Chain complexes of symmetric categorical groups

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Abstract. We define the cohomology categorical groups of a complex of symmetric categorical groups, and we construct a long 2-exact sequence from an extension of complexes. As special cases, we obtain Ulbrich cohomology of Picard categories and the Hattori-Villamayor-Zelinsky sequence associated with a ring homomorphism. Applications to simplicial cohomology with coefficients in a symmetric categorical group, and to derivations of categorical groups are also discussed.

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1 Introduction

In the late seventies, Villamayor and Zelinsky [18] and, independently, Hattori [9], discovered a long exact sequence connecting Amitsur cohomology groups of a commutative algebra with coefficients $\mathcal{U}$ (the group of units) and Pic (the Picard group). The search of a better understanding of the Hattori-Villamayor-Zelinsky sequence lead to a series of works by Takeuchi and Ulbrich culminating with a cohomology theory for Picard categories [13, 14, 15, 16, 17].

The aim of this work is to revisit the previous results using recent techniques developed in higher dimensional homological algebra. In fact, we will derive Hattori-Villamayor-Zelinsky sequence and Ulbrich cohomology as special instances of general results on the homology of symmetric categorical groups.

The plan of the paper is the following:

Sect. 2 The kernel and the cokernel of a morphism between symmetric categorical groups have been studied in [10, 19]. Here we refine these notions, introducing kernel and cokernel relative to a natural transformation φ, as

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in the following diagram

$$\begin{tikzcd}
Ker(F,\varphi) \arrow[r] & \mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\varphi}{\Uparrow}"] & \mathbb{C} \arrow[r] & Coker(\varphi,G) \\
 & & \mathbb{B} \arrow[ur, "G"'] & &
\end{tikzcd}$$

Sect. 3 Using relative kernels and cokernels, we define the cohomology categorical groups of a complex of symmetric categorical groups. As for abelian groups, there are two possible definitions, giving equivalent cohomology categorical groups.

Sect. 4 An extension of (symmetric) categorical groups is a diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\varphi}{\Uparrow}"] & \mathbb{C} \\
 & \mathbb{B} \arrow[ur, "G"'] &
\end{tikzcd}$$

which is 2-exact in the sense of [10, 19] and such that F is faithful and G is essentially surjective (see [1, 12]). Following the lines of [11], we associate a long 2-exact sequence of cohomology categorical groups to any extension of complexes of symmetric categorical groups.

Sect. 5 We specialize the previous result to get Ulbrich cohomology and Hattori-Villamayor-Zelinsky exact sequences. We discuss also simplicial cohomology with coefficients in a symmetric categorical group.

Sect. 6 In [7], a six term 2-exact sequence involving the low-dimensional cohomology of a categorical group $\mathbb{G}$ with coefficients in a symmetric $\mathbb{G}$-module is constructed. We obtain this sequence as a special case of the kernel-cokernel lemma for symmetric categorical groups, which is a special case of the long cohomology sequence obtained in Section 4.

2 Relative kernel and cokernel

A (symmetric) categorical group is a (symmetric) monoidal groupoid in which each object is invertible, up to isomorphism, with respect to the tensor product. We write CG for the 2-category of categorical groups, monoidal functors, and monoidal natural transformations (which always are natural isomorphisms); SCG is the 2-category of symmetric categorical groups, monoidal functors compatible with the symmetry, and monoidal natural transformations. For basic facts on (symmetric) categorical groups, we refer to [10, 19] and the references therein. As far as notations are concerned, if $\mathbb{G}$ is a (symmetric) categorical group, we write $\pi_0(\mathbb{G})$ for the (abelian) group of its connected components, and $\pi_1(\mathbb{G})$ for the abelian group of automorphisms of the unit object, that is

$\pi_1(\mathbb{G}) = \mathbb{G}(I, I)$. If G is a group, we write $G[0]$ for the discrete categorical group having the elements of G as objects. If G is abelian, we write $G[1]$ for the categorical group with just one object and having the elements of G as arrows (if G is not abelian, $G[1]$ is just a groupoid). If X is an object of a categorical group $\mathbb{G}$, we denote by X^* a fixed dual of X. (Note: composition is always written diagrammatically.)

