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Morphisms of 2-groupoids and low-dimensional cohomology of crossed modules

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Abstract. Given a morphism $P\colon \mathcal{G} \to \mathcal{H}$ of 2-groupoids, we construct a 6-term 2-exact sequence of cat-groups and pointed groupoids. We use this sequence to obtain an analogue for cat-groups (and, in particular, for crossed modules) of the fundamental exact sequence of non-abelian group cohomology. The link with simplicial topology is also explained.

Introduction

The aim of this paper is to obtain a basic result in low-dimensional cohomology of crossed modules. Homology and cohomology of crossed modules have been studied extensively, and a satisfactory theory has been developed (see [7] and the references therein, [14, 19, 20, 26]). The existing literature on this subject considers crossed modules and their morphisms as a category. Our point of view is that crossed modules are in a natural way the objects of a 2-category, and therefore they should be studied in a 2-dimensional context. This different point of view leads to

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a choice of limits and colimits which are the natural ones in our 2-categorical setting, that is bilimits, but which do not have a universal property in the underlying category of crossed modules. Accordingly, the notions of exactness and extension we consider are not equivalent to those studied in the previous papers devoted to this subject.

The result we look for to test our theory is a generalization to crossed modules of the fundamental exact sequence in non-abelian cohomology of groups [25]. To get this result, we adapt to crossed modules the method developed by Brown in [5], where the fundamental sequence is obtained as a special case of an exact sequence associated to a fibration of groupoids. In fact, to follow in a more transparent way the analogy with groups, we work with cat-groups instead of crossed modules, since the 2-category of (strict and small) cat-groups is biequivalent to the 2-category of crossed modules [6, 23].

The paper is organized as follows. In the first section we recall, for the reader's convenience and in a way convenient to be generalized to 2-groupoids, the result due to Brown. Section 2 is devoted to the construction of a 6-term 2-exact sequence of strict cat-groups and pointed groupoids from any morphism of 2-groupoids. For basic facts on cat-groups and 2-exact sequences we refer to [18, 27] and the bibliography therein; we recall in Section 2 the definitions we need. The idea of an higher-dimensional version of Brown's exact sequence comes from the paper [17] by Hardie, Kamps and the second author. The precise link between the main result of [17] and our 2-exact sequence is explained in Remark 2.7. In the third section we fix a cat-group $\mathbb{G}$ and an extension

$$\mathbb{A} \xrightarrow{i} \mathbb{B} \xrightarrow{j} \mathbb{C}$$

of $\mathbb{G}$-cat-groups. From such an extension we obtain, as a particular case of the sequence in Section 2, a 6-term 2-exact sequence of cat-groups and pointed groupoids

$$\mathbb{A}^{\mathbb{G}} \to \mathbb{B}^{\mathbb{G}} \to \mathbb{C}^{\mathbb{G}} \to H^1(\mathbb{G}, \mathbb{A}) \to H^1(\mathbb{G}, \mathbb{B}) \to H^1(\mathbb{G}, \mathbb{C})$$

which is the 2-dimensional generalization of the fundamental sequence in non-abelian group cohomology. As a corollary of the main result of Section 2, we also get a 9-term exact sequence of groups and pointed sets. Instead of exploiting the homological notion of 2-exactness, this 9-term sequence can also be obtained using classical results from simplicial topology. This is the content of Section 4.

1 Brown's exact sequence

As in the topological case, it is better to work with the homotopy fibre instead of the "set-theoretical" fibre. In this way, we can obtain an exact sequence from any functor between groupoids (and not only from a fibration). Moreover, we avoid some choices which would be quite hard to handle in the higher dimensional analogue developed in Section 2.

Recall that a groupoid $\mathbb{G}$ is a category in which each arrow is an isomorphism. Consider now a functor between groupoids

$$P \colon \mathbb{G} \to \mathbb{H}$$

and fix an object H in $\mathbb{H}$. The homotopy fibre $\mathbb{F}_H$ of P at the point H is the following comma groupoid:

- objects of $\mathbb{F}_H$ are pairs $(Y, y \colon P(Y) \to H)$, with Y an object of $\mathbb{G}$ and y an arrow in $\mathbb{H}$;

- an arrow $f\colon (Y_1, y_1) \to (Y_2, y_2)$ in $\mathbb{F}_H$ is an arrow $f\colon Y_1 \to Y_2$ in $\mathbb{G}$ such that $P(f) \cdot y_2 = y_1$ (composition denoted from left to right).

There is an obvious faithful functor $j\colon \mathbb{F}_H \to \mathbb{G}$. If X is an object of $\mathbb{G}$, we write $\mathbb{F}_X$ for $\mathbb{F}_{P(X)}$.

Now fix an object X in $\mathbb{G}$ and consider the following groups and pointed sets:

- $\pi_0(\mathbb{G})$, the set of isomorphism classes of objects of $\mathbb{G}$, pointed by the class of X ; $\pi_0(\mathbb{H})$, pointed by the class of $P(X)$; $\pi_0(\mathbb{F}_X)$, pointed by the class of $(X, 1_{P(X)})$;
- $\mathbb{G}(X) = \mathbb{G}(X, X)$, the group of automorphisms of the object X in $\mathbb{G}$; $\mathbb{H}(X) = \mathbb{H}(P(X), P(X))$, the group of automorphisms of the object $P(X)$ in $\mathbb{H}$; $\mathbb{F}_X(X) = \mathbb{F}_X((X, 1_{P(X)}), (X, 1_{P(X)}))$, the group of automorphisms of $(X, 1_{P(X)})$ in $\mathbb{F}_X$.

They can be connected by the following morphisms (square brackets are isomorphism classes):

- $j_X\colon \mathbb{F}_X(X) \to \mathbb{G}(X) \quad j_X(f\colon X \to X) = f$;
- $P_X\colon \mathbb{G}(X) \to \mathbb{H}(X) \quad P_X(f\colon X \to X) = P(f)$;
- $\pi_0(P)\colon \pi_0(\mathbb{G}) \to \pi_0(\mathbb{H}) \quad [X] \mapsto [P(X)]$;
- $\pi_0(j)\colon \pi_0(\mathbb{F}_X) \to \pi_0(\mathbb{G}) \quad [Y, y\colon P(Y) \to P(X)] \mapsto [Y]$;
- $\delta\colon \mathbb{H}(X) \to \pi_0(\mathbb{F}_X) \quad \delta(x\colon P(X) \to P(X)) = [X, x\colon P(X) \to P(X)]$.

Proposition 1.1 *With the previous notations, the sequence*

$$0 \to \mathbb{F}_X(X) \xrightarrow{j_X} \mathbb{G}(X) \xrightarrow{P_X} \mathbb{H}(X) \xrightarrow{\delta} \pi_0(\mathbb{F}_X) \xrightarrow{\pi_0(j)} \pi_0(\mathbb{G}) \xrightarrow{\pi_0(P)} \pi_0(\mathbb{H})$$

is exact.

