Multi-bimodels

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Résumé. Nous étudions les équivalences entre sous-catégories multi-réflectives de catégories de préfaisceaux covariants. En utilisant une notion convenable de multi-bimodle, nous obtenons une généralisation des classiques théorèmes de Eilenberg-Watts et de Morita concernants les catégories de modules. L'exemple motivant est donné par les catégories localement multi-présentables, c.-à-d. les catégories esquissables par des esquisses à limites et coproduits.

Introduction

In [1], Adamek and Borceux have established a very general Morita theory for sketches. Two sketches $\mathcal{S}$ and $\mathcal{T}$ are called Morita-equivalent if their categories Mod$\mathcal{S}$ and Mod$\mathcal{T}$ of Set-valued models are equivalent. In [1], Morita-equivalent sketches are classified by means of mutually inverse bimodels, where an $\mathcal{S}$-$\mathcal{T}$-bimodel is a model of $\mathcal{S}$ in a certain subcategory $\hat{\mathcal{T}}$ of the functor category [Mod$\mathcal{T}$, Set]. In [1] a great attention is devoted to (connected) limit-coproduct sketches, since in this case the category $\hat{\mathcal{T}}$ admets a more explicit description : it is equivalent to the dual of the product-completion $\prod$(Mod$\mathcal{T}$) of Mod$\mathcal{T}$.

The particular case of Morita-equivalent limit sketches was firstly studied in [4] following a different approach. In [4] a Morita theorem is established using *only* the fact that for a limit sketch $\mathcal{S}$, the category Mod$\mathcal{S}$ is reflective in the functor category Set$^{\mathbb{S}}$ (where $\mathbb{S}$ is the small category underlying the sketch $\mathcal{S}$).

The aim of this note is to improve the method used in [4] to recapture the case of limit-coproduct sketches, because for such a sketch $\mathcal{S}$, the category Mod$\mathcal{S}$ is multi-reflective in Set$^{\mathbb{S}}$. Even if we do not rise the level of generality of [1], the advantage of this method is that we obtain not only a Morita theorem (corollary 2.8 below), but also a theorem which is the direct generalisation of the Eilenberg-Watts theorem characterizing colimit-preserving functors between module categories. Moreover, since our definition of multi-bimodel is at a non-doctrinaire level, techniques are quite different from those used in [1].

Another approach to Morita theory for sketches, based on the so-called generic model of a sketch, is contained in [6].

To support intuition, we recall here the classical Morita theory (all details can be found in [3]). Let A and B be two unital rings, and A-mod and B-mod the corresponding categories of left modules. Any A-B-bimodul M induces a pair of adjoint functors

$$M \otimes_B -\colon B\text{-mod} \longrightarrow A\text{-mod} \qquad \mathrm{Lin}_A(M,-)\colon A\text{-mod} \longrightarrow B\text{-mod}$$

with $M \otimes_B -$ left adjoint to $\mathrm{Lin}_A(M,-)$. The Eilenberg-Watts theorem states that any colimit-preserving functor $F\colon B\text{-mod} \longrightarrow A\text{-mod}$ is isomorphic to one of the form $M \otimes_B -$ for a suitable bimodule M. As a consequence, the categories A-mod and B-mod are equivalent iff there exist a A-B-bimodule M and a B-A-bimodule N such that $M \otimes_B N$ is isomorphic to A and $N \otimes_A M$ is isomorphic to B.

1 Notations

For a category $\mathcal{A}$, we denote

$$\eta_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \prod(\mathcal{A})$$

its product-completion. If $\mathcal{B}$ is a category with products and $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an arbitrary functor, we write $F^*\colon \prod(\mathcal{A}) \longrightarrow \mathcal{B}$ for the $\prod$-extension (product-preserving extension) of F; it is the essentially unique product-preserving functor making commutative the following diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\eta_{\mathcal{A}}"] \arrow[dr, "F"'] & & \prod(\mathcal{A}) \arrow[dl, "F^*"] \\ & \mathcal{B} & \end{tikzcd}$$

(when we say that a diagram of functors is commutative, we mean commutative up to isomorphisms). Given a functor $G\colon \mathcal{A} \longrightarrow \mathcal{B}$, we write $\prod(G)\colon \prod(\mathcal{A}) \longrightarrow \prod(\mathcal{B})$ for the $\prod$-extension of $G \cdot \eta_{\mathcal{B}}\colon \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \prod(\mathcal{B})$.

