Morita equivalence for regular algebras

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Résumé: Nous étudions les catégories des modules réguliers sur les algèbres régulières, afin de généraliser certains résultats classiques de la théorie de Morita, faits dans le cas des algèbres unitaires, au cas des algèbres régulières.

Introduction: Classical Morita theory studies equivalences between categories of unital modules over unital R-algebras, for R a commutative unital ring. The key result is the Eilenberg-Watts theorem, which states that colimit preserving R-functors between module categories correspond to bimodules. Morita theory is also the base to define the Brauer group of the ring R, which is the group of Morita equivalence classes of Azumaya R-algebras.

The aim of this note is to study categories of *regular* modules over not necessarily unital R-algebras, where a module M over an R-algebra A is regular if the canonical morphism

$$A \otimes_A M \longrightarrow M$$

induced by the action of A over M, is an isomorphism (cf. [16]).

The first, simple but crucial fact is that, if A itself is regular as A-module, the category of regular A-modules is a colocalization of the category of all A-modules. This allows us to prove the analogous of the Eilenberg-Watts theorem and then to have a satisfactory Morita theory for regular R-algebras. Our second result is that the (classifying category of the bi-)category of regular R-algebras and regular bimodules is compact closed. This gives us a quick construction of a group in which the Brauer group of R embeds.

1 The Eilenberg-Watts theorem

We fix once for all a commutative unital ring R. Everything should be intended as enriched over the category of unital R-modules. In particular, the unlabelled tensor product $\otimes$ is the tensor product over R, and algebra means R-algebra. Modules and algebras are always associative, but not necessarily unital.

Definition 1.1 *Let A be an algebra and M a left A-module. We say that M is regular if the arrow*

$$A \otimes_A M \longrightarrow M$$

induced by the action $A \otimes M \longrightarrow M$, is an isomorphism.

We write A-mod for the category of left A-modules and A-mod^{reg} for its full subcategory of regular modules. In the same way one defines the category mod^{reg}-A of regular right A-modules and the category A-mod^{reg}-B of bimodules which are regular both as left A-modules and as right B-modules. An algebra A is regular if it is regular as left (equivalently, right) A-module.

Examples:

1) Clearly, if A is unital, then A is regular (a one-side unit is enough); moreover, in this case a module is regular iff it is unital (cf. proposition 3.2 in [17]).

2) Other examples of regular algebras are:

- rings with local units (cf. [1], [2], [3]) and, between them, rings of infinite matrices with a finite number of non-zero entries;

- left or right splitting algebras (cf. [17]);

- separable algebras and, in particular, Azumaya algebra without unit (cf. [9], [10], [11], [14], [16]).

3) In [16] and [10], regular bimodules over regular algebras are used to define (strict) Morita contexts and then to give an algebraic description of the second tale cohomology group of R.

4) A general argument on coequalizers shows that the dual of a regular algebra is regular; the fact that the tensor product of two regular algebras is regular will be proved in section 2.

Recall that each A-B-bimodule M induces a pair of adjoint functors

$$M \otimes_B - \ , \ \mathrm{Lin}_A(M, -) : A\text{-mod} \rightleftarrows B\text{-mod}$$

with $M \otimes_B -$ left adjoint to $\mathrm{Lin}_A(M, -)$. If M is regular as A-module, then the functor $M \otimes_B -$ factors through A-mod^{reg} and, since A-mod^{reg} is full in A-mod, we obtain an adjunction

$$M \otimes_B - \ , \ \mathrm{Lin}_A(M, -) : A\text{-mod}^{reg} \rightleftarrows B\text{-mod} \ .$$

In particular, if A is a regular algebra, we have an adjunction

$$A \otimes_A - \ , \ \mathrm{Lin}_A(A, -) : A\text{-mod}^{reg} \rightleftarrows A\text{-mod} \ .$$

