Multi-bimodels

Enrico M. Vitale


Introduction

In [1], Adamek and Borceux have established a very general Morita theory for sketches. Two sketches $\mathcal{S}$ and $\mathcal{T}$ are called Morita-equivalent if their categories $\text{Mod}\mathcal{S}$ and $\text{Mod}\mathcal{T}$ of Set-valued models are equivalent. In [1], Morita-equivalent sketches are classified by means of mutually inverse bimodels, where an $\mathcal{S}$-$\mathcal{T}$-bimodel is a model of $\mathcal{S}$ in a certain subcategory $\hat{\mathcal{T}}$ of the functor category $[\text{Mod}\mathcal{T}, \text{Set}]$. In [1] a great attention is devoted to (connected) limit-coproduct sketches, since in this case the category $\hat{\mathcal{T}}$ admits a more explicit description: it is equivalent to the dual of the product-completion $\prod (\text{Mod}\mathcal{T})$ of $\text{Mod}\mathcal{T}$.

The particular case of Morita-equivalent limit sketches was firstly studied in [4] following a different approach. In [4] a Morita theorem is established using only the fact that for a limit sketch $\mathcal{S}$, the category $\text{Mod}\mathcal{S}$ is reflective in the functor category $\text{Set}^\mathcal{S}$ (where $\mathcal{S}$ is the small category underlying the sketch $\mathcal{S}$).

The aim of this note is to improve the method used in [4] to recapture the case of limit-coproduct sketches, because for such a sketch $\mathcal{S}$, the category $\text{Mod}\mathcal{S}$ is multi-reflective in $\text{Set}^\mathcal{S}$. Even if we do not rise the level of generality of [1], the advantage of this method is that we obtain not only a Morita theorem (corollary 2.8 below), but also a theorem which is the direct generalisation of the Eilenberg-Watts theorem characterizing colimit-preserving functors between module categories. Moreover, since our definition of multi-bimodel is at a non-doctrinaire level, techniques are quite different from those used in [1].

Another approach to Morita theory for sketches, based on the so-called generic model of a sketch, is contained in [6].

To support intuition, we recall here the classical Morita theory (all details can be found in [3]). Let $A$ and $B$ be two unital rings, and $A$-mod and $B$-mod the corresponding categories of left modules. Any $A$-$B$-bimodul $M$ induces a
pair of adjoint functors

\[ M \otimes_B - : B\text{-mod} \longrightarrow A\text{-mod} \quad \text{Lin}_A(M, -) : A\text{-mod} \longrightarrow B\text{-mod} \]

with \( M \otimes_B - \) left adjoint to \( \text{Lin}_A(M, -) \). The Eilenberg-Watts theorem states that any colimit-preserving functor \( F : B\text{-mod} \longrightarrow A\text{-mod} \) is isomorphic to one of the form \( M \otimes_B - \) for a suitable bimodule \( M \). As a consequence, the categories \( A\text{-mod} \) and \( B\text{-mod} \) are equivalent if there exist a \( A\text{-}B \)-bimodule \( M \) and a \( B\text{-}A \)-bimodule \( N \) such that \( M \otimes_B N \) is isomorphic to \( A \) and \( N \otimes_A M \) is isomorphic to \( B \).

1 Notations

For a category \( A \), we denote

\[ \eta_A : A \longrightarrow \prod(A) \]

its product-completion. If \( B \) is a category with products and \( F : A \longrightarrow B \) is an arbitrary functor, we write \( F^* : \prod(A) \longrightarrow B \) for the \( \prod \)-extension (product-preserving extension) of \( F \); it is the essentially unique product-preserving functor making commutative the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & \prod(A) \\
\downarrow F & & \downarrow F^* \\
B & & \\
\end{array}
\]

(when we say that a diagram of functors is commutative, we mean commutative up to isomorphisms). Given a functor \( G : A \longrightarrow B \), we write \( \prod(G) : \prod(A) \longrightarrow \prod(B) \) for the \( \prod \)-extension of \( G \cdot \eta_B : A \longrightarrow B \longrightarrow \prod(B) \).