A functor $F\colon \mathcal{A} \longrightarrow \prod(\mathcal{B})$ is also called a multi-functor $F\colon \mathcal{A} \mapsto \mathcal{B}$. The composition of two multi-functors $F\colon \mathcal{A} \mapsto \mathcal{B}$ and $G\colon \mathcal{B} \mapsto \mathcal{C}$ is given by $F \cdot G^*\colon \mathcal{A} \longrightarrow \prod(\mathcal{B}) \longrightarrow \prod(\mathcal{C})$. Up to isomorphisms, this composition is associative and the unit $\eta_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \prod(\mathcal{A})$ acts as identity. For more details on multi-functors and multi-adjoints the reader can see [2] and [5].

In what follows $\mathbb{T}, \mathbb{S}, \ldots$ are small categories. Given a small category $\mathbb{T}$, $\mathrm{Mod}\mathcal{T}$ is a chosen multi-reflective subcategory of the functor category $\mathrm{Set}^{\mathbb{T}}$, $i_{\mathbb{T}}\colon \mathrm{Mod}\mathcal{T} \longrightarrow \mathrm{Set}^{\mathbb{T}}$ is the full inclusion and $R_{\mathbb{T}}\colon \mathrm{Set}^{\mathbb{T}} \mapsto \mathrm{Mod}\mathcal{T}$ its left multi-adjoint. If $\varphi\colon \mathbb{T}^{op} \longrightarrow \prod(\mathrm{Mod}^{\mathbb{T}})$ is a functor, in the following diagram

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[rr, "Y_{\mathbb{T}}"] \arrow[dr, "\varphi"'] & & \mathrm{Set}^{\mathbb{T}} \arrow[dl, "\hat{\varphi}"] \\
& \prod(\mathrm{Set}^{\mathbb{S}}) &
\end{tikzcd}$$

$\hat{\varphi}$ is the left Kan-extension of φ along the Yoneda embedding $Y_{\mathbb{T}}$, and $\mathrm{Hom}(\varphi, -)\colon \prod(\mathrm{Set}^{\mathbb{S}}) \longrightarrow \mathrm{Set}^{\mathbb{T}}$ is the right adjoint of $\hat{\varphi}$. If X is an object of $\prod(\mathrm{Set}^{\mathbb{S}})$ and T is an object of $\mathbb{T}$, $\mathrm{Hom}(\varphi, -)(X)(T)$ is given by the hom-set $\mathrm{Hom}[\varphi(T), X]$. Note that the Kan-extension $\hat{\varphi}$ exists because $\prod(\mathrm{Set}^{\mathbb{S}})$ is cocomplete.
(We will usually omit subscripts in $\eta_{\mathcal{A}}, i_{\mathbb{T}}, R_{\mathbb{T}}$ and $Y_{\mathbb{T}}$.)

2 Multi-bimodels

Definition 2.1 *Let $M\colon \mathbb{T}^{op} \mapsto Mod\mathcal{S}$ be a multi-functor. We say that M is a multi-bimodel if the functor*

$$Hom(M, -)\colon Mod\mathcal{S} \longrightarrow Set^{\mathbb{T}}$$

given by $Hom(M, -)(G)(T) = Hom[M(T), \eta(G)]$ for G in $Mod\mathcal{S}$ and T in $\mathbb{T}$, factors through the full inclusion $i\colon Mod\mathcal{T} \longrightarrow Set^{\mathbb{T}}$.