2.1 The relative kernel

Given a morphism $F\colon \mathbb{A} \to \mathbb{B}$ in CG, the notation for its kernel, see [10, 19], is

$$\begin{tikzcd} & \mathbb{A} \arrow[dr, "F"] \arrow[d, phantom, "\overset{\epsilon_F}{\Downarrow}"] & \\ KerF \arrow[ur, "e_F"] \arrow[rr, "0"'] & {} & \mathbb{B} \end{tikzcd}$$

We consider now two composable morphisms F and G in CG, such that the composite is naturally equivalent to the zero functor, and we construct the *relative kernel* as in the following diagram

$$\begin{tikzcd} Ker(F,\varphi) \arrow[r, "e_{(F,\varphi)}"] \arrow[rr, bend right, "0"', ""{name=L}] & \mathbb{A} \arrow[r, "F"] \arrow[rr, bend left, "0", ""{name=U}] \arrow[d, phantom, "\overset{\epsilon_{(F,\varphi)}}{\Downarrow}"] & \mathbb{B} \arrow[r, "G"] \arrow[u, phantom, "\overset{\varphi}{\Uparrow}"] & \mathbb{C} \end{tikzcd} \tag{1}$$

The relative kernel $Ker(F,\varphi)$ is in CG (in SCG if F, G and φ are in SCG), and it can be described as follows:

- an object is a pair $(A \in \mathbb{A}, a\colon FA \to I)$ such that the following diagram commutes

$$\begin{tikzcd} G(FA) \arrow[rr, "Ga"] \arrow[dr, "\varphi_A"'] & & GI \\ & I \arrow[ur, "G_I"'] & \end{tikzcd}$$

- an arrow $f\colon (A,a) \to (A',a')$ is an arrow $f\colon A \to A'$ such that the following diagram commutes

$$\begin{tikzcd} FA \arrow[rr, "Ff"] \arrow[dr, "a"'] & & FA' \arrow[dl, "a'"] \\ & I & \end{tikzcd}$$

- the faithful functor $e_{(F,\varphi)}$ is defined by $e_{(F,\varphi)}(A,a) = A$, and the natural transformation $\epsilon_{(F,\varphi)}$ by $\epsilon_{(F,\varphi)}(A,a) = a$.

The natural transformation $\epsilon_{(F,\varphi)}$ is compatible with φ, in the sense that the following diagram commutes

$$\begin{tikzcd} e_{(F,\varphi)} \cdot F \cdot G \arrow[r, Rightarrow, "e_{(F,\varphi)} \cdot \varphi"] \arrow[d, Rightarrow, "\epsilon_{(F,\varphi)} \cdot G"'] & e_{(F,\varphi)} \cdot 0 \arrow[d, Rightarrow, "can"] \\ 0 \cdot G \arrow[r, Rightarrow, "can"'] & 0 \end{tikzcd}$$

The relative kernel is a bi-limit, in the sense that it satisfies the following universal property (and it is determined by this property, up to equivalence): given a diagram in CG

$$\begin{tikzcd} \mathbb{K} \arrow[r, "E"] \arrow[rr, bend right, "0"'] & \mathbb{A} \arrow[r, "F"] \arrow[rr, bend left, "0"] \arrow[d, phantom, "\Downarrow\psi"] & \mathbb{B} \arrow[r, "G"] \arrow[u, phantom, "\Uparrow\varphi"] & \mathbb{C} \\ & {} & & \end{tikzcd} \tag{2}$$

with ψ compatible with φ, there is a factorization

$$(E' \colon \mathbb{K} \to Ker(F,\varphi)\,,\ \psi' \colon E' \cdot e_{(F,\varphi)} \Rightarrow E)$$

in CG of (E,ψ) through $(e_{(F,\varphi)}, \epsilon_{(F,\varphi)})$, that is the following diagram commutes

$$\begin{tikzcd} E' \cdot e_{(F,\varphi)} \cdot F \arrow[r, Rightarrow, "\psi' \cdot F"] \arrow[d, Rightarrow, "E' \cdot \epsilon_{(F,\varphi)}"'] & E \cdot F \arrow[d, Rightarrow, "\psi"] \\ E' \cdot 0 \arrow[r, Rightarrow, "can"'] & 0 \end{tikzcd}$$

and, if (E'', ψ'') is another factorization of (E,ψ) through $(e_{(F,\varphi)}, \epsilon_{(F,\varphi)})$, then there is a unique 2-cell $e \colon E' \Rightarrow E''$ such that