A functor \( F : A \longrightarrow \prod(B) \) is also called a multi-functor \( F : A \rightrightarrows B \). The composition of two multi-functors \( F : A \rightrightarrows B \) and \( G : B \rightrightarrows C \) is given by \( F \cdot G^* : A \longrightarrow \prod(B) \longrightarrow \prod(C) \). Up to isomorphisms, this composition is associative and the unit \( \eta_A : A \longrightarrow \prod(A) \) acts as identity. For more details on multi-functors and multi-adjoints the reader can see [2] and [5].

In what follows \( T, S, \ldots \) are small categories. Given a small category \( T \), \( \text{Mod}T \) is a chosen multi-reflective subcategory of the functor category \( \text{Set}^T \), \( i_T : \text{Mod}T \longrightarrow \text{Set}^T \) is the full inclusion and \( R_T : \text{Set}^T \rightrightarrows \text{Mod}T \) its left multi-adjoint. If \( \varphi : T^{op} \longrightarrow \prod(\text{Mod}T) \) is a functor, in the following diagram
\( \hat{\varphi} \) is the left Kan-extension of \( \varphi \) along the Yoneda embedding \( Y_T \), and \( \text{Hom}(\varphi, -) : \prod(\text{Set}^\mathbb{T}) \longrightarrow \text{Set}^\mathbb{T} \) is the right adjoint of \( \hat{\varphi} \). If \( X \) is an object of \( \prod(\text{Set}^\mathbb{T}) \) and \( T \) is an object of \( \mathbb{T} \), \( \text{Hom}(\varphi, -)(X)(T) \) is given by the hom-set \( \text{Hom}[\varphi(T), X] \). Note that the Kan-extension \( \hat{\varphi} \) exists because \( \prod(\text{Set}^\mathbb{T}) \) is cocomplete. (We will usually omit subscripts in \( \eta_A, i_T, R_T \) and \( Y_T \).)

2 Multi-bimodels

**Definition 2.1** Let \( M : \mathbb{T}^{\text{op}} \rightsquigarrow \text{Mod}S \) be a multi-functor. We say that \( M \) is a multi-bimodel if the functor 
\[ \text{Hom}(M, -) : \text{Mod}S \longrightarrow \text{Set}^\mathbb{T} \]
given by \( \text{Hom}(M, -)(G)(T) = \text{Hom}[M(T), \eta(G)] \) for \( G \) in \( \text{Mod}S \) and \( T \) in \( \mathbb{T} \), factors through the full inclusion \( i : \text{Mod}T \longrightarrow \text{Set}^\mathbb{T} \).

In other words, consider the composite functor
\[ \varphi_M : \mathbb{T}^{\text{op}} \longrightarrow \prod(\text{Set}^\mathbb{T}) \longrightarrow \prod(\text{Mod}S) \longrightarrow \text{Mod}T \]
we say that \( M \) is a multi-bimodel if the functor \( \text{Hom}(\varphi_M, -) \) factors as in the following diagram

We call \( \text{Lin}(M, -) : \text{Mod}S \longrightarrow \text{Mod}T \) the requested factorization. Note that if it exists, it is essentially unique. The key property of a multi-bimodel is the following one.
Proposition 2.2 With the previous notations, consider the composite functor

\[
\mathcal{M} \otimes - : \text{Mod}T \xrightarrow{i} \text{Set}^\mathcal{T} \xrightarrow{\varphi_M} \prod(\text{Set}^\mathcal{S}) \xrightarrow{R^*} \prod(\text{Mod}S)
\]

if \( \mathcal{M} \) is a multi-bimodel, then \( \mathcal{M} \otimes - \) is left multi-adjoint to \( \text{Lin}(\mathcal{M},-) \).