In other words, consider the composite functor

$$\varphi_M\colon \mathbb{T}^{op} \xrightarrow{M} \prod(\mathrm{Mod}\mathcal{S}) \xrightarrow{\prod(i)} \prod(\mathrm{Set}^{\mathbb{S}}) \quad ;$$

we say that M is a multi-bimodel if the functor $\mathrm{Hom}(\varphi_M, -)$ factors as in the following diagram

$$\begin{tikzcd}
\mathrm{Set}^{\mathbb{T}} & & \mathrm{Mod}\mathcal{T} \arrow[ll, "i"'] \\
\prod(\mathrm{Set}^{\mathbb{S}}) \arrow[u, "\mathrm{Hom}(\varphi_M, -)"] & \prod(\mathrm{Mod}\mathcal{S}) \arrow[l, "\prod(i)"] & \mathrm{Mod}\mathcal{S} \arrow[l, "\eta"] \arrow[u]
\end{tikzcd}$$

We call $\mathrm{Lin}(M, -)\colon \mathrm{Mod}\mathcal{S} \longrightarrow \mathrm{Mod}\mathcal{T}$ the requested factorization. Note that if it exists, it is essentially unique. The key property of a multi-bimodel is the following one.

Proposition 2.2 *With the previous notations, consider the composite functor*

$$M \otimes -\colon Mod\mathcal{T} \xrightarrow{\ i\ } Set^{\mathbb{T}} \xrightarrow{\hat{\varphi_M}} \prod(Set^{\mathbb{S}}) \xrightarrow{R^*} \prod(Mod\mathcal{S}) \quad ;$$

if M is a multi-bimodel, then $M \otimes -$ is left multi-adjoint to Lin($M, -$).

Proof: Consider the unit $\eta\colon \mathrm{Mod}\mathcal{S} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$, an object G in $\mathrm{Mod}\mathcal{S}$ and an object F in $\mathrm{Mod}\mathcal{T}$. The proof easily reduces to the following natural bijections :

$$M \otimes F = R^*(\varphi_M(i(F))) \longrightarrow \eta(G) \quad \text{iff}$$

$$\varphi_M(i(F)) \longrightarrow \prod(i)(\eta(G)) \quad \text{iff}$$

$$i(F) \longrightarrow \mathrm{Hom}[\varphi_M, \prod(i)(\eta(G))] = i(\mathrm{Lin}[M, G]) \quad \text{iff}$$

$$F \longrightarrow \mathrm{Lin}[M, G]$$

■

Remark : The previous proposition can equivalently stated saying that the $\prod$-extension $M^* \otimes -\colon \prod(\mathrm{Mod}\mathcal{T}) \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$ of $M \otimes -$ is left adjoint to $\prod(\mathrm{Lin}(M, -)$. In fact, the following general fact can be proved. Consider two functors $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ and $F\colon \mathcal{A} \longrightarrow \prod(\mathcal{B})$, and the $\prod$-extension $F^*\colon \prod(\mathcal{A}) \longrightarrow \prod(\mathcal{B})$; F is a left multi-adjoint of G iff F^* is a left adjoint of $\prod(G)$.

We need another preliminary fact on multi-bimodels.

Proposition 2.3 *Let $M\colon \mathbb{T}^{op} \mapsto Mod\mathcal{S}$ be a multi-bimodel ; then*

(i) the following diagram is commutative

$$\begin{tikzcd}
\mathbb{T}^{op} \arrow[r, "Y"] \arrow[dr, "M"'] & Set^{\mathbb{T}} \arrow[r, "R"] & \prod(Mod\mathcal{T}) \arrow[dl, "M^* \otimes -"] \\
& \prod(Mod\mathcal{S}) &
\end{tikzcd}$$

(ii) $M^* \otimes -$ preserves colimits and products ;

(iii) $M^* \otimes -$ is the unique (up to isomorphisms) functor which satisfies the two previous conditions.

To prove this proposition, we need an easy lemma.