Proof Consider an element $[Y, y]$ in $\pi_0(\mathbb{F}_X)$ and assume that $[Y] = [X]$ in $\pi_0(\mathbb{G})$. Then there is an arrow $y'\colon Y \to X$ in $\mathbb{G}$ and therefore $[Y, y] = [X, P(y')^{-1} \cdot y] = \delta(P(y')^{-1} \cdot y)$ because $y'\colon (Y, y) \to (X, P(y')^{-1} \cdot y)$ is an arrow in $\mathbb{F}_X$. The rest of the proof is straightforward. □

Now consider the strict fibre $\mathbb{S}_H$ of P at the point H :

- objects of $\mathbb{S}_H$ are the objects Y of $\mathbb{G}$ such that $P(Y) = H$;
- an arrow $f\colon Y_1 \to Y_2$ of $\mathbb{G}$ is in $\mathbb{S}_H$ if $P(f) = 1_H$.

There is, for each object H in $\mathbb{S}_H$, a full and faithful functor $i_H\colon \mathbb{S}_H \to \mathbb{F}_H$. Clearly, P is a fibration of groupoids [5] if and only if for each H the functor i_H is essentially surjective on objects. Therefore, if P is a fibration, we can replace $\mathbb{F}_X(X)$ and $\pi_0(\mathbb{F}_X)$ by $\mathbb{S}_X(X)$ and $\pi_0(\mathbb{S}_X)$ and we obtain Brown's exact sequence associated to a fibration of groupoids (Theorem 4.3 in [5]).

2 The 2-exact sequence

In this section we fix a morphism of 2-groupoids

$$P\colon \mathcal{G} \to \mathcal{H}$$

that is a 2-functor between 2-categories in which each arrow is an equivalence and each 2-cell is an isomorphism.

Fix an object H in $\mathcal{H}$; the homotopy fibre $\mathcal{F}_H$ of P at the point H is the following 2-groupoid:

- objects are pairs $(Y, y\colon P(Y) \to H)$, with Y an object in $\mathcal{G}$ and y an arrow in $\mathcal{H}$;

- arrows $(f, \varphi)\colon (Y_1, y_1) \to (Y_2, y_2)$ are pairs with $f\colon Y_1 \to Y_2$ an arrow in $\mathcal{G}$ and $\varphi\colon y_1 \Rightarrow P(f) \cdot y_2 \colon P(Y_1) \to H$ a 2-cell in $\mathcal{H}$;
- a 2-cell $\alpha\colon (f, \varphi) \Rightarrow (g, \psi)\colon (Y_1, y_1) \to (Y_2, y_2)$ is a 2-cell $\alpha\colon f \Rightarrow g$ in $\mathcal{G}$ such that the following diagram commutes

$$\begin{tikzcd} P(f) \cdot y_2 \arrow[rr, Rightarrow, "P(\alpha) \cdot y_2"] & & P(g) \cdot y_2 \\ & y_1 \arrow[ul, Rightarrow, "\varphi"] \arrow[ur, Rightarrow, "\psi"'] & \end{tikzcd}$$

There is a morphism $j\colon \mathcal{F}_H \to \mathcal{G}$ which sends $\alpha\colon (f, \varphi) \Rightarrow (g, \psi)\colon (Y_1, y_1) \to (Y_2, y_2)$ to $\alpha\colon f \Rightarrow g\colon Y_1 \to Y_2$. The morphism j is faithful on arrows and on 2-cells. If X is an object of $\mathcal{G}$, we write $\mathcal{F}_X$ for $\mathcal{F}_{P(X)}$.

We recall now the notion of 2-exact sequence for pointed groupoids (and, in particular, for cat-groups, i.e. monoidal groupoids in which each object is invertible, up to isomorphisms, w.r.t. the tensor product). Morphisms of pointed groupoids (cat-groups) are pointed functors (monoidal functors). A natural transformation between pointed (monoidal) functors is always assumed to be pointed (monoidal). Let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism of pointed groupoids; its homotopy kernel $kF\colon KerF \to \mathbb{G}$ is the homotopy fibre (in the sense of Section 1) of F on the base object I of $\mathbb{H}$. There is a natural transformation $\kappa F\colon kF \cdot F \Rightarrow 0$ (0 is the morphism which sends each arrow to the identity of I) given, for each object (Y, y) of $KerF$, by $y\colon F(Y) \to I$.

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "F"] \arrow[d, phantom, "\kappa F \Downarrow"] & \\ KerF \arrow[ur, "kF"] \arrow[rr, "0"'] & {} & \mathbb{H} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{G} \arrow[dr, "F"] \arrow[d, phantom, "\varphi \Downarrow"] & \\ \mathbb{K} \arrow[ur, "G"] \arrow[rr, "0"'] & {} & \mathbb{H} \end{tikzcd}$$

Moreover, given a pointed groupoid $\mathbb{K}$, a morphism G and a natural transformation φ as in the previous diagram, there is a unique comparison morphism $G'\colon \mathbb{K} \to KerF$, $G'(g\colon A_1 \to A_2) = G(g)\colon (G(A_1), \varphi_{A_1}) \to (G(A_2), \varphi_{A_2})$, such that $G' \cdot kF = G$ and $G' \cdot \kappa F = \varphi$ (compare with [15]). The universal property of $(KerF, kF, \kappa F)$ as a bilimit, discussed in [18, 27], determines it uniquely, up to equivalence.

Definition 2.1 Consider two morphisms G, F and a natural transformation φ of pointed groupoids as in the previous diagram; we say that the triple (G, φ, F) is 2-exact if the comparison $G'\colon \mathbb{K} \to KerF$ is full and essentially surjective on objects.

Now come back to the 2-functor between 2-groupoids $P\colon \mathcal{G} \to \mathcal{H}$ and fix an object X of $\mathcal{G}$. We can consider the following three hom-categories, which are in fact strict cat-groups: $\mathcal{G}(X) = \mathcal{G}(X, X)$, $\mathcal{H}(X) = \mathcal{H}(P(X), P(X))$ and $\mathcal{F}_X(X) = \mathcal{F}_X((X, 1_{P(X)}), (X, 1_{P(X)}))$. Moreover, we can consider the classifying groupoid $cl(\mathcal{G})$ of the 2-groupoid $\mathcal{G}$: $cl(\mathcal{G})$ has the same objects as $\mathcal{G}$ and 2-isomorphism classes of arrows of $\mathcal{G}$ as arrows. The groupoid $cl(\mathcal{G})$ is pointed by the object X. Similarly, we have the groupoid $cl(\mathcal{H})$ pointed by $P(X)$ and the groupoid $cl(\mathcal{F}_X)$ pointed by $(X, 1_{P(X)})$. These cat-groups and pointed groupoids can be connected by the following morphisms (square brackets are 2-isomorphism classes of arrows):