Proof: Consider the unit \( \eta: \text{Mod}S \rightarrow \prod(\text{Mod}S) \), an object \( G \) in \( \text{Mod}S \) and an object \( F \) in \( \text{Mod}T \). The proof easily reduces to the following natural bijections:

\[
\begin{align*}
M \otimes F &= R^*(\varphi_M(i(F))) \rightarrow \eta(G) \quad \text{iff} \\
\varphi_M(i(F)) &= \prod(i)(\eta(G)) \quad \text{iff} \\
i(F) &= \text{Hom}[\varphi_M, \prod(i)(\eta(G))] = i(\text{Lin}([M,G])) \quad \text{iff} \\
F &= \text{Lin}[M,G]
\end{align*}
\]

Remark: The previous proposition can equivalently stated saying that the \( \prod \)-extension \( M^* \otimes - : \prod(\text{Mod}T) \rightarrow \prod(\text{Mod}S) \) of \( \mathcal{M} \otimes - \) is left adjoint to \( \prod(\text{Lin}(\mathcal{M},-)) \). In fact, the following general fact can be proved. Consider two functors \( G: \mathcal{B} \rightarrow \mathcal{A} \) and \( F: \mathcal{A} \rightarrow \prod(\mathcal{B}) \), and the \( \prod \)-extension \( F^*: \prod(\mathcal{A}) \rightarrow \prod(\mathcal{B}) \).

We need another preliminary fact on multi-bimodels.

Proposition 2.3 Let \( \mathcal{M}: \mathcal{T}^{\text{op}} \rightarrow \text{Mod}S \) be a multi-bimodel; then

(i) the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{T}^{\text{op}} & \xrightarrow{Y} & \text{Set}^\mathcal{T} \\
\downarrow & & \downarrow R \\
M & \xrightarrow{M^* \otimes -} & \prod(\text{Mod}S)
\end{array}
\]

(ii) \( M^* \otimes - \) preserves colimits and products;

(iii) \( M^* \otimes - \) is the unique (up to isomorphisms) functor which satisfies the two previous conditions.

To prove this proposition, we need an easy lemma.
Lemma 2.4

1) Consider two functors $G: \prod(B) \to A$ and $F: A \to \prod(B)$, the $\prod$-extension $F^*$ and the composite $G \cdot \eta: \prod(B) \to A \to \prod(A)$; if $F$ is a left adjoint of $G$, then $F^*$ is a left adjoint of $G \cdot \eta$.

2) The unit $\eta: A \to \prod(A)$ preserves all colimits which turn out to exist in $A$.

3) If $I: A \to B$ is a full and faithful functor, then also $\prod(I)$ is full and faithful.

4) If $I: A \to B$ is full and faithful and has a left multi-adjoint $R: B \to A$, then $I \cdot R \simeq \eta: A \to \prod(A)$.

Proof of proposition 2.3: (i) : observe that the following diagram is commutative

\[
\begin{array}{ccc}
\prod(\text{Set}^T) & \xrightarrow{i \circ (\cdot \eta)} & \prod(\text{Mod}T) \\
\downarrow \text{Hom}(\varphi_M, -) \circ \eta & & \downarrow \prod(\text{Lin}(M, *)) \\
\prod(\text{Set}^S) & \xrightarrow{i \circ (\cdot \eta)} & \prod(\text{Mod}S)
\end{array}
\]

Since all the functors involved preserve products (three of them by definition, and $\text{Hom}(\varphi_M, -) \circ \eta$ by lemma 2.4), this commutativity can be checked precomposing with the unit $\eta: \text{Mod}S \to \prod(\text{Mod}S)$. Passing to left adjoints, we obtain the commutativity of the following diagram (use lemma 2.4 and proposition 2.2)

\[
\begin{array}{ccc}
\prod(\text{Set}^T) & \xrightarrow{R^*} & \prod(\text{Mod}T) \\
\downarrow \varphi_M^* & & \downarrow M^* \otimes - \\
\prod(\text{Set}^S) & \xrightarrow{R^*} & \prod(\text{Mod}S)
\end{array}
\]

and, precomposing with the unit $\eta: \text{Set}^T \to \prod(\text{Set}^T)$, we have the commutativity of the following diagram
Finally, precomposing with the Yoneda embedding $Y: \mathcal{T}^{op} \rightarrow \mathcal{S}$, we obtain the requested commutativity. In fact $Y \cdot \varphi_M \simeq \varphi_M$ because $Y$ is full and faithful, $\varphi_M = M \cdot \prod(i)$ by definition of $\varphi_M$, and $\prod(i) \cdot R^* \simeq \text{id}$ because $\prod(i)$ is a full and faithful right adjoint of $R^*$.