Lemma 2.4

1) *consider two functors* $G\colon \prod(\mathcal{B}) \longrightarrow \mathcal{A}$ *and* $F\colon \mathcal{A} \longrightarrow \prod(\mathcal{B})$, *the* $\prod$*-extension* F^* *and the composite* $G \cdot \eta\colon \prod(\mathcal{B}) \longrightarrow \mathcal{A} \longrightarrow \prod(\mathcal{A})$; *if* F *is a left adjoint of* G, *then* F^* *is a left adjoint of* $G \cdot \eta$;

2) *the unit* $\eta\colon \mathcal{A} \longrightarrow \prod(\mathcal{A})$ *preserves all colimits which turn out to exist in* $\mathcal{A}$;

3) *if* $I\colon \mathcal{A} \longrightarrow \mathcal{B}$ *is a full and faithful functor, then also* $\prod(I)$ *is full and faithful ;*

4) *if* $I\colon \mathcal{A} \longrightarrow \mathcal{B}$ *is full and faithful and has a left multi-adjoint* $R\colon \mathcal{B} \mapsto \mathcal{A}$, *then* $I \cdot R \simeq \eta\colon \mathcal{A} \longrightarrow \prod(\mathcal{A})$.

Proof of proposition 2.3: (i) : observe that the following diagram is commutative

$$\begin{tikzcd}
\prod(\mathrm{Set}^{\mathbb{T}}) & \prod(\mathrm{Mod}\mathcal{T}) \arrow[l, "\prod(i)"'] \\
\prod(\mathrm{Set}^{\mathbb{S}}) \arrow[u, "\mathrm{Hom}(\varphi_M, -)\cdot\eta"] & \prod(\mathrm{Mod}\mathcal{S}) \arrow[l, "\prod(i)"] \arrow[u, "\prod(\mathrm{Lin}(M,-))"']
\end{tikzcd}$$

Since all the functors involved preserve products (three of them by definition, and $\mathrm{Hom}(\varphi_M, -)\cdot\eta$ by lemma 2.4), this commutativity can be checked precomposing with the unit $\eta\colon \mathrm{Mod}\mathcal{S} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$.
Passing to left adjoints, we obtain the commutativity of the following diagram (use lemma 2.4 and proposition 2.2)

$$\begin{tikzcd}
\prod(\mathrm{Set}^{\mathbb{T}}) \arrow[r, "R^*"] \arrow[d, "\hat{\varphi_M}^*"'] & \prod(\mathrm{Mod}\mathcal{T}) \arrow[d, "M^* \otimes -"] \\
\prod(\mathrm{Set}^{\mathbb{S}}) \arrow[r, "R^*"'] & \prod(\mathrm{Mod}\mathcal{S})
\end{tikzcd}$$

and, precomposing with the unit $\eta\colon \mathrm{Set}^{\mathbb{T}} \longrightarrow \prod(\mathrm{Set}^{\mathbb{T}})$, we have the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{Set}^{\mathbb{T}} & \xrightarrow{R} & \prod(\mathrm{Mod}\mathcal{T}) \\ \downarrow{\scriptstyle \hat{\varphi_M}} & & \downarrow{\scriptstyle M^* \otimes -} \\ \prod(\mathrm{Set}^{\mathbb{S}}) & \xrightarrow[R^*]{} & \prod(\mathrm{Mod}\mathcal{S}) \end{array}$$