- $j_X \colon \mathcal{F}_X(X) \to \mathcal{G}(X)$
  $\alpha\colon (f,\varphi) \Rightarrow (g,\psi)\colon (X,1_{P(X)}) \to (X,1_{P(X)}) \ \mapsto \ \alpha\colon f \Rightarrow g\colon X \to X$
- $P_X \colon \mathcal{G}(X) \to \mathcal{H}(X)$
  $\alpha\colon f \Rightarrow g\colon X \to X \ \mapsto \ P(\alpha)\colon P(f) \Rightarrow P(g)\colon P(X) \to P(X)$
- $cl(j)\colon cl(\mathcal{F}_X) \to cl(\mathcal{H})$
  $[f,\varphi]\colon (Y_1,y_1) \to (Y_2,y_2) \ \mapsto \ [f]\colon Y_1 \to Y_2$
- $cl(P)\colon cl(\mathcal{G}) \to cl(\mathcal{H})$
  $[f]\colon Y_1 \to Y_2 \ \mapsto \ [P(f)]\colon P(Y_1) \to P(Y_2)$
- $\delta\colon \mathcal{H}(X) \to cl(\mathcal{F}_X)$
  $\beta\colon h \Rightarrow k\colon P(X) \to P(X) \ \mapsto \ [1_X,\beta]\colon (X,h) \to (X,k).$

Proposition 2.2 *With the previous notations, the sequence*

$$\mathcal{F}_X(X) \xrightarrow{j_X} \mathcal{G}(X) \xrightarrow{P_X} \mathcal{H}(X) \xrightarrow{\delta} cl(\mathcal{F}_X) \xrightarrow{cl(j)} cl(\mathcal{G}) \xrightarrow{cl(P)} cl(\mathcal{H})$$

with the obvious natural transformations $j_X \cdot P_X \Rightarrow 0\,,\ P_X \cdot \delta \Rightarrow 0\,,\ \delta \cdot cl(j) \Rightarrow 0\,,\ cl(j)\cdot cl(P) \Rightarrow 0\,,$ *is 2-exact.*

Proof 1) 2-exactness in $\mathcal{G}(X)$: it is straightforward to verify that the functor $j_X \colon \mathcal{F}_X(X) \to \mathcal{G}(X)$ is exactly the kernel of $P_X \colon \mathcal{G}(X) \to \mathcal{H}(X)$.

2) 2-exactness in $\mathcal{H}(X)$: consider the comparison $P'_X \colon \mathcal{G}(X) \to Ker\delta$
$\alpha\colon f \Rightarrow g\colon X \to X \ \mapsto \ P(\alpha)\colon (P(f),[f,1_{P(f)}]) \Rightarrow (P(g),[g,1_{P(g)}])$
- P'_X is essentially surjective: given an object $(h,[f,\varphi])$ in $Ker\delta$, we obtain an arrow $\varphi\colon (h,[f,\varphi]) \Rightarrow \delta(f)$ in $Ker\delta$;
- P'_X is full: given an arrow $\beta\colon \delta(f) \Rightarrow \delta(g)$ in $Ker\delta$, then $\delta(\beta)\cdot[g,1_{P(g)}] = [f,1_{P(f)}]$, but this means that there exists a 2-cell $\alpha\colon f \Rightarrow g$ such that $P(\alpha) = \beta$.

3) 2-exactness in $cl(\mathcal{G})$: consider the comparison $j'\colon cl(\mathcal{F}_X) \to Ker(cl(P))$
$[f,\varphi]\colon (Y_1,y_1) \to (Y_2,y_2) \ \mapsto \ [f]\colon (Y_1,[y_1]) \to (Y_2,[y_2])$
- j' is essentially surjective: obvious;
- j' is full: let $[f]\colon j'(Y_1,y_1) \to j'(Y_2,y_2)$ be an arrow in $Ker(cl(P))$, this means that there exists a 2-cell $\varphi\colon y_1 \Rightarrow P(f)\cdot y_2$ and then $[f,\varphi]\colon (Y_1,y_1) \to (Y_2,y_2)$ is an arrow in $cl(\mathcal{F}_X)$.

4) 2-exactness in $cl(\mathcal{F}_X)$: consider the comparison $\delta'\colon \mathcal{H}(X) \to Ker(cl(j))$
$\beta\colon h \Rightarrow k\colon P(X) \to P(X) \ \mapsto \ [1_X,\beta]\colon (X,h,[1_X]) \to (X,k,[1_X])$
- δ' is full: let $[f,\varphi]\colon \delta'(h) \to \delta'(k)$ be an arrow in $Ker(cl(j))$, then $[f]\cdot[1_X] = [1_X]$, that is there exists a 2-cell $\alpha\colon 1_X \Rightarrow f$. We obtain $\beta\colon h \Rightarrow k$ in $\mathcal{H}(X)$ in the following way :

$$\beta = (h \xRightarrow{\varphi} P(f)\cdot k \xRightarrow{P(\alpha)^{-1}\cdot k} k)\ ;$$

- δ' is essentially surjective: consider an object

$$(Y, y\colon P(Y) \to P(X), [x]\colon Y \to X)$$

in $Ker(cl(j))$, then $P(x)^{-1}\cdot y\colon P(X) \to P(X)$ is an object in $\mathcal{H}(X)$ and $[x,c]\colon (Y,y,[x]) \to \delta'(P(x)^{-1}\cdot y)$ is an arrow in $Ker(cl(j))$, where c is the canonical 2-cell $c\colon y \Rightarrow P(x)\cdot P(x)^{-1}\cdot y\,.$

□

As in Section 1, if $(\mathbb{G},I)$ is a pointed groupoid (a cat-group), we write $\pi_0(\mathbb{G})$ for the pointed set (the group) of isomorphism classes of objects and $\pi_1(\mathbb{G})$ for the (abelian) group of automorphisms $\mathbb{G}(I,I)$. π_0 and π_1 extend to morphisms and carry

2-exact sequences on exact sequences of pointed sets or groups. Finally, observe that if $\mathcal{G}$ is a 2-groupoid and X is a chosen object in $\mathcal{G}$, then $\pi_1(cl(\mathcal{G})) = \pi_0(\mathcal{G}(X))$. In a similar way, if $P\colon \mathcal{G} \to \mathcal{H}$ is a 2-functor, then $\pi_1(cl(P)) = \pi_0(P_X)$.

Corollary 2.3 *Let $P\colon \mathcal{G} \to \mathcal{H}$ be a 2-functor between 2-groupoids and fix an object X in $\mathcal{G}$; the following is an exact sequence of groups and pointed sets (the last three terms)*

$$0 \to \pi_1(\mathcal{F}_X(X)) \to \pi_1(\mathcal{G}(X)) \to \pi_1(\mathcal{H}(X)) \to$$

$$\pi_1(cl(\mathcal{F}_X)) = \pi_0(\mathcal{F}_X(X)) \to \pi_1(cl(\mathcal{G})) = \pi_0(\mathcal{G}(X)) \to \pi_1(cl(\mathcal{H})) = \pi_0(\mathcal{H}(X))$$

$$\to \pi_0(cl(\mathcal{F}_X)) \to \pi_0(cl(\mathcal{G})) \to \pi_0(cl(\mathcal{H}))\,.$$

Proof As far as exactness in $\pi_1(\mathcal{F}_X(X))$ is concerned, observe that j_X is the kernel of P_X, so it is faithful and then $\pi_1(j_X)$ is injective. The rest follows from the 2-exactness of the sequence in Proposition 2.2 and the previous remarks on π_0 and π_1. □

Remark 2.4 If $P\colon \mathbb{G} \to \mathbb{H}$ is a functor between groupoids, we can look at it as a 2-functor between discrete 2-groupoids (2-groupoids with no non-trivial 2-cells). The exact sequence of Corollary 2.3 reduces then to the exact sequence of Proposition 1.1, because the first non-trivial term is $\pi_0(\mathcal{F}_X(X))$.