$$\begin{tikzcd} E' \cdot e_{(F,\varphi)} \arrow[rr, Rightarrow, "e \cdot e_{(F,\varphi)}"] \arrow[dr, Rightarrow, "\psi'"'] & & E'' \cdot e_{(F,\varphi)} \arrow[dl, Rightarrow, "\psi''"] \\ & E & \end{tikzcd}$$

commutes. The relative kernel is also a standard homotopy kernel, in the sense that it satisfies the following universal property (and it is determined by this property, up to isomorphism): in the situation of diagram (2), there is a unique $E' \colon \mathbb{K} \to Ker(F,\varphi)$ in CG such that $E' \cdot e_{(F,\varphi)} = E$ and $E' \cdot \epsilon_{(F,\varphi)} = \psi$.

To prove the previous universal properties is a simple exercise. The next proposition expresses the kind of injectivity measured by the relative kernel (compare with the similar results stated in [10, 19] for the usual kernel).

Proposition 2.1 *With the notations of diagram (1).*

1. $\pi_1(Ker(F,\varphi)) = 0$ *if and only if* F *is faithful;*
2. $\pi_0(Ker(F,\varphi)) = 0$ *if and only if* F *is* φ*-full (this means full with respect to arrows* $g\colon FA_1 \to FA_2$ *such that* $G(g)\cdot\varphi_{A_2} = \varphi_{A_1}$*).*

Proof. 1) We know from [10, 19] that $\pi_1(KerF) = 0$ if and only if F is faithful. Moreover, the comparison between $Ker(F,\varphi)$ and $KerF$ is full and faithful, so that it induces an isomorphism between $\pi_1(Ker(F,\varphi))$ and $\pi_1(KerF)$.

2) Let (A,a) be an object of $Ker(F,\varphi)$. The arrow $a\cdot F_I\colon FA \to I \to FI$ is such that $G(a\cdot F_I)\cdot\varphi_I = \varphi_A$. If F is φ-full, there exists $\alpha\colon A \to I$ such that $F(\alpha) = a\cdot F_I$. This means that α realizes an isomorphism between (A,a) and the unit object of $Ker(F,\varphi)$.

Conversely, if $g\colon FA_1 \to FA_2$ is such that $G(g)\cdot\varphi_{A_2} = \varphi_{A_1}$, then the following is an object of $Ker(F,\varphi)$

$$(A_1\otimes A_2^*, g\otimes 1\colon F(A_1\otimes A_2^*) \simeq FA_1\otimes FA_2^* \to FA_2\otimes FA_2^* \simeq I)$$

If $\pi_0(Ker(F,\varphi)) = 0$, there is a morphism $h\colon (A_1\otimes A_2^*, g\otimes 1) \to (I, F_I^{-1})$ in $Ker(F,\varphi)$. Now, if we call $f\colon A_1 \to A_2$ the following composition

$$A_1 \simeq A_1\otimes I \simeq\ A_1\otimes A_2^*\otimes A_2 \xrightarrow{h\otimes 1} I\otimes A_2\ \simeq A_2$$

we have that $F(f) = g$. □

A direct consequence of the universal property (as a bi-limit) of the relative kernel is the following cancellation property.

Proposition 2.2 *In the situation of diagram (1), consider the following diagram in* CG

$$\begin{tikzcd}
 & Ker(F,\varphi) \arrow[dr, "e_{(F,\varphi)}"] \arrow[d, phantom, "\overset{\alpha}{\Downarrow}"] & \\
\mathbb{K} \arrow[ur, "H"] \arrow[rr, "0"'] & {} & \mathbb{A}
\end{tikzcd}$$

If α *and* $\epsilon_{(F,\varphi)}$ *are compatible, then there is a unique 2-cell* $\overline{\alpha}\colon H \Rightarrow 0$ *such that* $\overline{\alpha}\cdot e_{(F,\varphi)} = \alpha$.

Proof. This is because (H,α) and $(0, can\colon 0\cdot e_{(F,\varphi)} \Rightarrow 0)$ provide two factorizations of $(0, can\colon 0\cdot F \Rightarrow 0)$ through the relative kernel. □

To finish, let us observe that the usual kernel is a special case of the relative one. Indeed, given a morphism $F\colon \mathbb{A} \to \mathbb{B}$ in CG, we can consider the canonical natural isomorphism

$$\begin{tikzcd}
 & \mathbb{B} \arrow[dr, "0"] \arrow[d, phantom, "\overset{can}{\Downarrow}"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & 0
\end{tikzcd}$$

and the relative kernel $Ker(F, can)$ is nothing but the usual kernel $KerF$. In particular, *can*-full just means full.