Finally, precomposing with the Yoneda embedding $Y\colon \mathbb{T}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}}$, we obtain the requested commutativity. In fact $Y \cdot \hat{\varphi}_M \simeq \varphi_M$ because Y is full and faithful, $\varphi_M = M \cdot \prod(i)$ by definition of φ_M, and $\prod(i) \cdot R^* \simeq \mathrm{id}$ because $\prod(i)$ is a full and faithful right adjoint of R^*.
(ii) : $M^* \otimes -$ preserves products by definition and colimits because, by proposition 2.2, it has a right adjoint.
(iii) : let $G\colon \prod(\mathrm{Mod}\mathcal{T}) \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$ be a functor which preserves colimits and products and such that $Y \cdot R \cdot G$ is isomorphic to M. We have $Y \cdot R \cdot G \simeq Y \cdot R \cdot (M^* \otimes -)$, but R preserves colimits (because it factors as $R = \eta \cdot R^*\colon \mathrm{Set}^{\mathbb{T}} \longrightarrow \prod(\mathrm{Set}^{\mathbb{T}}) \longrightarrow \prod(\mathrm{Mod}\mathcal{T})$, R^* preserves colimits because it is a left adjoint, and η preserves colimits by lemma 2.4) and Y is dense, so that we can deduce $R \cdot G \simeq R \cdot (M^* \otimes -)$. (Here we have used that $\prod(\mathrm{Mod}\mathcal{S})$ is cocomplete, which is the case because it is reflective in the cocomplete category $\prod(\mathrm{Set}^{\mathbb{S}})$.) This implies $i \cdot R \cdot G \simeq i \cdot R \cdot (M^* \otimes -)$, that is $\eta \cdot G \simeq \eta \cdot (M^* \otimes -)$ (lemma 2.4). Since both G and $M^* \otimes -$ preserve products, this implies that G and $M^* \otimes -$ are isomorphic. ■

Now we can give two basic examples of multi-bimodels.

Proposition 2.5

1) *consider the composite*

$$M = Y \cdot R : \mathbb{T}^{op} \longrightarrow \mathit{Set}^{\mathbb{T}} \longrightarrow \prod(\mathit{Mod}\mathcal{T}) \ ;$$

M is a multi-bimodel and $M^* \otimes -$ is isomorphic to the identity functor on $\prod(\mathit{Mod}\mathcal{T})$;

2) *let $M\colon \mathbb{T}^{op} \mapsto \mathit{Mod}\mathcal{S}$ be a multi-bimodel; consider two functors $\alpha\colon \mathit{Mod}\mathcal{S} \longrightarrow \prod(\mathit{Mod}\mathcal{R})$ and $\beta\colon \mathit{Mod}\mathcal{R} \longrightarrow \mathit{Mod}\mathcal{S}$, with α left multi-adjoint to β; the composite*

$$N = M \cdot \alpha^*\colon \mathbb{T}^{op} \longrightarrow \prod(\mathit{Mod}\mathcal{S}) \longrightarrow \prod(\mathit{Mod}\mathcal{R})$$

is a multi-bimodel and $N^* \otimes -$ is isomorphic to $(M^* \otimes -) \cdot \alpha^*$.

Proof: 1) : we will prove that the diagram

$$\begin{tikzcd}
\mathrm{Set}^{\mathbb{T}} & & \mathrm{Mod}\mathcal{T} \arrow[ll, "i"'] \\
\prod(\mathrm{Set}^{\mathbb{T}}) \arrow[u, "\mathrm{Hom}(\varphi_M,-)"] & \prod(\mathrm{Mod}\mathcal{T}) \arrow[l, "\prod(i)"] & \mathrm{Mod}\mathcal{T} \arrow[l, "\eta"] \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

is commutative. This means that $\mathrm{Lin}(M,-)$ is the identity functor and then $M^*\otimes -$, being left adjoint to $\prod(\mathrm{Lin}(M,-))$, is isomorphic to the identity functor. Let F be an object of $\mathrm{Mod}\mathcal{T}$ and T an object of $\mathbb{T}$:
$\mathrm{Hom}[\varphi_M(T),\prod(i)(\eta(F))] = \mathrm{Hom}[\prod(i)(R(Y(T))),\prod(i)(\eta(F))] =$
$= \mathrm{Hom}[R(Y(T)),\eta(F)] = \mathrm{Hom}[Y(T),i(F)] = (iF)(T)$.
2) : we will prove that the diagram