Remark 2.5 Brown's exact sequence of Proposition 1.1 satisfies a strong exactness condition in $\pi_0(\mathbb{F}_X)$, which is the transition point between groups and pointed groupoids. The 2-dimensional analogue of strong exactness has been formulated in [16]. It is not difficult to prove that the sequence of Proposition 2.2 is strongly 2-exact in $cl(\mathcal{F}_X)$, let us just observe that the needed action $\mathcal{H}(X) \times cl(\mathcal{F}_X) \to cl(\mathcal{F}_X)$ sends $(f\colon P(X) \to P(X), (Y, y\colon P(Y) \to P(X)))$ into $(Y, y \cdot f\colon P(Y) \to P(X) \to P(X))$.

Remark 2.6 Proposition 2.2 and Corollary 2.3 hold also for a morphism $P\colon \mathcal{G} \to \mathcal{H}$ of bigroupoids, that is a pseudo-functor between bicategories [2, 3] in which each arrow is an equivalence and each 2-cell is an isomorphism. The generalization is strightforward: just observe that the homotopy fibre $\mathcal{F}_H$ inherits a structure of bicategory from that of $\mathcal{G}$. Clearly, if $\mathcal{G}$ and $\mathcal{H}$ are bigroupoids, the cat-groups $\mathcal{F}_X(X), \mathcal{G}(X)$ and $\mathcal{H}(X)$ of Proposition 2.2 are no longer strict. In Section 3 we will use this more general version of Proposition 2.2.

Remark 2.7 Recall that a morphism of bigroupoids $P\colon \mathcal{G} \to \mathcal{H}$ is a fibration if the functor $cl(P)\colon cl(\mathcal{G}) \to cl(\mathcal{H})$ is a fibration of groupoids and for each Y_1, Y_2 in $\mathcal{G}$ the functor $P_{Y_1,Y_2}\colon \mathcal{G}(Y_1, Y_2) \to \mathcal{H}(P(Y_1), P(Y_2))$ is a fibration of groupoids [17, 22]. This is equivalent to ask that for each object X in $\mathcal{G}$, the induced functor $St_P(X)\colon St_{\mathcal{G}}(X) \to St_{\mathcal{H}}(P(X))$ (where the "star-groupoid" $St_{\mathcal{G}}(X)$ is the groupoid having morphisms $y\colon X \to Y$ as objects and 2-cells $\alpha\colon y_1 \Rightarrow y_2\colon X \to Y$ as arrows) is an essentially surjective fibration. When P is a fibration, one easily checks that for each object H of $\mathcal{H}$, the homotopy fibre $\mathcal{F}_H$ is biequivalent to the strict fibre $\mathcal{S}_H$ (i.e. the sub-bigroupoid of $\mathcal{G}$ having as 2-cells the 2-cells $\alpha\colon f \Rightarrow g\colon Y_1 \to Y_2$ such that $P(\alpha)$ is the identity 2-cell of 1_H). Therefore, if P is a fibration, $\mathcal{F}_X(X)$ and $cl(\mathcal{F}_X)$ are equivalent to $\mathcal{S}_X(X)$ and $cl(\mathcal{S}_X)$ and the sequence of Corollary 2.3 is exactly the Hardie-Kamps-Kieboom 9-term exact sequence associated to a fibration of bigroupoids (Theorem 2.4 in [17]).

Remark 2.8 Proposition 2.2 can be also used to construct a Picard-Brauer 2-exact sequence from a homomorphism of unital commutative rings. In fact such a morphism induces a pseudo-functor between the bigroupoids having Azumaya algebras as objects, invertible bimodules as arrows and bimodule isomorphisms as 2-cells. Compare with [27], where a similar 2-exact sequence is obtained using homotopy cokernels instead of homotopy fibres.

3 The cohomology sequence

Let us fix a cat-group $\mathbb{G}$. A $\mathbb{G}$-cat-group is a pair $(\mathbb{C}, \gamma)$ where $\mathbb{C}$ is a cat-group and $\gamma\colon \mathbb{G} \to Aut\mathbb{C}$ is a monoidal functor with codomain the cat-group of monoidal auto-equivalences of $\mathbb{C}$. $\mathbb{G}$-cat-groups are the objects of a 2-category, having equivariant monoidal functors as arrows and compatible monoidal transformations as 2-cells (see [9, 12] for more details and for an equivalent definition of $\mathbb{G}$-cat-group in terms of an action $\mathbb{G} \times \mathbb{C} \to \mathbb{C}$). Observe that homotopy kernels in the 2-category of $\mathbb{G}$-cat-groups are computed as in the 2-category of cat-groups (in other words, if $j\colon (\mathbb{B}, \beta) \to (\mathbb{C}, \gamma)$ is a morphism of $\mathbb{G}$-cat-groups and $i\colon \mathbb{A} \to \mathbb{B}$ is its kernel as a morphism of cat-groups, then $\mathbb{A}$ inherits from $\mathbb{B}$ a structure $\alpha\colon \mathbb{G} \to Aut\mathbb{A}$ of $\mathbb{G}$-cat-group such that $i\colon \mathbb{A} \to \mathbb{B}$ is a morphism of $\mathbb{G}$-cat-groups).

If $(\mathbb{C}, \gamma)$ is a $\mathbb{G}$-cat-group, a *derivation* is a pair $\langle M\colon \mathbb{G} \to \mathbb{C}, \mathbf{m}\rangle$ where M is a functor and

$$\mathbf{m} = \{m_{X,Y}\colon M(X) \otimes \gamma(X)(M(Y)) \to M(X \otimes Y)\}_{X,Y \in \mathbb{G}}$$

is a natural family of coherent isomorphisms (for more details, see [13], where $\mathbb{C}$ is assumed to be braided, or [11], where $\mathbb{G}$ is discrete). Derivations are the objects of a bigroupoid $\mathcal{Z}^1(\mathbb{G}, \mathbb{C})$:

- an arrow is a pair $\langle C, \mathbf{c}\rangle\colon \langle M, \mathbf{m}\rangle \to \langle N, \mathbf{n}\rangle$ with $C \in \mathbb{C}$ and

  $$\mathbf{c} = \{c_X\colon M(X) \otimes \gamma(X)(C) \to C \otimes N(X)\}_{X \in \mathbb{G}}$$

  is a natural family of isomorphisms, compatible with $\mathbf{m}$ and $\mathbf{n}$;
- a 2-cell $f\colon \langle C, \mathbf{c}\rangle \Rightarrow \langle C', \mathbf{c}'\rangle$ is an arrow $f\colon C \to C'$ in $\mathbb{C}$ compatible with $\mathbf{c}$ and $\mathbf{c}'$.

