A Morita theorem in topology

Francis Borceux - Enrico M. Vitale
Institut de Mathématique Pure et Appliquée
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium

This paper is a contribution, for a particular aspect, to the problem of how the study of topology through locales is similar to linear algebra.

Our starting-point is that if $(X, \mathcal{O}(X))$ is a space with its topology, what counts is the locale of opens $\mathcal{O}(X)$, from which it is possible to rediscover the space X through the morphisms of locales $\mathcal{O}(X) \longrightarrow \mathbf{2}$ under the condition that the space X is sober; such condition is not a restriction in this context essentially because the categories of sheaves on a space and on its soberification are equivalent categories (see [1]).

So it is necessary to study the notion of sheaf on a locale L and it has been proved that

$$\text{Sup-lattices } (Sh(L)) \simeq Mod(L)$$

where $Sh(L)$ is the category of sheaves on a locale L and $Mod(L)$ is the category of modules on L (see [2]).

We can therefore wonder what we can say about two locales if we know that they have equivalent categories of modules.

Remembering that a locale is a particular commutative idempotent monoid in the category of sup-lattices, the right approach to such problem is, as in linear algebra, to consider monoids in general.

We will show that results inspired from the deepest theorems in classical Morita theory (see [3]) can be proved if we replace the category of Abelian groups with the category $\mathbb{S}$ of sup-lattices.

Let's look at this category:

a sup-lattice is a partially ordered set with arbitrary suprema; a morphism between sup-lattices is a sup-preserving function; the hom-set $\mathbb{S}(M, N)$ is a sup-lattice with the point-wise order; $\mathbb{S}$ is a symmetric monoidal closed category with the tensor product given by the codomain of the universal bimorphism

$$M \times N \longrightarrow M \otimes N$$

and with the hom as right adjoint to the tensor

$$M \otimes - \;\dashv\; \mathbb{S}(M, -);$$

so we can define what is a monoid in $\mathbb{S}$ with the corresponding categories of modules and bimodules: for example a monoid in $\mathbb{S}$ is a triple (A, m_A, e_A)

where A is a sup-lattice, $A \otimes A \xrightarrow{m_A} A$ and $\mathbf{2} \xrightarrow{e_A} A$ are sup-lattice morphisms for which the usual equations hold ($\mathbf{2}$ is the sup-lattice $\{0,1\}$ with $0 \leq 1$); analogously for the notions of module, bimodule and relative morphisms; we write $A - mod$ for the category of left modules over the monoid A, and so on.

Furthermore, given three monoids A, B and C and two bimodules

$$A \xrightarrow{M} B \text{ and } B \xrightarrow{N} C,$$

$\mathbb{S}$ is rich enough to define the bimodule

$$M \otimes_B N \in \ A - mod - C$$

($M \otimes_B N$ is exactly the coequalizer $M \otimes B \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_B N$) and such a tensor is biclosed

$$M \otimes_B - \ \dashv \ A - mod(M, -) \qquad - \otimes_B N \ \dashv \ mod - C(N, -)$$

(for details we refer to the first two chapters of [2] where the monoids are always commutative; it is not difficult the extension to the non-commutative case).

The problem is: given two Morita equivalent monoids A and B, that is two monoids such that $mod - A$ and $mod - B$ are equivalent categories, what can we say about A and B?

The first thing to do is the study of the functors

$$t : mod - A \longrightarrow mod - B$$

under the condition that the functions between the hom-sets

$$t_{X,Y} : mod - A(X, Y) \longrightarrow mod - B(tX, tY)$$

are sup-lattice functions (because of the good isomorphism between arbitrary products and coproducts holding in $\mathbb{S}$, this condition is guaranteed if the functor preserves products or coproducts; such isomorphism will be explicited in the following).