$$\begin{tikzcd}
\mathrm{Set}^{\mathbb{T}} & & \mathrm{Mod}\mathcal{T} \arrow[ll, "i"'] \\
& & \mathrm{Mod}\mathcal{S} \arrow[u, "\mathrm{Lin}(M,-)"'] \\
\prod(\mathrm{Set}^{\mathbb{R}}) \arrow[uu, "\mathrm{Hom}(\varphi_N,-)"] & \prod(\mathrm{Mod}\mathcal{R}) \arrow[l, "\prod(i)"] & \mathrm{Mod}\mathcal{R} \arrow[l, "\eta"] \arrow[u, "\beta"']
\end{tikzcd}$$

is commutative. This means that $\mathrm{Lin}(N,-) = \beta \cdot \mathrm{Lin}(M,-)$ and then $N^* \otimes -$, being left adjoint to $\prod(\mathrm{Lin}(N,-))$, is isomorphic to $(M^* \otimes -)\cdot \alpha^*$. Let F be an object of $\mathrm{Mod}\mathcal{R}$ and T an object of $\mathbb{T}$:
$\mathrm{Hom}[\varphi_N(T),\prod(i)(\eta(F))] = \mathrm{Hom}[\prod(i)(\alpha^*(M(T))),\prod(i)(\eta(F))] =$
$= \mathrm{Hom}[\alpha^*(M(T)),\eta(F)] = \mathrm{Hom}[M(T),\prod(\beta)(\eta(F))] =$
$= \mathrm{Hom}[M(T),\eta(\beta(F))] = \mathrm{Hom}[\prod(i)(M(T)),\prod(i)(\eta(\beta(F)))] =$
$= \mathrm{Hom}[\varphi_M(T),\prod(i)(\eta(\beta(F)))] = \mathrm{Lin}[M(T),\beta(F)]$. ■

We are ready to construct the composition of multi-bimodels.

Proposition 2.6 *Let $M\colon \mathbb{T}^{op} \mapsto Mod\mathcal{S}$ and $N\colon \mathbb{S}^{op} \mapsto Mod\mathcal{R}$ be two multi-bimodels; the composite*

$$M\cdot(N^*\otimes -)\colon \mathbb{T}^{op} \longrightarrow \prod(Mod\mathcal{S}) \longrightarrow \prod(Mod\mathcal{R})$$

is a multi-bimodel. We call this multi-bimodel the composition $M \otimes N$ of M and N. Composition of multi-bimodels is associative and the multi-bimodel

$$Y \cdot R : \mathbb{T}^{op} \longrightarrow Set^{\mathbb{T}} \longrightarrow \prod(Mod\mathcal{T})$$

acts as identity (all up to isomorphisms).

Proof: Since $N^* \otimes -$ is left adjoint to $\prod(\mathrm{Lin}(N, -))$, we can use the second part of proposition 2.5 and we have that $P = M \cdot (N^* \otimes -)$ is a multi-bimodel and $P^* \otimes - \simeq (M^* \otimes -) \cdot (N^* \otimes -)$. The rest of the statement easily follows from proposition 2.3. ∎

The announced generalizations of the Eilenberg-Watts theorem and of the Morita theorem are now two simple corollaries of the previous analysis.

Corollary 2.7 *There is a bijection between isomorphism classes of multi-bimodels*

$$\mathbb{T}^{op} \mapsto Mod\mathcal{S}$$

and isomorphism classes of left multi-adjoints

$$Mod\mathcal{T} \mapsto Mod\mathcal{S} \ \ .$$

This bijection preserves composition and identities.