In $\mathcal{Z}^1(\mathbb{G}, \mathbb{C})$ there is a trivial derivation

$$\theta_{\mathbb{C}} = \langle 0\colon \mathbb{G} \to \mathbb{C}, \{I \otimes \gamma(X)(I) \simeq I\}\rangle$$

and the cat-group $\mathcal{Z}^1(\mathbb{G}, \mathbb{C})(\theta_{\mathbb{C}}, \theta_{\mathbb{C}})$ is the cat-group $\mathbb{C}^{\mathbb{G}}$ of $\mathbb{G}$-*invariant objects*. Explicitly:

- an object of $\mathbb{C}^{\mathbb{G}}$ is a pair $\langle C, \mathbf{c}\rangle$ with $C \in \mathbb{C}$ and

  $$\mathbf{c} = \{c_X\colon \gamma(X)(C) \to C\}_{X \in \mathbb{G}}$$

  a natural family of isomorphisms compatible with the monoidal structure of $\mathbb{G}$;
- an arrow $f\colon \langle C, \mathbf{c}\rangle \to \langle D, \mathbf{d}\rangle$ in $\mathbb{C}^{\mathbb{G}}$ is an arrow $f\colon C \to D$ in $\mathbb{C}$ such that the following diagram commutes for each $X \in \mathbb{G}$

$$\begin{tikzcd} \gamma(X)(C) \arrow[r, "c_X"] \arrow[d, "\gamma(X)(f)"'] & C \arrow[d, "f"] \\ \gamma(X)(D) \arrow[r, "d_X"'] & D \end{tikzcd}$$

A morphism $j\colon (\mathbb{B},\beta) \to (\mathbb{C},\gamma)$ of $\mathbb{G}$-cat-groups induces a pseudo-functor

$$j_*\colon \mathcal{Z}^1(\mathbb{G},\mathbb{B}) \to \mathcal{Z}^1(\mathbb{G},\mathbb{C}).$$

This pseudo-functor j_* sends a derivation $\langle H\colon \mathbb{G} \to \mathbb{B}, \mathbf{h}\rangle$ into the derivation $\langle H \cdot j\colon \mathbb{G} \to \mathbb{B} \to \mathbb{C}, j(\mathbf{h})\rangle$, where $j(\mathbf{h})$ is defined by the following composition

$$\begin{array}{c} j(H(X)) \otimes \gamma(X)(j(H(Y))) \\ \big\downarrow \simeq \\ j(H(X)) \otimes j(\beta(X)(H(Y))) \\ \big\downarrow \simeq \\ j(H(X) \otimes \beta(X)(H(Y))) \\ \big\downarrow j(h_{X,Y}) \\ j(H(X \otimes Y)) \end{array}$$

(the first isomorphism is the equivariant structure of j, the second one is its monoidal structure) and is defined in an obvious way on arrows and 2-cells.

In the next lemma we need the homotopy fibre $\mathbb{F}$ of j_* at the point $\theta_{\mathbb{B}}$. Let us describe explicitly the objects of $\mathbb{F}$ (without loosing in generality, we can assume that $j(I) = I$, so that $j_*(\theta_{\mathbb{B}}) = \theta_{\mathbb{C}}$). An object of $\mathbb{F}$ is a 4-tuple

$$\langle\, \mathcal{D} = \langle H\colon \mathbb{G} \to \mathbb{B}, \mathbf{h}\rangle \in \mathcal{Z}^1(\mathbb{G},\mathbb{B}),\ \langle \overline{H} \in \mathbb{C}, \overline{\mathbf{h}}\rangle \in \mathcal{Z}^1(\mathbb{G},\mathbb{C})(j_*(\mathcal{D}),\theta_{\mathbb{C}})\,\rangle$$

with

$$\mathbf{h} = \{h_{X,Y}\colon H(X) \otimes \beta(X)(H(Y)) \to H(X \otimes Y)\}_{X,Y\in\mathbb{G}}$$
$$\overline{\mathbf{h}} = \{\overline{h}_X\colon j(H(X)) \otimes \gamma(X)(\overline{H}) \to \overline{H}\}_{X\in\mathbb{G}}$$

Lemma 3.1 *Consider an essentially surjective morphism $j\colon \mathbb{B} \to \mathbb{C}$ of $\mathbb{G}$-cat-groups and its homotopy kernel*

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "j"] \arrow[d, phantom, "\kappa j \Downarrow"] & \\ \mathbb{A} = Ker j \arrow[ur, "i"] \arrow[rr, "0"'] & \ & \mathbb{C} \end{tikzcd}$$

The homotopy fibre $\mathbb{F}$ of $j_*\colon \mathcal{Z}^1(\mathbb{G},\mathbb{B}) \to \mathcal{Z}^1(\mathbb{G},\mathbb{C})$ at the point $\theta_{\mathbb{B}}$ is biequivalent to the bigroupoid of derivations $\mathcal{Z}^1(\mathbb{G},\mathbb{A})$.

Proof Given an object in $\mathcal{Z}^1(\mathbb{G},\mathbb{A})$

$$\langle F\colon \mathbb{G} \to \mathbb{A}, \mathbf{f} = \{f_{X,Y}\colon F(X) \otimes \alpha(X)(F(Y)) \to F(X \otimes Y)\}_{X,Y\in\mathbb{G}}\rangle$$

we get an object in $\mathbb{F}$

$$\langle\, \langle F \cdot i\colon \mathbb{G} \to \mathbb{A} \to \mathbb{B}, i(\mathbf{f})\rangle,\ \langle I \in \mathbb{C}, \{\kappa j_{F(X)}\colon j(i(F(X))) \to I\}_{X\in\mathbb{G}}\rangle\,\rangle$$

This construction extends to a 2-functor $\epsilon\colon \mathcal{Z}^1(\mathbb{G},\mathbb{A}) \to \mathbb{F}$ which is always locally an equivalence (even if $j\colon \mathbb{B} \to \mathbb{C}$ is not essentially surjective). Let us check that ϵ is surjective on objects up to equivalence. Let $\langle\, \langle H\colon \mathbb{G} \to \mathbb{B}, \mathbf{h}\rangle, \langle \overline{H} \in \mathbb{C}, \overline{\mathbf{h}}\rangle\,\rangle$ be an object of $\mathbb{F}$. Since $j\colon \mathbb{B} \to \mathbb{C}$ is essentially surjective, there is an object $Z \in \mathbb{B}$ and an arrow $z\colon \overline{H} \to j(Z)$. Now we can construct a functor

$$D\colon \mathbb{G} \to \mathbb{B} \qquad X \mapsto Z^* \otimes H(X) \otimes \beta(X)(Z)$$