Proposition 1.2 *Let A be a regular algebra; the functor*

$$A \otimes_A - : A\text{-mod} \longrightarrow A\text{-mod}^{reg}$$

is right adjoint to the full inclusion

$$i : A\text{-}mod^{reg} \longrightarrow A\text{-}mod\ .$$

Proof: let X be in A-mod^{reg} and Y in A-mod. Given an arrow $X \longrightarrow A \otimes_A Y$, we obtain an arrow $X \longrightarrow A \otimes_A Y \longrightarrow Y$, where the second component is the arrow induced by the action of A on Y. Conversely, given an arrow $g: X \longrightarrow Y$, we obtain an arrow

$$X \longrightarrow A \otimes_A X \xrightarrow{1 \otimes_A g} A \otimes_A Y \quad ,$$

where the first component is the inverse of the arrow induced by the action of A on X. Precomposing with the isomorphism $A \otimes_A X \longrightarrow X$, one checks that these constructions are a bijection of hom-sets. The naturality is obvious. ■

From the previous proposition, it follows that A-mod^{reg} is a *colocalization* of A-mod (that is, the full inclusion has a right exact right adjoint). Since A-mod is a complete and cocomplete abelian category, standard arguments on localizations (cf. [7] vol.1, ch.3; vol.2, ch.1) give us the following:

Corollary 1.3 *Let A be a regular algebra; the category A-mod^{reg} is complete, cocomplete and abelian.*

This corollary allows us to have the "free " presentation of a regular module, which is used to prove the Eilenberg-Watts theorem.

Lemma 1.4 *Let A be a regular algebra and let X be in A-mod^{reg}; then X is the coequalizer in A-mod^{reg} of a pair of arrows between copowers of A.*

Proof: consider the copower of A indexed by the elements of X and the canonical A-linear arrow

$$\varphi: \coprod_X A \longrightarrow X\ .$$

Since X is regular, the action $A \otimes X \longrightarrow X$ is surjective, and then also φ is surjective; since A-mod^{reg} is abelian, φ is the coequalizer of its kernel pair

$$N \underset{\varphi_1}{\overset{\varphi_0}{\rightrightarrows}} \coprod_X A\ .$$

Now, repeat the argument starting from N

$$\psi: \coprod_N A \longrightarrow N\ .$$

We obtain the following diagram, which is a coequalizer in A-mod^{reg}

$$\coprod_N A \underset{\varphi_1 \cdot \psi}{\overset{\varphi_0 \cdot \psi}{\rightrightarrows}} \coprod_X A \xrightarrow{\varphi} X\ .$$

■

Consider now a functor $F\colon\ B\text{-mod} \longrightarrow A\text{-mod}$, with A and B two arbitrary algebras. The A-module FB can be provided with a structure of right B-module (compatible with its structure of left A-module) taking as action

$$\mu_F \colon FB \otimes B \longrightarrow FB$$

the arrow corresponding, by adjunction, to the composite

$$B \longrightarrow \mathrm{Lin}_B(B,B) \longrightarrow \mathrm{Lin}_A(FB,FB) \longrightarrow \mathrm{Lin}_R(FB,FB)$$

where the first component is induced by the multiplication of B, the second is the action of F and the third is the inclusion.

Definition 1.5 *A functor $F\colon\ B$-mod $\longrightarrow A$-mod is regular if the right B-module (FB, μ_F) is regular.*

Now we are ready to state the Eilenberg-Watts theorem for regular algebras.

Proposition 1.6 *Let A and B be two regular algebras and*

$$F\colon\ B\text{-mod}^{reg} \longrightarrow A\text{-mod}^{reg}$$

a functor. The assignments

$$F \mapsto (FB, \mu_F) \quad \text{and} \quad M \mapsto M \otimes_B -$$

give rise to an adjoint equivalence between the category of regular and colimit preserving functors

$$\vec{Funct}^{reg}(B\text{-mod}^{reg}, A\text{-mod}^{reg})$$

and the category of regular bimodules

$$A\text{-mod}^{reg}\text{-}B$$

Proof: given M in A-mod^{reg}-B, the functor

$$M \otimes_B -\colon\ B\text{-mod}^{reg} \longrightarrow A\text{-mod}^{reg}$$

preserves colimits because it factors as

$$(M \otimes_B -) \cdot i\colon\ B\text{-mod}^{reg} \longrightarrow B\text{-mod} \longrightarrow A\text{-mod}^{reg}$$

and then it is left adjoint to the functor

$$(B \otimes_B -) \cdot \mathrm{Lin}_A(M, -)\colon\ A\text{-mod}^{reg} \longrightarrow B\text{-mod} \longrightarrow B\text{-mod}^{reg}\ .$$