2.2 The relative cokernel

Given a morphism $G\colon \mathbb{B} \to \mathbb{C}$ in SCG, the notation for its cokernel, see [10, 19], is

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "P_G"] \arrow[d, phantom, "\overset{\pi_G}{\Downarrow}"] & \\ \mathbb{B} \arrow[ur, "G"] \arrow[rr, "0"'] & & CokerG \end{tikzcd}$$

The picture for the *relative cokernel* is the following one (everything is in SCG):

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[rr, bend right, "0"'] & \mathbb{B} \arrow[r, "G"] \arrow[rr, bend left, "0"] \arrow[d, phantom, "\overset{\varphi}{\Downarrow}"] & \mathbb{C} \arrow[r, "P_{(\varphi,G)}"] \arrow[u, phantom, "\overset{\pi_{(\varphi,G)}}{\Uparrow}"] & Coker(\varphi, G) \end{tikzcd} \qquad (3)$$

The relative cokernel $Coker(\varphi, G)$ can be described as follows:

- objects are those of $\mathbb{C}$;
- pre-arrows are pairs $(B, f)\colon X \to Y$ with $B \in \mathbb{B}$ and $f\colon X \to GB \otimes Y$;
- an arrow is a class of pre-arrows, two pre-arrows $(B, f), (B', f')\colon X \to Y$ are equivalent if there is $A \in \mathbb{A}$ and $a\colon B \to F(A) \otimes B'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & GB \otimes Y \arrow[d, "Ga\otimes 1"] \\ GB' \otimes Y \arrow[d, "\simeq"'] & G(FA \otimes B') \otimes Y \arrow[d, "\simeq"] \\ I \otimes GB' \otimes Y & G(FA) \otimes GB' \otimes Y \arrow[l, "\varphi_A \otimes 1 \otimes 1"] \end{tikzcd}$$

- the essentially surjective functor $P_{(\varphi,G)}$ and the natural transformation $\pi_{(\varphi,G)}$ are defined as for the usual cokernel.

Once again, the natural transformation $\pi_{(\varphi,G)}$ is compatible with φ, in the sense that the following diagram commutes

$$\begin{tikzcd} F \cdot G \cdot P_{(\varphi,G)} \arrow[r, Rightarrow, "\varphi \cdot P_{(\varphi,G)}"] \arrow[d, Rightarrow, "F\cdot\pi_{(\varphi,G)}"'] & 0 \cdot P_{(\varphi,G)} \arrow[d, Rightarrow, "can"] \\ F \cdot 0 \arrow[r, Rightarrow, "can"'] & 0 \end{tikzcd}$$

Like the relative kernel, the relative cokernel is both a bi-limit and a standard homotopy cokernel with respect to diagrams in SCG of the following kind

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[rr, bend right, "0"'] & \mathbb{B} \arrow[r, "G"] \arrow[d, phantom, "\Downarrow\varphi"] \arrow[rr, bend left, "0"] & \mathbb{C} \arrow[r, "P"] \arrow[u, phantom, "\Uparrow\psi"] & \mathbb{K} \end{tikzcd} \tag{4}$$

where ψ is compatible with φ in the obvious sense. We leave to the reader to state the universal properties and the cancellation property for the relative cokernel.

In the next proposition, we fix the kind of surjectivity measured by the relative cokernel.

Proposition 2.3 *With the notations of diagram (3).*

1. $\pi_0(Coker(\varphi, G)) = 0$ *if and only if* G *is essentially surjective;*
2. $\pi_1(Coker(\varphi, G)) = 0$ *if and only if* G *is* φ*-full (this means that, given* $h\colon GB_1 \to GB_2$*, there is* $A \in \mathbb{A}$ *and* $g\colon B_1 \to FA \otimes B_2$ *such that* $h = G(g) \cdot (\varphi_A \otimes 1_{GB_2})$ *).*

Proof. 1) From [10, 19], we know that $\pi_0(CokerG) = 0$ if and only if G is essentially surjective. Moreover, the comparison between $CokerG$ and $Coker(\varphi, G)$ is full and essentially surjective, so that it induces an isomorphism between $\pi_0(CokerG)$ and $\pi_0(Coker(\varphi, G))$.