(Z^* is a dual of Z in the cat-group $\mathbb{B}$) which has a structure of derivation $\mathbf{d} = \{d_{X,Y}\colon D(X)\otimes\beta(X)(D(Y)) \to D(X\otimes Y)\}$ obtained from that of H in the following way

$$\begin{array}{c}
D(X)\otimes\beta(X)(D(Y)) \\
\Big\downarrow{=} \\
Z^*\otimes H(X)\otimes\beta(X)(Z)\otimes\beta(X)(Z^*\otimes H(Y)\otimes\beta(Y)(Z)) \\
\Big\downarrow{\simeq} \\
Z^*\otimes H(X)\otimes\beta(X)(Z)\otimes\beta(X)(Z^*)\otimes\beta(X)(H(Y)\otimes\beta(Y)(Z)) \\
\Big\downarrow{\simeq} \\
Z^*\otimes H(X)\otimes\beta(X)(H(Y))\otimes\beta(X)(\beta(Y)(Z)) \\
\Big\downarrow{\simeq} \\
Z^*\otimes H(X)\otimes\beta(X)(H(Y))\otimes\beta(X\otimes Y)(Z) \\
\Big\downarrow{1\otimes h_{X,Y}\otimes 1} \\
Z^*\otimes H(X\otimes Y)\otimes\beta(X\otimes Y)(Z) = D(X\otimes Y)
\end{array}$$

Observe now that the functor $D\colon \mathbb{G}\to\mathbb{B}$ factors through the kernel of j. Indeed, if $X\in\mathbb{G}$, we have

$$\begin{array}{c}
j(D(X)) = j(Z^*\otimes H(X)\otimes\beta(X)(Z)) \\
\Big\downarrow{\simeq} \\
j(Z^*)\otimes j(H(X))\otimes j(\beta(X)(Z)) \\
\Big\downarrow{\simeq} \\
j(Z^*)\otimes j(H(X))\otimes\gamma(X)(j(Z)) \\
\Big\downarrow{z^*\otimes 1\otimes z^{-1}} \\
\overline{H}^*\otimes j(H(X))\otimes\gamma(X)(\overline{H}) \\
\Big\downarrow{1\otimes\overline{h}_X} \\
\overline{H}^*\otimes\overline{H}\simeq I
\end{array}$$

Let us call $\widetilde{D}\colon \mathbb{G}\to\mathbb{A}$ the factorization of $D\colon\mathbb{G}\to\mathbb{B}$ through the kernel $\mathbb{A}$. The structure $\mathbf{d}$ of the derivation D pass to $\widetilde{D}$ because $\overline{\mathbf{h}}$ is compatible with $j(\mathbf{h})$ and with the structure of the trivial derivation $\theta_{\mathbb{C}}$. In this way, we have built up an object $\langle\widetilde{D}\colon\mathbb{G}\to\mathbb{A},\tilde{\mathbf{d}}\rangle$ of $\mathcal{Z}^1(\mathbb{G},\mathbb{A})$. Finally, an arrow

$$\langle\,\langle H\colon\mathbb{G}\to\mathbb{B},\mathbf{h}\rangle,\,\langle\overline{H}\in\mathbb{C},\overline{\mathbf{h}}\rangle\,\rangle\to\epsilon\langle\widetilde{D}\colon\mathbb{G}\to\mathbb{A},\tilde{\mathbf{d}}\rangle$$

in $\mathbb{F}$ is provided by $Z\in\mathbb{B}$, $z\colon\overline{H}\to j(Z)$ and by the family of canonical isomorphisms

$$\{H(X)\otimes\beta(X)(Z)\simeq Z\otimes Z^*\otimes H(X)\otimes\beta(X)(Z) = Z\otimes D(X)\}_{X\in\mathbb{G}}$$

□

An essentially surjective morphism with its homotopy kernel

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "j"] \arrow[d, phantom, "\kappa j \Downarrow"] & \\ \mathbb{A} \arrow[ur, "i"] \arrow[rr, "0"'] & & \mathbb{C} \end{tikzcd}$$

is called in [4, 8, 24] an *extension*. Putting together Proposition 2.2 and the previous lemma, we get our generalization of the fundamental sequence in non-abelian group cohomology. We write $H^1(\mathbb{G},\mathbb{C})$ for $cl(\mathcal{Z}^1(\mathbb{G},\mathbb{C}))$.

Corollary 3.2 *Consider an extension of* $\mathbb{G}$*-cat-groups*

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "j"] \arrow[d, phantom, "\kappa j \Downarrow"] & \\ \mathbb{A} \arrow[ur, "i"] \arrow[rr, "0"'] & & \mathbb{C} \end{tikzcd}$$

There is a 2-exact sequence of cat-groups and pointed groupoids

$$\mathbb{A}^{\mathbb{G}} \to \mathbb{B}^{\mathbb{G}} \to \mathbb{C}^{\mathbb{G}} \to H^1(\mathbb{G},\mathbb{A}) \to H^1(\mathbb{G},\mathbb{B}) \to H^1(\mathbb{G},\mathbb{C})\,.$$

Remark 3.3 Observe that if the cat-groups $\mathbb{G},\mathbb{B},\mathbb{C}$ and the monoidal functor $j\colon \mathbb{B} \to \mathbb{C}$ are strict, then $j_*\colon \mathcal{Z}^1(\mathbb{G},\mathbb{B}) \to \mathcal{Z}^1(\mathbb{G},\mathbb{C})$ is a 2-functor between 2-groupoids, and the cat-groups involved in the previous corollary are strict, that is they are crossed modules.

To end, we sketch an equivalent description of the bigroupoid $\mathcal{Z}^1(\mathbb{G},\mathbb{C})$ of derivations using the semi-direct product. Let us start with a general construction: if $\mathcal{G}$ and $\mathcal{H}$ are bicategories, $[\mathcal{G},\mathcal{H}]$ is the bicategory of pseudo-functors $\mathcal{G} \to \mathcal{H}$, pseudo-natural transformations and modifications [2, 3]. If $\mathbb{G}$ and $\mathbb{H}$ are cat-groups, we can see them as bicategories with only one object, and $[\mathbb{G},\mathbb{H}]$ is now a bigroupoid. Explicitly:

- an object of $[\mathbb{G},\mathbb{H}]$ is a monoidal functor $F\colon \mathbb{G} \to \mathbb{H}$;
- an arrow $(H,\varphi)\colon F \to G\colon \mathbb{G} \to \mathbb{H}$ is an object H of $\mathbb{H}$ and a natural transformation

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "-\otimes H"] & \\ \mathbb{G} \arrow[ur, "F"] \arrow[dr, "G"'] \arrow[rr, phantom, "\varphi \Downarrow"] & & \mathbb{H} \\ & \mathbb{H} \arrow[ur, "H\otimes -"'] & \end{tikzcd}$$

making commutative the following diagrams

$$\begin{tikzcd} F(X)\otimes F(Y)\otimes H \arrow[r, "1\otimes\varphi_Y"] \arrow[d, "\simeq"'] & F(X)\otimes H\otimes G(Y) \arrow[r, "\varphi_X\otimes 1"] & H\otimes G(X)\otimes G(Y) \arrow[d, "\simeq"] \\ F(X\otimes Y)\otimes H \arrow[rr, "\varphi_{X\otimes Y}"'] & & H\otimes G(X\otimes Y) \end{tikzcd}$$

$$\begin{tikzcd}
F(I)\otimes H \arrow[d, "\varphi_I"'] & I\otimes H \simeq H \arrow[l, "\simeq"'] \arrow[d, "1"] \\
H\otimes G(I) & H\otimes I \simeq H \arrow[l, "\simeq"]
\end{tikzcd}$$

Observe that composition of arrows and parallel composition of 2-cells are defined using the tensor product in $\mathbb{H}$. Finally, a morphism of cat-groups $P\colon \mathbb{H} \to \mathbb{K}$ induces a pseudo-functor $P_*\colon [\mathbb{G},\mathbb{H}] \to [\mathbb{G},\mathbb{K}]$.