Via lemma 1.4, the rest of the proof runs as in the classical case of unital algebras (cf. [5] or [15]). ∎

The previous proposition is a proper generalization of the Eilenberg-Watts theorem for unital algebras. In fact, the condition of regularity on F always holds if B is unital. (More in general, it holds if F preserves coproducts and if there exists a left B-linear arrow $\varphi\colon B \longrightarrow \coprod_B B$ such that

$$B \xrightarrow{\varphi} \coprod_B B \xrightarrow{\theta} B \otimes B$$

is a section for the multiplication, where θ is the left B-linear arrow induced by tensoring with a fixed element of B.)

2 The bicategory of regular algebras

In this section we generalize to regular algebras some classical consequences of the Eilenberg-Watts theorem. First of all, let us restate it in the more meaningful language of bicategories (cf. [6]). We write Alg^{reg} for the bicategory whose objects are regular algebras, whose 1-arrows are regular bimodules and whose 2-arrows are morphisms of bimodules. The composition of two bimodules $M\colon A \mapsto B$ and $N\colon B \mapsto C$ is their tensor product $M \otimes_B N\colon A \mapsto C$. The identity on an object A is A itself seen as A-A-bimodule. Proposition 1.6 can be now expressed in the following way:

Proposition 2.1 *The assignments of proposition 1.6 give rise to a biequivalence between the 2-category of regular algebras and regular and colimit-preserving functors, and the bicategory Alg^{reg}. This biequivalence is the identity on the objects.*

Since any biequivalence preserves and reflects invertible 1-arrows, we have the following:

Corollary 2.2 *The biequivalence of proposition 2.1 restricts to a biequivalence between regular equivalences and invertible regular bimodules.*

In other words, two regular algebras are Morita equivalent (= there exists a regular equivalence $B\text{-mod}^{reg} \longrightarrow A\text{-mod}^{reg}$) iff they are equivalent in the bicategory Alg^{reg} (= there exist two regular bimodules $M\colon A \mapsto B$, $N\colon B \mapsto A$ and two isomorphisms of bimodules $M \otimes_B N \simeq A$, $N \otimes_A M \simeq B$).

As in the classical case, several Morita invariants can be now easily deduced (cf. [5] and [15]).

Proposition 2.3 *Consider two Morita equivalent regular algebras A and B. Let F: $B\text{-mod}^{reg} \longrightarrow A\text{-mod}^{reg}$ be the given regular equivalence, $M = FB$ the corresponding invertible regular A-B-bimodule, and N the inverse of M.*

1) there is a regular equivalence

$$\text{mod}^{reg}\text{-}B \longrightarrow \text{mod}^{reg}\text{-}A \;;$$

2) there is a strict monoidal equivalence

$$T\colon\ B\text{-mod}^{reg}\text{-}B \longrightarrow A\text{-mod}^{reg}\text{-}A\ ;$$

3) the regular Picard groups of A and B are isomorphic;

4) the centers of A and B are isomorphic;

5) the lattice of left regular ideals of B is isomorphic to the lattice of regular A-submodules of M;

6) the lattices of two-sided regular ideals of A and B are isomorphic.

Proof: 1) and 2): the invertible 1-arrows $M\colon A \mapsto B$ and $N\colon B \mapsto A$ induce equivalences between hom-categories:

$$- \otimes_A M\colon\ \mathrm{Alg}^{reg}(R,A) \longrightarrow \mathrm{Alg}^{reg}(R,B)$$

$$T = M \otimes_B - \otimes_B N\colon\ \mathrm{Alg}^{reg}(B,B) \longrightarrow \mathrm{Alg}^{reg}(A,A)\ .$$

