aligned}$$

and

$$\begin{aligned} M \cdot J_{\mathscr{K}} &= \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}}^{-1} \cdot \operatorname{Sind}\eta_{\mathscr{K}}^{\mathrm{Lim}} \cdot J_{\mathscr{K}} \\ &= \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}} \cdot J_{\operatorname{Lim}\mathscr{K}}^{-1} \cdot J_{\operatorname{Lim}\mathscr{K}} \cdot \operatorname{Ind}\operatorname{Rec}\eta_{\mathscr{K}} \\ &= \operatorname{Ind}\operatorname{Rec} L_{\mathscr{K}} \cdot \operatorname{Ind}\operatorname{Rec}\eta_{\mathscr{K}} \\ &= \mathrm{Id}. \end{aligned}$$

Hence $\operatorname{Ind}\operatorname{Rec}\mathscr{K} \cong \operatorname{Sind}\mathscr{K}$.

Since $E(\operatorname{Ind}\mathscr{K}) \subseteq \operatorname{Sind}\mathscr{K}$ and the previous argument shows that this inclusion is onto on objects, we also have

$$E(\operatorname{Ind}\mathscr{K}) \cong \operatorname{Sind}\mathscr{K}.$$

∎

COROLLARY 5.2. *Let $\mathscr{K}$ be an exact precontinuous category having sifted colimits and product-stable regular epimorphisms. Then the functor $C_{\mathscr{K}}^{\mathrm{Sind}} : \mathrm{Sind}\,\mathscr{K} \to \mathscr{K}$ preserves products.*

Proof. Consider the functor

$$U : \mathbf{Set}^{(\mathrm{Ind}\,\mathscr{K})^{\mathrm{op}}} \to \mathbf{Set}^{\mathscr{K}^{\mathrm{op}}}$$

given by precomposing with

$$\left(\eta_{\mathscr{K}}^{\mathrm{Ind}}\right)^{\mathrm{op}} : \mathscr{K}^{\mathrm{op}} \to (\mathrm{Ind}\,\mathscr{K})^{\mathrm{op}}.$$

A domain–codomain restriction of U yields the functor

$$U^* : (\mathrm{Ind}\,\mathscr{K})_{\mathrm{ex}} \to E(\mathrm{Ind}\,\mathscr{K}) \cong \mathrm{Sind}\,\mathscr{K}.$$

Following Lemma 4.3, the functor

$$\widehat{C_{\mathscr{K}}^{\mathrm{Ind}}} : (\mathrm{Ind}\,\mathscr{K})_{\mathrm{ex}} \to \mathscr{K}$$

preserves limits. Since U^* is onto on objects and

$$U^* \cdot C_{\mathscr{K}}^{\mathrm{Sind}} \cong \widehat{C_{\mathscr{K}}^{\mathrm{Ind}}},$$

$C_{\mathscr{K}}^{\mathrm{Sind}}$ preserves products. ∎

Remark 5.3. The above result shows that Problem 3.1 lies in the preservation of equalizers.

PROPOSITION 5.4. *Let $\mathscr{K}$ be a complete category with reflexive coequalizers. Then* $\mathrm{Ind}\,\mathscr{K}$ *is algebraically exact iff the functor* $\mathrm{Ind}\,C_{\mathscr{K}}^{\mathrm{Rec}}$ *preserves limits.*

Proof. *Sufficiency*. Since $\mathscr{K}$ has reflexive coequalizers, we have

$$\eta_{\mathscr{K}}^{\mathrm{Rec}} \vdash C_{\mathscr{K}}^{\mathrm{Rec}}$$

and therefore

$$\mathrm{Ind}\,\mathscr{K} \underset{\mathrm{Ind}\,\eta_{\mathscr{K}}^{\mathrm{Rec}}}{\overset{\mathrm{Ind}\,C_{\mathscr{K}}^{\mathrm{Rec}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Ind}\,\mathrm{Rec}\,\mathscr{K}.$$

Following Proposition 5.1, $\mathrm{Ind}\,\mathscr{K}$ is a complete localization of an algebraically exact category $\mathrm{Sind}\,\mathscr{K}$. Therefore $\mathrm{Ind}\,\mathscr{K}$ is algebraically exact.