Now we apply the previous construction to a particular case. Consider a cat-group $\mathbb{G}$ and a $\mathbb{G}$-cat-group $(\mathbb{C},\gamma\colon \mathbb{G} \to Aut\mathbb{C})$. Following [12], we can construct the semi-direct product $\mathbb{G}\times_\gamma \mathbb{C}$ together with the projection $P\colon \mathbb{G}\times_\gamma\mathbb{C} \to \mathbb{G}$, which is a monoidal functor. Therefore, we have a pseudo-functor

$$P_*\colon [\mathbb{G},\mathbb{G}\times_\gamma \mathbb{C}] \to [\mathbb{G},\mathbb{G}]$$

and it is possible to construct a biequivalence from $\mathcal{Z}^1(\mathbb{G},\mathbb{C})$ to the homotopy fibre of P_* at the point $Id_{\mathbb{G}} \in [\mathbb{G},\mathbb{G}]$. This biequivalence is another way to formulate the universal property of the semi-direct product studied in [13].

4 The simplicial topological point of view

As with the original paper [5] of Brown, we are motivated by homotopy theory and the classical exact sequences of homotopy groups and pointed sets which occur there, although in the previous sections the linkage to 2-types of topological spaces is far in the background. Indeed, because our bicategorical notion of 2-exact sequence, the presentation has been more homological/algebraic in feeling.

Nevertheless, the link to simplicial homotopy theory as pioneered by Daniel Kan, John Moore, and John Milnor in the fifties is quite direct. To briefly review the relevant parts of this theory [1], recall that as observed by Kan, the property of the singular complex of a topological space which permits one to combinatorially define *all* of the homotopy groups at any base point is that the singular complex has a simple simplicial horn-lifting property which makes it a “Kan complex”, and that a corresponding “Kan fibration” property (that corresponds essentially to the lifting properties of a fibration of spaces) is all that is needed to associate to a pointed simplicial Kan fibration f with fiber F :

$$F \subset E \longrightarrow B$$

[1]For more detail see [21], [10], or [1]

a long exact sequence of pointed sets, groups and abelian groups:

$$\begin{tikzcd}
\pi_n(F) \arrow[r, "\pi_n(i)"] & \pi_n(E) \arrow[r, "\pi_n(f)"] & \pi_n(B) \arrow[dll, dotted, no head] \\
\pi_2(F) \arrow[r, "\pi_2(i)"'] & \pi_2(E) \arrow[r, "\pi_2(f)"] & \pi_2(B) \arrow[dll, "\delta"'] \\
\pi_1(F) \arrow[r, "\pi_1(i)"'] & \pi_1(E) \arrow[r, "\pi_1(f)"] & \pi_1(B) \arrow[dll, "\delta"'] \\
\pi_0(F) \arrow[r, "\pi_0(i)"'] & \pi_0(E) \arrow[r, "\pi_0(f)"] & \pi_0(B).
\end{tikzcd}$$

Now associated with any pointed simplicial complex X is the contractible complex $\mathcal{P}(X)$ of "based paths" of X which has as its 0-simplices the 1-simplices of X of the form $x \longrightarrow 0$, where $0 \in X_0$ is the base point. It is supplied with a canonical pointed simplicial map (last face) $d_n(X)\colon \mathcal{P}(X) \longrightarrow X$ which is a Kan fibration provided that X is a Kan complex. The fiber of this simplicial map is then also a Kan complex and is, of course, the complex $\Omega(X)$ of loops of X at the base point of X.

$$\Omega(X) \subset \mathcal{P}(X) \longrightarrow X$$

The long exact sequence associated with this fibration (since $\pi_i(\mathcal{P}(X)) = \{0\}$) just recapitulates the familiar sequence of isomorphisms

$$\pi_i(\Omega(X)) \simeq \pi_{i+1}(X) \quad i \geq 0.$$

Thus if $f\colon X \longrightarrow Y$ is a pointed simplicial map of Kan complexes, one can form the pullback along f of the fibration $d_n(Y)\colon \mathcal{P}(Y) \longrightarrow Y$

$$\begin{tikzcd}
\Omega(Y) \arrow[r, "\cong"] \arrow[d, hook] & \Omega(Y) \arrow[d, hook] \\
\Gamma(f) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{P}(Y) \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The fibers at the base point are then isomorphic and pr_1, as a pullback of a fibration, is itself a fibration. We then obtain a long exact sequence associated with the fibration pr_1,

$$\begin{tikzcd}
\pi_n(\Omega(Y)) \arrow[r, "\pi_n(i)"] & \pi_n(\Gamma(f)) \arrow[r, "\pi_n(f)"] & \pi_n(X) \arrow[dll, dotted, no head] \\
\pi_2(\Omega(Y)) \arrow[r, "\pi_2(i)"'] & \pi_2(\Gamma(f)) \arrow[r, "\pi_2(pr_1)"] & \pi_2(X) \arrow[dll, "\delta"'] \\
\pi_1(\Omega(Y)) \arrow[r, "\pi_1(i)"'] & \pi_1(\Gamma(f)) \arrow[r, "\pi_1(pr_1)"] & \pi_1(X) \arrow[dll, "\delta"'] \\
\pi_0(\Omega(Y)) \arrow[r, "\pi_0(i)"'] & \pi_0(\Gamma(f)) \arrow[r, "\pi_0(pr_1)"] & \pi_0(X)
\end{tikzcd}$$

which then becomes the *long exact sequence of the pointed simplicial mapping* $f\colon X \longrightarrow Y$:

$$\begin{tikzcd}
\pi_{n+1}(Y) \arrow[r, "\pi_n(i)"] & \pi_n(\Gamma(f)) \arrow[r, "\pi_n(f)"] & \pi_n(X) \arrow[dll, dotted, no head] \\
\pi_3(Y) \arrow[r, "\pi_2(i)"'] & \pi_2(\Gamma(f)) \arrow[r, "\pi_2(pr_1)"] & \pi_2(X) \arrow[dll, "\delta"'] \\
\pi_2(Y) \arrow[r, "\pi_1(i)"'] & \pi_1(\Gamma(f)) \arrow[r, "\pi_1(pr_1)"] & \pi_1(X) \arrow[dll, "\delta"'] \\
\pi_1(Y) \arrow[r, "\pi_0(i)"'] & \pi_0(\Gamma(f)) \arrow[r, "\pi_0(pr_1)"] & \pi_0(X) \arrow[dll, "\pi_0(f)"] \\
\pi_0(Y) & &
\end{tikzcd}$$