For such functors, tA has also an A-module structure given by

$$t_{A,A} : mod - A(A, A) \longrightarrow mod - B(tA, tA)$$

which, via the closure of the tensor and the monoid-isomorphism

$$A \simeq mod - A(A, A)$$

gives rise to an action

$$A \otimes tA \longrightarrow tA.$$

What allows us to classify functors between categories of modules is the following lemma:

Lemma: A is a cogenerator for $mod - A$, more exactly for every M in $mod - A$, M is the coker of a pair of arrows between modules like $\coprod_I A$ (the coproduct of A's copies indexed over an arbitrary set I).

Proof: It follows from the fact (proved in [2]) that every $M \in mod - A$ is a quotient of a free module, that is there exists an epimophism

$$\coprod_I A \longrightarrow M,$$

and that every epimorphism is surjective. ■

Now it is clear that a colimit-preserving functor

$$t : mod - A \longrightarrow mod - B$$

must be completely determined by tA; in fact

$$t \simeq -\otimes_A tA.$$

Proof: We need a natural isomorphism

$$F : -\otimes_A tA \longrightarrow t;$$

$$F_M : M \otimes_A tA \longrightarrow tM$$

is given, via the isomorphism

$$M \simeq mod - A(A, M),$$

by

$$mod - A(A, M) \otimes_A tA \longrightarrow tM$$

that is the trasformed of

$$t_{A,M} : mod - A(A, M) \longrightarrow mod - B(tA, tM)$$

in the adjunction

$$-\otimes_A tA \dashv mod - B(tA, -);$$

such $F : -\otimes_A tA \longrightarrow t$ is natural because it is built with natural steps; now using the fact that both the functors are colimit-preserving, we prove that it is an isomorphism:

1) $F_A : A \otimes_A tA \longrightarrow tA$ is the canonical isomorphism,

2) for a free module $\coprod_I A$ it follows from the commutativity of the diagram

$$\begin{tikzcd}
(\coprod_I A) \otimes_A tA \arrow[r, "F_{\coprod_I A}"] \arrow[d, phantom, "\simeq"] & t(\coprod_I A) \arrow[d, phantom, "\simeq"] \\
\coprod_I (A \otimes_A tA) \arrow[r, "\coprod_I F_A"'] & \coprod_I (tA)
\end{tikzcd}$$

3) if M is an arbitrary module, for the Lemma we can write M as a coker

$$\coprod_J A \rightrightarrows \coprod_I A \longrightarrow M$$

then

$$\begin{tikzcd}
(\coprod_J A)\otimes_A tA \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F_{\coprod_J A}"'] & (\coprod_I A)\otimes_A tA \arrow[r] \arrow[d, "F_{\coprod_I A}"'] & M\otimes_A tA \arrow[d, "F_M"] \\
t(\coprod_J A) \arrow[r, shift left] \arrow[r, shift right] & t(\coprod_I A) \arrow[r] & tM
\end{tikzcd}$$

and, as the naturality of F makes these diagrams commutative, we have finished our proof. ■

Corollary: Two monoids A and B are Morita-equivalent if and only if there are two bimodules

$$A \xrightarrow{M} B \text{ and } B \xrightarrow{N} A$$

such that

$$M\otimes_B N \simeq A \text{ and } N\otimes_A M \simeq B$$

as bimodules.

Proof: It suffices to notice that

$$\begin{tikzcd}
mod - A \arrow[r, shift left, "-\otimes_A X"] \arrow[r, shift right, "-\otimes_A Y"'] & mod - B
\end{tikzcd}$$

are isomorphic as functors if and only if

$$A \xrightarrow{X} B \text{ and } A \xrightarrow{Y} B$$

are isomorphic as bimodules. ■

Now we want to say something about these bimodules M and N : to this end it is important to notice (see [2]) that if $(M_i)_{i\in I}$ is an arbitrary family of modules, then

$$\prod_I M_i \simeq \coprod_I M_i$$

and the following diagram commutes:

$$\begin{tikzcd}
M_i \arrow[r, "id"] \arrow[d] & M_i \\
\coprod_I M_i \arrow[r, "\simeq"] & \prod_I M_i \arrow[u]
\end{tikzcd}$$

(that is not surprising if we remember that the fundamental difference between an abelian group and a sup-lattice is that the internal operation is respectively finitary in the first case and infinitary in the second one).