3): the regular Picard group $\mathrm{Pic}^{reg}(A)$ of A is the group of isomorphism classes of regular autoequivalences of A-mod^{reg}. The equivalence

$$T\colon\ \mathrm{Alg}^{reg}(B,B) \longrightarrow \mathrm{Alg}^{reg}(A,A)$$

restricts to an equivalence between invertible regular B-B-bimodules and invertible regular A-A-bimodules. We obtain then the isomorphism $\mathrm{Pic}^{reg}(B) \simeq \mathrm{Pic}^{reg}(A)$ localizing to B and A the biequivalence of Corollary 2.2.
4): consider once again the strict monoidal equivalence

$$T\colon\ \mathrm{Alg}^{reg}(B,B) \longrightarrow \mathrm{Alg}^{reg}(A,A)\ ;$$

in particular, we have an isomorphism

$$T_{B,B}\colon\ B\text{-mod-}B(B,B) \longrightarrow A\text{-mod-}A(TB,TB)\ ,$$

that is an isomorphism between the centers of B and A, because TB is isomorphic to A.
5) and 6): any equivalence induces an isomorphism between the lattice of subobjects of an object and of its image. Considering the regular equivalence

$$F\colon\ B\text{-mod}^{reg} \longrightarrow A\text{-mod}^{reg}$$

and the object B in B-mod^{reg}, we have point 5) of the statement. Considering the monoidal equivalence

$$T\colon\ B\text{-mod}^{reg}\text{-}B \longrightarrow A\text{-mod}^{reg}\text{-}A$$

and the object B in B-mod^{reg}-B, we have point 6) of the statement. ■

The next two propositions, proved in [11], contain the main facts to endow Alg^{reg} of a compact closed structure. They are quite obvious if the algebras have units, but they need some more attention for regular algebras.

Proposition 2.4 *Let A and B be two algebras and consider M in mod^{reg}-A, M' in A-mod^{reg}, N in mod^{reg}-B and N' in B-mod^{reg}. The obvious isomorphism*

$$M \otimes M' \otimes N \otimes N' \simeq M \otimes N \otimes M' \otimes N'$$

induces an isomorphism

$$(M \otimes_A M') \otimes (N \otimes_B N') \simeq (M \otimes N) \otimes_{A\otimes B} (M' \otimes N') .$$

Proof: recall that $M \otimes_A M'$ is given by the following quotient

$$M \otimes A \otimes M' \xrightarrow{\lambda_A} M \otimes M' \xrightarrow{p_A} M \otimes_A M' = \frac{M \otimes M'}{\mathrm{Im}\lambda_A}$$

where $\lambda_A(m \otimes a \otimes m') = ma \otimes m' - m \otimes am'$; analogously, we have

$$N \otimes B \otimes N' \xrightarrow{\lambda_B} N \otimes N' \xrightarrow{p_B} N \otimes_B N' = \frac{N \otimes N'}{\mathrm{Im}\lambda_B}$$

$$M \otimes N \otimes A \otimes B \otimes M' \otimes N' \xrightarrow{\lambda_{A\otimes B}} M \otimes N \otimes M' \otimes N' \xrightarrow{p_{A\otimes B}} \dots$$
$$\dots (M \otimes N) \otimes_{A\otimes B} (M' \otimes N') = \frac{M \otimes N \otimes M' \otimes N'}{\mathrm{Im}\lambda_{A\otimes B}} .$$

We need a pair of arrows α, β making commutative the following diagram, where $\tau\colon M' \otimes N \longrightarrow N \otimes M'$ is the twist

$$\begin{tikzcd}
M \otimes M' \otimes N \otimes N' \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "p_A \otimes p_B"'] & M \otimes N \otimes M' \otimes N' \arrow[d, "p_{A\otimes B}"] \\
\frac{M\otimes M'}{\mathrm{Im}\lambda_A} \otimes \frac{N\otimes N'}{\mathrm{Im}\lambda_B} \arrow[r, shift right, "\alpha"'] & \frac{M\otimes N\otimes M'\otimes N'}{\mathrm{Im}\lambda_{A\otimes B}} \arrow[l, shift right, "\beta"']
\end{tikzcd}$$

For this, consider the inclusions

$$i\colon \mathrm{Im}\lambda_A \hookrightarrow M \otimes M' \quad j\colon \mathrm{Im}\lambda_B \hookrightarrow N \otimes N'$$

and take the images Λ_A and Λ_B of

$$i \otimes 1 \otimes 1\colon \mathrm{Im}\lambda_A \otimes N \otimes N' \longrightarrow M \otimes M' \otimes N \otimes N'$$

$$1 \otimes 1 \otimes j \colon M \otimes M' \otimes \mathrm{Im}\lambda_B \longrightarrow M \otimes M' \otimes N \otimes N' \ .$$