(ii) : $M^* \odot -$ preserves products by definition and colimits because, by proposition 2.2, it has a right adjoint.

(iii) : let $G: \prod(\mathcal{M}) \rightarrow \prod(\mathcal{N})$ be a functor which preserves colimits and products and such that $Y \cdot R \cdot G \simeq Y \cdot R \cdot (M^* \odot -)$, but $R$ preserves colimits (because it factors as $R = \eta \cdot R^*: \mathcal{T} \rightarrow \prod(\mathcal{M})$, $R^*$ preserves colimits because it is a left adjoint, and $\eta$ preserves colimits by lemma 2.4) and $Y$ is dense, so that we can deduce $R \cdot G \simeq R \cdot (M^* \odot -)$. (Here we have used that $\prod(\mathcal{M})$ is cocomplete, which is the case because it is reflective in the cocomplete category $\prod(\mathcal{S})$.) This implies $i \cdot R \cdot G \simeq i \cdot R \cdot (M^* \odot -)$, that is $\eta \cdot G \simeq \eta \cdot (M^* \odot -)$ (lemma 2.4). Since both $G$ and $M^* \odot -$ preserve products, this implies that $G$ and $M^* \odot -$ are isomorphic.

Proof: 1) : we will prove that the diagram

\[
\begin{array}{ccc}
\text{Set}^\mathcal{T} & \xrightarrow{R} & \prod(\mathcal{M}) \\
\varphi_M \downarrow & & \downarrow M^* \odot - \\
\prod(\mathcal{S}) & \xrightarrow{R^*} & \prod(\mathcal{N})
\end{array}
\]

Now we can give two basic examples of multi-bimodels.

**Proposition 2.5**

1) consider the composite

$$M = Y \cdot R : \mathcal{T}^{op} \rightarrow \mathcal{S} \rightarrow \prod(\mathcal{M})$$

$M$ is a multi-bimodel and $M^* \odot -$ is isomorphic to the identity functor on $\prod(\mathcal{M})$;

2) let $M: \mathcal{T}^{op} \rightarrow \mathcal{M}$ be a multi-bimodel; consider two functors $\alpha: \mathcal{M} \rightarrow \prod(\mathcal{N})$ and $\beta: \mathcal{N} \rightarrow \mathcal{M}$, with a left multi-adjoint to $\beta$; the composite

$$N = M \cdot \alpha^*: \mathcal{T}^{op} \rightarrow \prod(\mathcal{N}) \rightarrow \prod(\mathcal{M})$$

is a multi-bimodel and $N^* \odot -$ is isomorphic to $(M^* \odot -) \cdot \alpha^*$.

Proof: 1) : we will prove that the diagram
is commutative. This means that $\text{Lin}(M, -)$ is the identity functor and then $M^* \otimes -$, being left adjoint to $\prod(\text{Lin}(M, -))$, is isomorphic to the identity functor. Let $F$ be an object of $\text{Mod} T$ and $T$ an object of $\mathbb{T}$:

$\text{Hom}[[i(F), \eta(F)]] = \text{Hom}[\prod(i(M(T))), \prod(i(\eta(F)))] =
\text{Hom}[R(Y(T)), \eta(F)] = \text{Hom}[Y(T), i(F)] = (iF)(T)$.