From this it is possible to show that if

$$M \otimes_B N \simeq A \quad \text{and} \quad N \otimes_A M \simeq B,$$

then M and N are both faithfully projective with respect to A and with respect to B, where, for example, "M faithfully projective with respect to A" means that in $A - mod$ there exist two commutative diagrams of this kind:

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr] & & A \\ & M^I \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "id"] \arrow[dr] & & M \\ & A^J \arrow[ur] & \end{tikzcd}$$

Proof: We omit a detailed proof, for which we refer to the classical case (see [3]); we limit ourselves to notice that the condition over A and B implies immediately that

$$B \simeq A - mod(M, M) \quad \text{and} \quad N \simeq A - mod(M, A)$$

and that the two triangular diagrams mean that the two composition-morphisms

$$A - mod(A, M) \otimes_B A - mod(M, A) \longrightarrow A - mod(A, A)$$

$$A - mod(M, A) \otimes_A A - mod(A, M) \longrightarrow A - mod(M, M)$$

are surjective. ■

We write $mod_{f.p.} - A$ for the full subcategory of $mod - A$ whose objects are the faithfully projective modules; we have

Corollary: $mod - A$ is equivalent to $mod - B$ if and only if $mod_{f.p.} - A$ is equivalent to $mod_{f.p.} - B$.

Proof: If

$$t : mod_{f.p.} - A \longrightarrow mod_{f.p.} - B$$

is an equivalence, we can write $t \simeq - \otimes_A tA$ by the same argument developed above and then extend t to an equivalence

$$- \otimes_A tA : mod - A \longrightarrow mod - B;$$

conversely it suffices to notice that if $M \in mod_{f.p.} - A$, $N \in A - mod - B$ and $N \in mod_{f.p.} - B$, then $M \otimes_A N \in mod_{f.p.} - B$. ■

Let us conclude with the commutative case: let $mod - A$ and $mod - B$ be equivalent categories of modules. We know that the functors realizing such an equivalence must be of the form

$$\begin{tikzcd} mod - A \arrow[r, shift left, "- \otimes_A M"] & mod - B \arrow[l, shift left, "- \otimes_B N"] \end{tikzcd}$$

with $\quad M\otimes_B N \simeq A \quad$ and $\quad N\otimes_A M \simeq B, \quad$ but then also the functors

$$A - mod \underset{M\otimes_B -}{\overset{N\otimes_A -}{\rightleftarrows}} B - mod$$

are an equivalence; so we have two monoid isomorphisms

$$mod - A(A,A) \simeq mod - B(M,M)$$

$$B - mod(B,B) \simeq A - mod(M,M).$$

Such isomorphisms can be restricted to two isomorphisms

$$A - mod - A(A,A) \simeq A - mod - B(M,M)$$

$$B - mod - B(B,B) \simeq A - mod - B(M,M)$$

and, considering that $A - mod - A(A,A)$ is a monoid isomorphic to the center $\mathbb{Z}(A)$ of A (see [3]), we have that if A and B are Morita-equivalent, then $\mathbb{Z}(A) \simeq \mathbb{Z}(B)$, that is if A and B are Morita-equivalent commutative monoids, then $A \simeq B$.

In particular, if X and Y are sober topological spaces and their locales $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are Morita-equivalent, then X and Y are omeomorphic.

REFERENCES

[1] Borceux F., Fasci, Logica e Topoi, Quaderni dell'Unione Matematica Italiana 34, Pitagora Editrice, Bologna (1989).

[2] Joyal A. and Tierney M., An extension of the Galois theory of Grothendieck, Memoirs American Mathematical Society 51 (1984).

[3] Bass H., Algebraic K-theory, W.A.Benjamin Inc., New York (1968).