Recall that

$$\pi \colon \frac{M \otimes M'}{\mathrm{Im}\lambda_A} \otimes \frac{N \otimes N'}{\mathrm{Im}\lambda_B} \longrightarrow \frac{M \otimes M' \otimes N \otimes N'}{\Lambda_A + \Lambda_B}$$
$$[m \otimes m'] \otimes [n \otimes n'] \rightsquigarrow [m \otimes m' \otimes n \otimes n']$$

(square brackets are equivalence classes) is an isomorphism (cf. [8], chap.2, 3, n.6). Now, instead of α and β, we can look for two arrows α', β' making commutative the following diagram

$$\begin{tikzcd}
M \otimes M' \otimes N \otimes N' \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "\pi \cdot (p_A \otimes p_B)"'] & M \otimes N \otimes M' \otimes N' \arrow[d, "p_{A \otimes B}"] \\
\frac{M \otimes M' \otimes N \otimes N'}{\Lambda_A + \Lambda_B} \arrow[r, shift right, "\alpha'"'] & \frac{M \otimes N \otimes M' \otimes N'}{\mathrm{Im}\lambda_{A \otimes B}} \arrow[l, shift right, "\beta'"']
\end{tikzcd}$$

To define α', we have to show that, for each element p in $\Lambda_A + \Lambda_B$, the element $(1 \otimes \tau \otimes 1)(p)$ is in $\mathrm{Im}\lambda_{A \otimes B}$. Suppose first p of the form $p = (ma \otimes m' - m \otimes am') \otimes n \otimes n'$ (that is, p is in Λ_A). Since M and N are regular, m and n can be written as

$$m = \sum_i m_i a_i \qquad n = \sum_j n_j b_j \ ,$$

so that

$$\begin{aligned}
(1 \otimes \tau \otimes 1)(p) &= ma \otimes n \otimes m' \otimes n' - m \otimes n \otimes am' \otimes n' \\
&= \sum_{i,j} (m_i a_i a \otimes n_j b_j \otimes m' \otimes n' - m_i a_i \otimes n_j b_j \otimes am' \otimes n')
\end{aligned}$$

Finally, one checks that this is an element of $\mathrm{Im}\lambda_{A \otimes B}$ adding and substracting the term

$$\sum_{i,j} m_i \otimes n_j \otimes a_i am' \otimes b_j n' \ .$$

If p is in Λ_B, one works in the same way using the regularity of M' and N'. Conversely, given an element $y = ma \otimes nb \otimes m' \otimes n' - m \otimes n \otimes am' \otimes bn'$ in $\mathrm{Im}\lambda_{A \otimes B}$, we have

$$\begin{aligned}
(1 \otimes \tau^{-1} \otimes 1)(y) &= ma \otimes m' \otimes nb \otimes n' - m \otimes am' \otimes n \otimes bn' \\
&= (ma \otimes m' - m \otimes am') \otimes nb \otimes n' + \\
&\quad + m \otimes am' \otimes (nb \otimes n' - n \otimes bn')
\end{aligned}$$

which is in $\Lambda_A + \Lambda_B$. This allows us to define β' and the proof is complete. ■

Corollary 2.5 *Let A and B be two regular algebras and consider M in mod^{reg}-A and N in mod^{reg}-B. Then $A \otimes B$ is a regular algebra and $M \otimes N$ is in mod^{reg}-$A \otimes B$.*

Proposition 2.6 *Let A and B be two regular algebras. The categories A-mod^{reg}-B and $A \otimes B^{op}$-mod^{reg} are isomorphic.*