2) we will prove that the diagram

\[
\begin{array}{ccc}
\text{Set}^\mathbb{T} & \xrightarrow{i} & \text{Mod} T \\
\downarrow \text{Hom}(\varphi_M, -) & & \downarrow \text{id} \\
\prod(\text{Set}^\mathbb{T}) & \xrightarrow{\prod(i)} & \prod(\text{Mod} T) \\
\downarrow \eta & & \downarrow \text{Mod} T \\
\text{Mod} T & & \\
\end{array}
\]

is commutative. This means that $\text{Lin}(M, -)$ is the identity functor and then $M^* \otimes -$, being left adjoint to $\prod(\text{Lin}(M, -))$, is isomorphic to the identity functor. Let $F$ be an object of $\text{Mod} R$ and $T$ an object of $\mathbb{T}$:

$\text{Hom}[[\varphi_N(T), \prod(i)(\eta(F))]] = \text{Hom}[\prod(\alpha^*(M(T))), \prod(i)(\eta(F))] =
\text{Hom}[R(Y(T)), \eta(F)] = \text{Hom}[Y(T), i(F)] = (iF)(T)$.

We are ready to construct the composition of multi-bimodels.

**Proposition 2.6** Let $M : \mathbb{T}^{\text{op}} \rightarrow \text{Mod} S$ and $N : S^{\text{op}} \rightarrow \text{Mod} R$ be two multi-bimodels; the composite

$M \cdot (N^* \otimes -) : \mathbb{T}^{\text{op}} \rightarrow \prod(\text{Mod} S) \rightarrow \prod(\text{Mod} R)$

is commutative. This means that $\text{Lin}(N, -) = \beta \cdot \text{Lin}(M, -)$ and then $N^* \otimes -$, being left adjoint to $\prod(\text{Lin}(N, -))$, is isomorphic to $(M^* \otimes -) \cdot \alpha^*$. Let $F$ be an object of $\text{Mod} R$ and $T$ an object of $\mathbb{T}$:

$\text{Hom}[[\varphi_N(T), \prod(i)(\eta(F))]] = \text{Hom}[\prod(i(\alpha^*(M(T))), \prod(i)(\eta(F))] =
\text{Hom}[[\alpha^*(M(T)), \eta(F)] = \text{Hom}[M(T), \prod(i(\eta(F))] =
\text{Hom}[M(T), \eta(\beta(F))] = \text{Hom}[\prod(i(M(T)), \prod(i)(\eta(\beta(F)))] =
\text{Hom}[[\varphi_M(T), \prod(i)(\eta(\beta(F)))] = \text{Lin}[M(T), \beta(F)].$
is a multi-bimodel. We call this multi-bimodel the composition $M \otimes N$ of $M$
and $N$. Composition of multi-bimodels is associative and the multi-bimodel
\[
Y \cdot R : T^{op} \longrightarrow \text{Set}^{T} \longrightarrow \prod (\text{Mod}T)
\]
acts as identity (all up to isomorphisms).

**Proof:** Since $N^{*} \otimes -$ is left adjoint to $\prod (\text{Lin}(N, -))$, we can use the second
part of proposition 2.5 and we have that $P = M \cdot (N^{*} \otimes -)$ is a multi-bimodel
and $P^{*} \otimes - \simeq (M^{*} \otimes -) \cdot (N^{*} \otimes -)$. The rest of the statement easily follows
from proposition 2.3.

The announced generalizations of the Eilenberg-Watts theorem and of the
Morita theorem are now two simple corollaries of the previous analysis.

**Corollary 2.7** There is a bijection between isomorphism classes of multi-
bimodels
\[
T^{op} \mapsto \text{Mod}S
\]
and isomorphism classes of left multi-adjoints
\[
\text{Mod}T \mapsto \text{Mod}S.
\]
This bijection preserves composition and identities.