or, equivalently,

$$\begin{tikzcd}
\pi_n(\Gamma(f)) \arrow[r, "\pi_n(pr_1)"] & \pi_n(X) \arrow[r, "\pi_n(f)"] & \pi_n(Y) \arrow[dll, dotted, no head] \\
\pi_2(\Gamma(f)) \arrow[r, "\pi_2(pr_1)"'] & \pi_2(X) \arrow[r, "\pi_2(f)"] & \pi_2(Y) \arrow[dll, "\pi_1(i)"'] \\
\pi_1(\Gamma(f)) \arrow[r, "\pi_1(pr_1)"'] & \pi_1(X) \arrow[r, "\pi_1(f)"] & \pi_1(Y) \arrow[dll, "\pi_0(i)"'] \\
\pi_0(\Gamma(f)) \arrow[r, "\pi_0(pr_1)"'] & \pi_0(X) \arrow[r, "\pi_0(f)"] & \pi_0(Y)
\end{tikzcd}$$

and thus another justification for calling $\Gamma(f)$ the *homotopy fiber* of $f\colon X \longrightarrow Y$. If f is already a fibration with fiber F, then $pr_2\colon \Gamma(f) \longrightarrow \mathcal{P}(Y)$ is a fibration and its long exact sequence combined with the contractibility of $\mathcal{P}(Y)$ then gives that $\pi_n(F) \simeq \pi_n(\Gamma(f))$, as expected. If f is an inclusion $f\colon X \subseteq Y$ then $\Gamma(f)$ defines the *homotopy groups of* Y *relative to* X with $\pi_{n-1}(\Gamma(f)) = \pi_n(Y;X)$.

In the previous sections, we have used bigroupoids, that is bicategories in which every 2-cell is an isomorphism and every 1-cell is an equivalence (*i.e.*, invertible up to isomorphism with respect to the (horizontal)"tensor product " composition). The link to simplicial topology now comes from the fact that every bicategory $\mathcal{G}$ has a simplicial set *nerve*, $\mathbf{Ner}(\mathcal{G})$, and *this simplicial set is is a Kan complex, precisely when the bicategory is a bigroupoid*, *i.e.*, has exactly the same invertibility requirements as those required for the bicategory to be a bigroupoid [10]. This nerve is minimal in dimensions ≥ 2 and if one chooses a base point $0 \in \mathbf{Ner}(\mathcal{G})_0$, which is the set of objects or 0-cells of $\mathcal{G}$, then $\mathbf{Ner}(\mathcal{G})$ has at that basepoint:

- $\pi_0(\mathbf{Ner}(\mathcal{G}))$ = the pointed set of categorical equivalence classes of the objects of $\mathcal{G}$, pointed by the class of 0.
- $\pi_1(\mathbf{Ner}(\mathcal{G}))$ = the group of homotopy classes of 1-cells of the form $f : 0 \longrightarrow 0$ under (horizontal) tensor composition of 1-cells
- = $\pi_0(\mathcal{G}(0,0))$, the set of connected components of the groupoid $\mathcal{G}(0,0) =_{\mathrm{DEF}} \mathcal{G}(0)$, whose objects are 1-cells $f\colon 0 \longrightarrow 0$ and whose arrows are the 2-cell isomorphisms $\alpha\colon f \Longrightarrow g$ and whose nerve is $\Omega(\mathbf{Ner}(\mathcal{G}))$ at the basepoint 0.
- = $\pi_1(cl(\mathcal{G}))$, where $cl(\mathcal{G})$ denotes, as in Section 2, the groupoid which has the same objects as $\mathcal{G}$ but has 2-cell isomorphism classes of 1-cells for arrows and is the *fundamental groupoid* $\Pi_1(\mathbf{Ner}(\mathcal{G}))$ of the Kan complex $\mathbf{Ner}(\mathcal{G})$.
- $\pi_2(\mathbf{Ner}(\mathcal{G}))$ = the abelian group of 2-simplices all of whose 1-simplex faces are at the base point $s_0(0)\colon 0 \longrightarrow 0$[2]
- = $\mathrm{Aut}(1_0)$ in the groupoid $\mathcal{G}(0,0))$, where $1_0 = s_0(0)\colon 0 \longrightarrow 0$ is the pseudo-identity 1-cell for 0 under tensor composition
- = $\pi_1(\mathcal{G}(0))$ in the notation of Section 2, and equivalently, $\pi_1(\Omega(\mathbf{Ner}(\mathcal{G}))$ in conventional simplicial notation.
- For $i \geq 3$, $\pi_i(\mathbf{Ner}(\mathcal{G})) = \mathbf{0}$, since, by definition, the canonical map

$$\mathbf{Ner}(\mathcal{G}) \longrightarrow \mathrm{Cosk}^3(\mathbf{Ner}(\mathcal{G}))$$

  is an isomorphism and this forces all higher dimensional homotopy groups of the pointed Kan complex $\mathbf{Ner}(\mathcal{G})$ to be trivial.

Now it is easy to verify that simplicial maps between nerves of bigroupoids correspond exactly to strictly unitary homomorphisms $P\colon \mathcal{G} \longrightarrow \mathcal{H}$ of bigroupoids. With this in mind and choosing $P(0) = 0$ as the base point of $\mathcal{H}$, we obtain a pointed simplicial mapping of Kan complexes $\mathbf{Ner}(P)\colon \mathbf{Ner}(\mathcal{G}) \longrightarrow \mathbf{Ner}(\mathcal{H})$. Thus all one need note is that the nerve of the "homotopy fiber bigroupoid" $\mathcal{F}_0$ of Section 2 is just $\Gamma(\mathbf{Ner}(P))$, $\mathcal{F}_0(0) \simeq \Omega(\Gamma(\mathbf{Ner}(P)))$, and that the long exact sequence of the pointed simplicial mapping $\mathbf{Ner}(P)$ is precisely the nine term sequence of Corollary 2.3.

Similar remarks apply to Brown's original paper: every category $\mathcal{G}$ has canonically associated to a simplicial set, its Grothendieck nerve, whose n-simplices can be identified with "composable sequences of length n of arrows of the category". The resulting simplicial set is a Kan complex if, and only if, every arrow of $\mathcal{G}$ is invertible, *i.e.*, $\mathcal{G}$ is a groupoid. For any object 0 in $\mathbf{Ner}(\mathcal{G})$ as a base point, $\mathbf{Ner}(\mathcal{G})$ has only the pointed set of isomorphism classes of its objects as π_0 and $Aut(0)$ as π_1. The long exact sequence above then reduces to a six term one of exactly the same form.

[2]The complex is minimal in this dimension (and higher), so homotopic 2-simplices are equal.

Note that in both cases, the weakest tenable notion of $f\colon X \longrightarrow Y$ is a *fibration* is that which guarantees that for any choice of basepoint, the canonical simplicial mapping from the true fiber $Fib(f)$ of the mapping f to the "homotopy fiber" $\Gamma(f)$ of the same mapping be a weak equivalence.

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