Proof: we sketch the proof, more details can be found in [11]. Let M be in A-mod^{reg}-B; we define an action

$$A \otimes B^{op} \otimes M \longrightarrow M \qquad (a \otimes b) \cdot m = amb$$

and we have to show that M is regular with respect to this action. This essentially amount to show that

$$\alpha\colon (A \otimes B^{op}) \otimes_{A \otimes B^{op}} M \longrightarrow A \otimes_A M \otimes_B B$$

$$a \otimes b \otimes m \rightsquigarrow a \otimes m \otimes b$$

is an isomorphism. The crucial point is to prove that its inverse

$$\beta\colon A \otimes_A M \otimes_B B \longrightarrow (A \otimes B^{op}) \otimes_{A \otimes B^{op}} M$$

$$a \otimes m \otimes b \rightsquigarrow a \otimes b \otimes m$$

is well defined. This means

$$\beta(a \otimes a'm \otimes b) = \beta(aa' \otimes m \otimes b) \ \ , \ \ \beta(a \otimes mb' \otimes b) = \beta(a \otimes m \otimes b'b) \ .$$

We check the first condition, the second is similar: since A and B are regular, we can write

$$a = \sum_i a_i a'_i \qquad b = \sum_j b_j b'_j$$

so that

$$\begin{aligned}
\beta(a \otimes a'm \otimes b) &= (a \otimes b) \otimes a'm \\
&= \sum_{i,j} (a_i a'_i \otimes b_j b'_j) \otimes a'm \\
&= \sum_{i,j} (a_i \otimes b'_j) \cdot (a'_i \otimes b_j) \otimes a'm \\
&= \sum_{i,j} (a_i \otimes b'_j) \otimes a'_i a' m b_j \\
&= \sum_{i,j} (a_i \otimes b'_j) \cdot (a'_i a' \otimes b_j) \otimes m \\
&= \sum_{i,j} (a_i a'_i a' \otimes b_j b'_j) \otimes m \\
&= (aa' \otimes b) \otimes m \\
&= \beta(aa' \otimes m \otimes b) \ .
\end{aligned}$$

Clearly, a morphism $f\colon M \longrightarrow M'$ in A-mod^{reg}-B is also $A \otimes B^{op}$-linear with respect to the action of $A \otimes B^{op}$ on M and M' just defined.

Conversely, consider N in $A \otimes B^{op}$-mod^{reg} and let n be in N. Since N is regular, we can write

$$n = \sum_i (a_i \otimes b_i) \cdot n_i$$

and we define

$$A \otimes N \longrightarrow N \quad a \otimes n \rightsquigarrow \sum_i (aa_i \otimes b_i) \cdot n_i$$

$$N \otimes B \longrightarrow N \quad n \otimes b \rightsquigarrow \sum_i (a_i \otimes b_i b) \cdot n_i$$

In other words, the action $A \otimes N \longrightarrow N$ is given by the following composition, where m_A is the multiplication of A :

$$A \otimes N \simeq A \otimes [(A \otimes B^{op}) \otimes_{A \otimes B^{op}} N] \simeq \dots$$

$$\dots [A \otimes (A \otimes B^{op})] \otimes_{A \otimes B^{op}} N \xrightarrow{m_A \otimes 1 \otimes 1} (A \otimes B^{op}) \otimes_{A \otimes B^{op}} N \simeq N$$

Tensorizing over A and using that A is regular, one has that N is regular as left A-module. The argument for the action $N \otimes B \longrightarrow N$ is similar. As far as the compatibility between the two actions is concerned, we have

$$(an)b = (\sum_i (aa_i \otimes b_i) \cdot n_i) \cdot b = \sum_i (aa_i \otimes b_i b) \cdot n_i = a(nb) \ .$$

Now consider a morphism $f\colon N \longrightarrow N'$ in $A \otimes B^{op}$-mod. Since N is regular as A-B-bimodule, we can write

$$n = \sum_i a_i n_i b_i$$

so that

$$\begin{aligned} f(an) &= \sum_i f(aa_i n_i b_i) \\ &= \sum_i f((aa_i \otimes b_i) \cdot n_i) \\ &= \sum_i (aa_i \otimes b_i) \cdot f(n_i) \\ &= a \cdot \sum_i (a_i \otimes b_i) \cdot f(n_i) \\ &= a \cdot f(\sum_i (a_i \otimes b_i) \cdot n_i) \\ &= af(n) \ . \end{aligned}$$

In the same way one proves that $f(nb) = f(n)b$.