2) Let $[B \in \mathbb{B}, b\colon I \to GB \otimes I]\colon I \to I$ be a morphism in $Coker(\varphi, G)$. The arrow b gives rise to an arrow $h\colon GI \simeq I \to GB \otimes I \simeq GB$ in $\mathbb{C}$. If G is φ-full, there is $A \in \mathbb{A}$ and $g\colon I \to FA \otimes B$ such that $h = G(g) \cdot (\varphi_A \otimes 1_{GB})$. The pair (A, g) attests that the morphism $[B, b]$ is equal to the identity on I in $Coker(\varphi, G)$.

Conversely, let $g\colon GB_1 \to GB_2$ be a morphism in $\mathbb{C}$. We get the following morphism in $Coker(\varphi, G)$

$$I \xrightarrow{\pi_G(B_1)^{-1}} GB_1 \xrightarrow{P_G(h)} GB_2 \xrightarrow{\pi_G(B_2)} I$$

If $\pi_1(Coker(\varphi, G)) = 0$, the previous morphism is equal to the identity. This means that there is $A \in \mathbb{A}$ and $a\colon B_1 \to FA \otimes B_2$ such that $G(a) \cdot (\varphi_A \otimes 1) = h$. $\square$

The usual cokernel is a particular case of the relative cokernel. Indeed, given a morphism $G\colon \mathbb{B} \to \mathbb{C}$ in SCG, its cokernel is the relative cokernel $Coker(can, G)$ as in the following diagram

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow can"] & \\ 0 \arrow[ur, "0"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd}$$

Once again, *can*-full just means full.

2.3 2-exactness and relative 2-exactness

Let us recall from [10, 19] the notion of 2-exactness. Consider a sequence (F, φ, G) in SCG together with the canonical factorizations through the kernel and the cokernel

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "F'"'] \arrow[rrrr, bend left, "0"] \arrow[rrrr, phantom, bend left=15, "\Uparrow\varphi"] & & \mathbb{B} \arrow[rr, "G"] \arrow[dr, "P_F"] & & \mathbb{C} \\
& KerG \arrow[ur, "e_G"] \arrow[r] & CokerF' \simeq KerG' \arrow[r] & CokerF \arrow[ur, "G'"'] &
\end{tikzcd}
$$

We say that the sequence (F, φ, G) is *2-exact* if the functor F' is full and essentially surjective on objects or, equivalently, if the functor G' is full and faithful. This is also equivalent to say that $CokerF'$ (or $KerG'$) is equivalent to 0.

Consider now the following diagram in SCG

$$
\begin{tikzcd}
\mathbb{A}' \arrow[r, "L"] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow\alpha"] & \mathbb{A} \arrow[r, "F"] \arrow[rr, bend right=60, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow\varphi"] & \mathbb{B} \arrow[r, "G"] \arrow[rr, bend left=60, "0"] \arrow[rr, phantom, bend left=30, "\Uparrow\gamma"] & \mathbb{C} \arrow[r, "M"] & \mathbb{C}'
\end{tikzcd}
$$

with α compatible with φ and φ compatible wit γ. By the universal property of the relative kernel $Ker(G, \gamma)$, we get a factorization (F', φ') of (F, φ) through $(e_{(G,\gamma)}, \epsilon_{(G,\gamma)})$. By the cancellation property of $e_{(G,\gamma)}$, we have a 2-cell $\overline{\alpha}$ as in the following diagram

$$
\begin{tikzcd}
\mathbb{A}' \arrow[r, "L"] \arrow[dr, "0"'] \arrow[dr, phantom, shift left=3, "\Downarrow\overline{\alpha}"] & \mathbb{A} \arrow[r, "F"] \arrow[d, "F'"] \arrow[dr, phantom, "\Uparrow\varphi'"] & \mathbb{B} \arrow[r, "G"] & \mathbb{C} \\
& Ker(G, \gamma) \arrow[ur, "e_{(G,\gamma)}"'] \arrow[r] & Coker(\overline{\alpha}, F') &
\end{tikzcd}
$$