Proof: Given a multi-bimodel $M\colon \mathbb{T}^{op} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$ we obtain the functor $M \otimes - \colon \mathrm{Mod}\mathcal{T} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$ left multi-adjoint to the functor $\mathrm{Lin}(M, -)\colon \mathrm{Mod}\mathcal{S} \longrightarrow \mathrm{Mod}\mathcal{T}$. Conversely, given a left multi-adjoint $\alpha\colon \mathrm{Mod}\mathcal{T} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$, then the composite

$$Y \cdot R \cdot \alpha^*\colon \mathbb{T}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}} \longrightarrow \prod(\mathrm{Mod}\mathcal{T}) \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$$

is a multi-bimodel. The rest of the statement easily follows from propositions 2.3 and 2.5. ∎

Corollary 2.8 *The categories $Mod\mathcal{T}$ and $Mod\mathcal{S}$ are equivalent if and only if there exist two multi-bimodels $M\colon \mathbb{T}^{op} \mapsto Mod\mathcal{S}$ and $N\colon \mathbb{S}^{op} \mapsto Mod\mathcal{T}$ such that $M \otimes N \simeq Y_{\mathbb{T}} \cdot R_{\mathbb{T}}$ and $N \otimes M \simeq Y_{\mathbb{S}} \cdot R_{\mathbb{S}}$.*

Proof: It follows from the previous corollary using the following general fact : two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent iff their product-completions $\prod(\mathcal{A})$ and $\prod(\mathcal{B})$ are equivalent. ∎

3 Comparison with related results

I - We want to compare the classification given in corollary 2.8 with that given in theorem 5.6 of [4]. In [4] a bimodel is a functor $M\colon \mathbb{T}^{op} \longrightarrow \mathrm{Mod}\mathcal{S}$ such that the functor $\mathrm{Hom}(M,-)\colon \mathrm{Mod}\mathcal{S} \longrightarrow \mathrm{Set}^{\mathbb{T}}$, given by $\mathrm{Hom}(M,-)(G)(T) = \mathrm{Hom}[M(T),G]$ for G in $\mathrm{Mod}\mathcal{S}$ and T in $\mathbb{T}$, factors through the full inclusion $i\colon \mathrm{Mod}\mathcal{T} \longrightarrow \mathrm{Set}^{\mathbb{T}}$.

We will use the following fact.

Lemma 3.1 *Given two functors $i\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $r\colon \mathcal{B} \longrightarrow \mathcal{A}$, r is a left adjoint of i iff $r \cdot \eta\colon \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \prod(\mathcal{A})$ is a left multi-adjoint of i.*

When the chosen subcategory $\mathrm{Mod}\mathcal{T}$ is reflective in $\mathrm{Set}^{\mathbb{T}}$, and not only multi-reflective, we obtain theorem 5.6 of [4] from corollary 2.8 via the following proposition.

Proposition 3.2

1) *there is a bijection between bimodels $\mathbb{T}^{op} \longrightarrow \mathrm{Mod}\mathcal{S}$ in the sense of definition 4.1 in [4] and multi-bimodels $\mathbb{T}^{op} \mapsto \mathrm{Mod}\mathcal{S}$ which factor through the unit $\eta\colon \mathrm{Mod}\mathcal{S} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$;*

2) *if $\mathrm{Mod}\mathcal{T}, \mathrm{Mod}\mathcal{S} \ldots$ are reflective in the appropriate categories of functors, then the previous bijection preserves composition and identities.*

Proof: 1) : it is easy to check that, given the composite functor $M \cdot \eta\colon \mathbb{T}^{op} \longrightarrow \mathrm{Mod}\mathcal{S} \longrightarrow \prod(\mathrm{Mod}\mathcal{S})$, M is a bimodel iff $M \cdot \eta$ is a multi-bimodel (in both directions one uses that the unit η is full and faithful).
2) : let $r\colon \mathrm{Set}^{\mathbb{T}} \longrightarrow \mathrm{Mod}\mathcal{T}$ the reflector. The previous lemma says that the identity bimodel $Y \cdot r\ \colon \mathbb{T}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}} \longrightarrow \mathrm{Mod}\mathcal{T}$ corresponds to the identity multi-bimodel $Y \cdot R : \mathbb{T}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}} \longrightarrow \prod(\mathrm{Mod}\mathcal{T})$.
The key to prove that the bijection of point 1) also preserves composition is to observe that the following diagram is commutative (where $N = M \cdot \eta$ is the multi-bimodel corresponding to a bimodel M and $M \otimes -$ is the left Kan-extension of M along $Y \cdot r$)