It is simple to verify that the two constructions just described are mutually inverse, using once again that, if M is in A-mod^{reg}-B and N is in $A \otimes B^{op}$-mod^{reg}, we can write

$$m = \sum_i a_i m_i b_i \ \ , \ \ n = \sum_i (a_i \otimes b_i) n_i \ .$$

■

The classifying category cl($\mathbb{B}$) of a bicategory $\mathbb{B}$ has been introduced in [6]: it has the same objects as $\mathbb{B}$ and, as arrows, 2-isomorphism classes of 1-arrows of $\mathbb{B}$. For an introduction to compact closed categories, the reader can see [12]: they are symmetric monoidal categories in which each object has a left adjoint.

Corollary 2.7 *The category cl(Alg^{reg}) is compact closed.*

Proof: by proposition 2.4, the tensor product of R-modules induces a tensor product in cl(Alg^{reg}). Moreover, given three regular algebras A, B and C, by proposition 2.6 we have a bijection

$$\mathrm{cl}(\mathrm{Alg}^{reg})(A \otimes B^{op}, C) \simeq \mathrm{cl}(\mathrm{Alg}^{reg})(A, C \otimes B)$$

which in fact is natural in A and C. This implies that B^{op} is left adjoint to B (cf. [12]). ■

Recall that the Brauer group $\mathcal{B}(R)$ of the unital commutative ring R is the group of Morita equivalence classes of unital Azumaya algebras. Moreover, it is a known fact that a unital algebra is Azumaya if and only if $A \otimes A^{op}$ is Morita equivalent to R (cf. [13], [20]). The previous corollary allows us to embed the Brauer group into a bigger group built up using regular algebras.

Proposition 2.8

1) *Morita equivalence classes of invertible regular algebras constitute an abelian group in which $\mathcal{B}(R)$ embeds.*

2) *A regular algebra A is invertible iff $A \otimes A^{op}$ is Morita equivalent to R.*

Proof: 1): by corollary 2.2, Morita equivalence classes of invertible regular algebras are exactly the invertible elements of the commutative monoid of isomorphism classes of objects of cl(Alg^{reg}) (that is, the monoid cl(cl(Alg^{reg}))). The fact that $\mathcal{B}(R)$ embeds in this group follows from the fact, already quoted, that a module on a unital algebra is regular iff it is unital.
2): if there exists a regular algebra B such that $A \otimes B$ is isomorphic, in cl(Alg^{reg}), to R, then B is left adjoint to A. But A^{op} is left adjoint to A, so that B is isomorphic, in cl(Alg^{reg}), to A^{op}. ■

Remarks:

1) In classical Morita theory, one can prove that if a bimodule M induces an equivalence
$$M \otimes_B -\colon\ B\text{-mod} \longrightarrow A\text{-mod}$$
(everything is unital), then M is a faithfully projective A-module. Because of the lack of projectivity of a not necessarily unital algebra, what remains true in our more general context is that M is a generator, in the sense that the evaluation
$$M \otimes \mathrm{Lin}_A(M, A) \longrightarrow A$$
is surjective.

2) In the first section we have proved that, if A is a regular algebra, then A-mod^{reg} is a colocalization of A-mod. This implies that A-mod^{reg} is abelian and then exact (in the sense of Barr, cf. [4]). Moreover, by lemma 1.4, A is a (regular) generator in A-mod^{reg}. By proposition 2.1 in [18], we deduce that A-mod^{reg} is a localization of the category of algebras of the monad $\mathbb{T}$ induced by the adjunction
$$\coprod_{-} A\ ,\ \mathrm{Lin}_A(A, -) : A\text{-mod}^{reg} \rightleftarrows \mathcal{SET}$$
(where, for each set S, $\coprod_S A$ is the S-indexed copower of A). The infinitary algebraic theory $\mathcal{T}$, corresponding to the monad $\mathbb{T}$, fails to be an annular theory only for cardinality reasons (A is not abstractly finite in A-mod^{reg}). In fact, the category of algebras of $\mathbb{T}$ is the exact completion of the full subcategory of A-mod^{reg} spanned by copowers of A, and then it is abelian. This implies that $\mathcal{T} \otimes \mathcal{Z} \simeq \mathcal{T}$, where $\otimes$ is here the tensor product of theories and $\mathcal{Z}$ is the theory of abelian groups (cf. [19]).

3 *

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