**Proof:** Given a multi-bimodel $M : T^{op} \longrightarrow \prod (\text{Mod}S)$ we obtain the functor
$M \otimes - : \text{Mod}T \longrightarrow \prod (\text{Mod}S)$ left multi-adjoint to the functor $\text{Lin}(M, -) : \text{Mod}S \longrightarrow \text{Mod}T$.
Conversely, given a left multi-adjoint $\alpha : \text{Mod}T \longrightarrow \prod (\text{Mod}S)$, then the com-
posite
\[
Y \cdot R \cdot \alpha^{*} : T^{op} \longrightarrow \text{Set}^{T} \longrightarrow \prod (\text{Mod}T) \longrightarrow \prod (\text{Mod}S)
\]
is a multi-bimodel. The rest of the statement easily follows from propositions
2.3 and 2.5.

**Corollary 2.8** The categories $\text{Mod}T$ and $\text{Mod}S$ are equivalent if and only if
there exist two multi-bimodels $M : T^{op} \mapsto \text{Mod}S$ and $N : S^{op} \mapsto \text{Mod}T$ such that
$M \otimes N \simeq Y_{T} \cdot R_{T}$ and $N \otimes M \simeq Y_{S} \cdot R_{S}$.

**Proof:** It follows from the previous corollary using the following general fact :
two categories $A$ and $B$ are equivalent iff their product-completions $\prod (A)$ and
$\prod (B)$ are equivalent.
3 Comparison with related results

We want to compare the classification given in corollary 2.8 with that given in theorem 5.6 of [4]. In [4] a bimodel is a functor $M: \mathcal{T}^{op} \to \text{Mod} \mathcal{S}$ such that the functor $\text{Hom}(M, -): \text{Mod} \mathcal{S} \to \text{Set}^T$, given by $\text{Hom}(M, -)(G)(T) = \text{Hom}[M(T), G]$ for $G$ in $\text{Mod} \mathcal{S}$ and $T$ in $\mathcal{T}$, factors through the full inclusion $i: \text{Mod} \mathcal{T} \to \text{Set}^T$.

We will use the following fact.

Lemma 3.1 Given two functors $i: \mathcal{A} \to \mathcal{B}$ and $r: \mathcal{B} \to \mathcal{A}$, $r$ is a left adjoint of $i$ iff $r \cdot \eta: \mathcal{B} \to \mathcal{A}$ is a left multi-adjoint of $i$.

When the chosen subcategory $\text{Mod} \mathcal{T}$ is reflective in $\text{Set}^T$, and not only multi-reflective, we obtain theorem 5.6 of [4] from corollary 2.8 via the following proposition.

Proposition 3.2

1) there is a bijection between bimodels $\mathcal{T}^{op} \to \text{Mod} \mathcal{S}$ in the sense of definition 4.1 in [4] and multi-bimodels $\mathcal{T}^{op} \to \text{Mod} \mathcal{S}$ which factor through the unit $\eta: \text{Mod} \mathcal{S} \to \prod(\text{Mod} \mathcal{S})$;

2) if $\text{Mod} \mathcal{T}, \text{Mod} \mathcal{S} \ldots$ are reflective in the appropriate categories of functors, then the previous bijection preserves composition and identities.

Proof: 1) : it is easy to check that, given the composite functor $M \cdot \eta: \mathcal{T}^{op} \to \text{Mod} \mathcal{S} \to \prod(\text{Mod} \mathcal{S})$, $M$ is a bimodel iff $M \cdot \eta$ is a multi-bimodel (in both directions one uses that the unit $\eta$ is full and faithful).

2) : let $r: \text{Set}^T \to \text{Mod} \mathcal{T}$ the reflector. The previous lemma says that the identity bimodel $Y \cdot r: \mathcal{T}^{op} \to \text{Set}^T \to \text{Mod} \mathcal{T}$ corresponds to the identity multi-bimodel $Y \cdot R: \mathcal{T}^{op} \to \text{Set}^T \to \prod(\text{Mod} \mathcal{T})$.