The dual construction gives rise to the following diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] & \mathbb{B} \arrow[rr, "G"] \arrow[drr, "P_{(\alpha,F)}"'] & & \mathbb{C} \arrow[rr, "M"] \arrow[dl, phantom, "\overset{\varphi''}{\Uparrow}"] \arrow[dr, phantom, "\overset{\overline{\gamma}}{\Downarrow}"] & & \mathbb{C}' \\
& Ker(G',\overline{\gamma}) \arrow[rr] & & Coker(\alpha,F) \arrow[u, "G'"'] \arrow[urr, "0"'] & &
\end{tikzcd}$$

A direct calculation shows that $Coker(\overline{\alpha},F') \simeq Ker(G',\overline{\gamma})$. We say that the sequence $(L,\alpha,F,\varphi,G,\gamma,M)$ is *relative 2-exact* if the functor F' is essentially surjective and $\overline{\alpha}$-full or, equivalently, if the functor G' is faithful and $\overline{\gamma}$-full. This is also equivalent to say that $Coker(\overline{\alpha},F')$ (or $Ker(G',\overline{\gamma})$) is equivalent to 0.

Since the comparison $Ker(G,\gamma) \to KerG$ is full and faithful, 2-exactness always implies relative 2-exactness. To make clear the difference between 2-exactness and relative 2-exactness, let us consider two basic examples.

Example 2.4

1. Consider the following sequence in SCG

$$0 \xrightarrow{0} \mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{P_F} CokerF$$

together with $can\colon 0\cdot F \Rightarrow 0$ and $\pi_F\colon F\cdot P_F \Rightarrow 0$. It is always 2-exact in $\mathbb{B}$. It is 2-exact in $\mathbb{A}$ if and only if F is full and faithful. Moreover, it is relative 2-exact in $\mathbb{A}$ if and only if F is faithful. Indeed, $\pi_0(Ker(F,\pi_F)) = 0$, so that any functor F is π_F-full.

2. Consider the following sequence in SCG

$$KerG \xrightarrow{e_G} \mathbb{B} \xrightarrow{G} \mathbb{C} \xrightarrow{0} 0$$

together with $\epsilon_G\colon e_G\cdot G \Rightarrow 0$ and $can\colon G\cdot 0 \Rightarrow 0$. It is always 2-exact in $\mathbb{B}$. It is 2-exact in $\mathbb{C}$ if and only if G is full and essentially surjective. Moreover, it is relative 2-exact in $\mathbb{C}$ if and only if G is essentially surjective. Indeed, $\pi_1(Coker(can,G)) = 0$, so that any functor G is ϵ_G-full.

3 The cohomology of a complex

From [11, 13], recall that a complex of symmetric categorical groups is a diagram in SCG of the form

$$\mathbb{A}_\bullet = \mathbb{A}_0 \xrightarrow{L_0} \mathbb{A}_1 \xrightarrow{L_1} \mathbb{A}_2 \xrightarrow{L_2} \cdots \xrightarrow{L_{n-1}} \mathbb{A}_n \xrightarrow{L_n} \mathbb{A}_{n+1} \xrightarrow{L_{n+1}} \cdots$$

together with a family of 2-cells $\{\alpha_n \colon L_n \cdot L_{n+1} \Rightarrow 0\}_{n \geq 0}$ such that, for all n, the following diagram commutes

$$\begin{tikzcd}
L_{n-1} \cdot L_n \cdot L_{n+1} \arrow[r, Rightarrow, "L_{n-1} \cdot \alpha_n"] \arrow[d, Rightarrow, "\alpha_{n-1} \cdot L_{n+1}"'] & L_{n-1} \cdot 0 \arrow[d, Rightarrow, "can"] \\
0 \cdot L_{n+1} \arrow[r, Rightarrow, "can"'] & 0
\end{tikzcd}$$

To define the n-th cohomology categorical group of the complex $\mathbb{A}_\bullet$, we use the following part of the complex

$$\begin{tikzcd}
\mathbb{A}_{n-2} \arrow[r, "L_{n-2}"] \arrow[rr, bend left, "0"] & \mathbb{A}_{n-1} \arrow[r, "L_{n-1}"] \arrow[rr, bend right, "0"'] \arrow[phantom, r, "\Uparrow \alpha_{n-2}" very near start, shift left=5ex] & \mathbb{A}_n \arrow[r, "L_n"] \arrow[rr, bend left, "0"] \arrow[phantom, r, "\Downarrow \alpha_{n-1}" near start, shift right=5ex] & \mathbb{A}_{n+1} \arrow[r, "L_{n+1}"] \arrow[phantom, l, "\Uparrow \alpha_n", shift left=5ex] & \mathbb{A}_{n+2}
\end{tikzcd}$$