Conversely, assume that $\mathrm{Ind}\,\mathscr{K}$ is algebraically exact. It suffices to prove that

$$\mathrm{Ind}\,C_{\mathscr{K}}^{\mathrm{Rec}} \cong C_{\mathrm{Ind}\,\mathscr{K}}^{\mathrm{Sind}} \cdot \mathrm{Sind}\,\eta_{\mathscr{K}}^{\mathrm{Ind}} : \mathrm{Sind}\,\mathscr{K} \to \mathrm{Ind}\,\mathscr{K} \tag{16}$$

since the right-hand functors preserve limits. Since both $\operatorname{Ind} C^{\mathrm{Rec}}_{\mathscr{K}}$ and $C^{\mathrm{Sind}}_{\operatorname{Ind}\mathscr{K}}$. $\operatorname{Sind} \eta^{\mathrm{Ind}}_{\mathscr{K}}$ preserve sifted colimits, we only need to show that they have the same precomposition with $\eta^{\mathrm{Sind}}_{\mathscr{K}}$. In fact,

$$\begin{aligned}
\operatorname{Ind} C^{\mathrm{Rec}}_{\mathscr{K}} \cdot \eta^{\mathrm{Sind}}_{\mathscr{K}} &\cong \operatorname{Ind} C^{\mathrm{Rec}}_{\mathscr{K}} \cdot \eta^{\mathrm{Ind}}_{\mathrm{Rec}\,\mathscr{K}} \cdot \eta^{\mathrm{Rec}}_{\mathscr{K}} && \text{by 5.1}\\
&\cong \eta^{\mathrm{Ind}}_{\mathscr{K}} \cdot C^{\mathrm{Rec}}_{\mathscr{K}} \cdot \eta^{\mathrm{Rec}}_{\mathscr{K}} && \text{by naturality}\\
&\cong \eta^{\mathrm{Ind}}_{\mathscr{K}} && \text{by (14)}\\
&\cong C^{\mathrm{Sind}}_{\operatorname{Ind}\mathscr{K}} \cdot \eta^{\mathrm{Sind}}_{\operatorname{Ind}\mathscr{K}} \cdot \eta^{\mathrm{Ind}}_{\mathscr{K}} && \text{by naturality and (10).}
\end{aligned}$$

∎

COROLLARY 5.5. *Let $\mathscr{K}$ be a precontinuous category with reflexive coequalizers and such that $\operatorname{Ind} C^{\mathrm{Rec}}_{\mathscr{K}}$ preserves limits. Then $\mathscr{K}$ is algebraically exact.*

Proof. We have

$$\mathscr{K} \underset{\eta^{\mathrm{Ind}}_{\mathscr{K}}}{\overset{C^{\mathrm{Ind}}_{\mathscr{K}}}{\underset{\perp}{\rightleftarrows}}} \operatorname{Ind}\mathscr{K} \underset{\operatorname{Ind}\eta^{\mathrm{Rec}}_{\mathscr{K}}}{\overset{\operatorname{Ind} C^{\mathrm{Rec}}_{\mathscr{K}}}{\underset{\perp}{\rightleftarrows}}} \operatorname{Ind}\operatorname{Rec}\mathscr{K} \cong \operatorname{Sind}\mathscr{K}.$$

Since by Proposition 5.1

$$\eta^{\mathrm{Sind}}_{\mathscr{K}} \cong \eta^{\mathrm{Ind}}_{\mathrm{Rec}\,\mathscr{K}} \cdot \eta^{\mathrm{Rec}}_{\mathscr{K}} \cong \operatorname{Ind}\eta^{\mathrm{Rec}}_{\mathscr{K}} \cdot \eta^{\mathrm{Ind}}_{\mathscr{K}}$$

and $\eta^{\mathrm{Sind}}_{\mathscr{K}} \vdash C^{\mathrm{Sind}}_{\mathscr{K}}$, we conclude

$$C^{\mathrm{Sind}}_{\mathscr{K}} \cong C^{\mathrm{Ind}}_{\mathscr{K}} \cdot \operatorname{Ind}\operatorname{Rec}\mathscr{K}.$$

Therefore $C^{\mathrm{Sind}}_{\mathscr{K}}$ preserves limits. ∎

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