The key to prove that the bijection of point 1) also preserves composition is to observe that the following diagram is commutative (where $N = M \cdot \eta$ is the multi-bimodel corresponding to a bimodel $M$ and $M \otimes -$ is the left Kan-extension of $M$ along $Y \cdot r$)

\[
\begin{array}{ccc}
\text{Mod} \mathcal{T} & \xrightarrow{\eta} & \prod(\text{Mod} \mathcal{T}) \\
M \otimes - & \searrow & N^* \otimes - \\
\text{Mod} \mathcal{S} & \xrightarrow{\eta} & \prod(\text{Mod} \mathcal{S})
\end{array}
\]

that is $N^* \otimes - = \prod(M \otimes -)$.
II (Adamek, Borceux) - Let $\mathcal{T}$ and $\mathcal{S}$ be two limit sketches, $\mathcal{T}$ the small category underlying the sketch $\mathcal{T}$ and $\text{Mod}\mathcal{S}$ the usual category of Set-valued model of $\mathcal{S}$. A functor $\mathcal{T}^{\text{op}} \rightarrow \text{Mod}\mathcal{S}$ is a bimodel in the sense of definition 4.1 in [4] iff it is a $\mathcal{T}$-model in $(\text{Mod}\mathcal{S})^{\text{op}}$ (cf. section 7 in [4]). Categories sketchable by limit sketches are exactly locally presentable categories.

More in general, locally multipresentable categories are exactly categories sketchable by limit-coproduct sketches. The category of Set-valued models of such a sketch is multi-reflective in the category of Set-valued functors defined on the small category underlying the sketch. To end this note, we show that our notion of multi-bimodel specializes to the notion of bimodel used in [1] to classify Morita-equivalences between limit-coproduct sketches. If $\mathcal{T}$ and $\mathcal{S}$ are two such sketches, in [1] a $\mathcal{T}$-$\mathcal{S}$-bimodel is a $\mathcal{T}$-model in $(\prod(\text{Mod}\mathcal{S}))^{\text{op}}$, where $\text{Mod}\mathcal{S}$ is the category of Set-valued models of $\mathcal{S}$.

**Proposition 3.3** Let $\mathcal{T}$ and $\mathcal{S}$ be two limit-coproduct sketches. A functor $\mathcal{T}^{\text{op}} \rightarrow \prod(\text{Mod}\mathcal{S})$ is a multi-bimodel iff it is a $\mathcal{T}$-model in $(\prod(\text{Mod}\mathcal{S}))^{\text{op}}$.

**Proof:** The proof is a "$\prod$-ification" of the proof of the first proposition in section 7 of [4], making use of the following general fact: let $A$ be an object of a category $\mathcal{A}$ and consider the unit $\eta: A \rightarrow \prod(A)$; the hom-functor $\text{Hom}(\cdot, \eta(A)): \prod(A)^{\text{op}} \rightarrow \text{Set}$ preserves coproducts.

**Example 3.4**

Let $\mathcal{T}$ be the sketch over the following poset $\mathcal{T}$:

$$
\begin{align*}
x_1 &\rightarrow y &\leftarrow x_2
\end{align*}
$$

with no cones and with the discrete cocone over $\{x_1, x_2\}$. Then $\text{Mod}\mathcal{T} \simeq \text{Set} \times \text{Set}$, so $\mathcal{T}$ is Morita-equivalent to the sketch $\mathcal{S}$ with two objects, no nonidentity maps, no cones and no cocones.

The multi-bimodel $M: \mathcal{T}^{\text{op}} \rightarrow \text{Mod}\mathcal{S}$ inducing the equivalence $\text{Mod}\mathcal{T} \simeq \text{Mod}\mathcal{S}$ is given by $M(x_1) = \eta((*, \emptyset)), M(x_2) = \eta((\emptyset, *))$ and $M(y) = M(x_1) \times M(x_2)$. This equivalence can not be induced by a $\mathcal{T}$-model in $(\text{Mod}\mathcal{S})^{\text{op}}$.

4 *

References


Enrico M. Vitale
Département de Mathématique
Université catholique de Louvain
2, ch. du Cyclotron
1348 Louvain-la-Neuve, Belgium
vitale@agel.ucl.ac.be