$$\begin{tikzcd}
\mathrm{Mod}\mathcal{T} \arrow[r, "\eta"] \arrow[d, "M \otimes -"'] & \prod(\mathrm{Mod}\mathcal{T}) \arrow[d, "N^* \otimes -"] \\
\mathrm{Mod}\mathcal{S} \arrow[r, "\eta"'] & \prod(\mathrm{Mod}\mathcal{S})
\end{tikzcd}$$

that is $N^* \otimes - = \prod(M \otimes -)$. ∎

II (Adamek, Borceux) - Let $\mathcal{T}$ and $\mathcal{S}$ be two limit sketches, $\mathbb{T}$ the small category underlying the sketch $\mathcal{T}$ and Mod$\mathcal{S}$ the usual category of Set-valued model of $\mathcal{S}$. A functor $\mathbb{T}^{op} \longrightarrow \mathrm{Mod}\mathcal{S}$ is a bimodel in the sense of definition 4.1 in [4] iff it is a $\mathcal{T}$-model in $(\mathrm{Mod}\mathcal{S})^{op}$ (cf. section 7 in [4]). Categories sketchable by limit sketches are exactly locally presentable categories.

More in general, locally multipresentable categories are exactly categories sketchable by limit-coproduct sketches. The category of Set-valued models of such a sketch is multi-reflective in the category of Set-valued functors defined on the small category underlying the sketch. To end this note, we show that our notion of multi-bimodel specializes to the notion of bimodel used in [1] to classify Morita-equivalences between limit-coproduct sketches. If $\mathcal{T}$ and $\mathcal{S}$ are two such sketches, in [1] a $\mathcal{T}$-$\mathcal{S}$-bimodel is a $\mathcal{T}$-model in $(\prod(\mathrm{Mod}\mathcal{S}))^{op}$, where Mod$\mathcal{S}$ is the category of Set-valued models of $\mathcal{S}$.

Proposition 3.3 *Let $\mathcal{T}$ and $\mathcal{S}$ be two limit-coproduct sketches. A functor $\mathbb{T}^{op} \longrightarrow \prod(Mod\mathcal{S})$ is a multi-bimodel iff it is a $\mathcal{T}$-model in $(\prod(Mod\mathcal{S}))^{op}$.*

Proof: The proof is a "$\prod$-fication" of the proof of the first proposition in section 7 of [4], making use of the following general fact : let A be an object of a category $\mathcal{A}$ and consider the unit $\eta\colon \mathcal{A} \longrightarrow \prod(\mathcal{A})$; the hom-functor $\mathrm{Hom}(-, \eta(A))\colon \prod(\mathcal{A})^{op} \longrightarrow \mathrm{Set}$ preserves coproducts. ∎

Example 3.4

Let $\mathcal{T}$ be the sketch over the following poset $\mathbb{T}$:

$$x_1 \longrightarrow y \longleftarrow x_2$$

with no cones and with the discrete cocone over $\{x_1, x_2\}$. Then $\mathrm{Mod}\mathcal{T} \simeq \mathrm{Set} \times \mathrm{Set}$, so $\mathcal{T}$ is Morita-equivalent to the sketch $\mathcal{S}$ with two objects, no nonidentity maps, no cones and no cocones.
The multi-bimodel $M\colon \mathbb{T}^{op} \mapsto \mathrm{Mod}\mathcal{S}$ inducing the equivalence $\mathrm{Mod}\mathcal{T} \simeq \mathrm{Mod}\mathcal{S}$ is given by $M(x_1) = \eta((*, \emptyset)), M(x_2) = \eta((\emptyset, *))$ and $M(y) = M(x_1) \times M(x_2)$. This equivalence can not be induced by a $\mathcal{T}$-model in $(\mathrm{Mod}\mathcal{S})^{op}$.

4 *

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