and we repeat the construction given in 2.3: by the universal property of the relative kernel $Ker(L_n, \alpha_n)$, we get a factorization $(L'_{n-1}, \alpha'_{n-1})$ of (L_{n-1}, α_{n-1}) through $(e_{(L_n,\alpha_n)}, \epsilon_{(L_n,\alpha_n)})$. By the cancellation property of $e_{(L_n,\alpha_n)}$, we have a 2-cell $\overline{\alpha}_{n-2}$ as in the following diagram

$$\begin{tikzcd}
\mathbb{A}_{n-2} \arrow[r, "L_{n-2}"] \arrow[dr, "0"'] & \mathbb{A}_{n-1} \arrow[rr, "L_{n-1}"] \arrow[d, "L'_{n-1}"'] \arrow[phantom, dl, "\Downarrow \overline{\alpha}_{n-2}"] \arrow[phantom, dr, "\Uparrow \alpha'_{n-1}"] & & \mathbb{A}_n \arrow[r, "L_n"] & \mathbb{A}_{n+1} \\
& Ker(L_n, \alpha_n) \arrow[urr, "e_{(L_n,\alpha_n)}"'] \arrow[r] & Coker(\overline{\alpha}_{n-2}, L'_{n-1}) & &
\end{tikzcd}$$

Definition 3.1 With the previous notations, we define the n-th cohomology categorical group of the complex $\mathbb{A}_\bullet$ as the following relative cokernel

$$H^n(\mathbb{A}_\bullet) = Coker(\overline{\alpha}_{n-2}, L'_{n-1}).$$

Note that, as in Section 2.3, there is a dual construction of $H^n(\mathbb{A}_\bullet)$ starting with the relative cokernel $Coker(\alpha_{n-2}, L_{n-1})$ and ending with a convenient relative kernel. The resulting categorical groups are equivalent. Note also that, to get $H^0(\mathbb{A}_\bullet)$ and $H^1(\mathbb{A}_\bullet)$, we have to complete the complex $\mathbb{A}_\bullet$ on the left with two zero-morphisms and two canonical 2-cells

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{A}_0 \xrightarrow{L_0} \mathbb{A}_1 \ldots \ , \ \ can \colon 0 \cdot 0 \Rightarrow 0 \ , \ \ can \colon 0 \cdot L_0 \Rightarrow 0$$

We give now an explicit description of $H^n(\mathbb{A}_\bullet)$:

- an object of $H^n(\mathbb{A}_\bullet)$ is an object of the relative kernel $Ker(L_n, \alpha_n)$, that is a pair
$$(A_n \in \mathbb{A}_n, a_n \colon L_n(A_n) \to I)$$
such that $L_{n+1}(a_n) = \alpha_n(A_n)$;

- a pre-arrow $(A_n, a_n) \to (A'_n, a'_n)$ is a pair
$$(X_{n-1} \in \mathbb{A}_{n-1}, x_{n-1} \colon A_n \to L_{n-1}(X_{n-1}) \otimes A'_n)$$
such that the following diagram commutes
$$\begin{tikzcd}
L_n(A_n) \arrow[r, "L_n(x_{n-1})"] \arrow[d, "a_n"'] & L_n(L_{n-1}(X_{n-1}) \otimes A'_n) \arrow[d, "\simeq"] \\
I & L_n(L_{n-1}(X_{n-1})) \otimes L_n(A'_n) \arrow[d, "\alpha_{n-1}(X_{n-1}) \otimes 1"] \\
L_n(A'_n) \arrow[u, "a'_n"] & I \otimes L_n(A'_n) \arrow[l, "\simeq"']
\end{tikzcd}$$

- an arrow is a class of pre-arrows; two parallel pre-arrows
$$(X_{n-1}, x_{n-1}), (X'_{n-1}, x'_{n-1}) \colon (A_n, a_n) \to (A'_n, a'_n)$$
are equivalent if there is a pair
$$(P_{n-2} \in \mathbb{A}_{n-2}, p_{n-2} \colon X_{n-1} \to L_{n-2}(P_{n-2}) \otimes X'_{n-1})$$
such that the following diagram commutes
$$\begin{tikzcd}
A_n \arrow[r, "x_{n-1}"] \arrow[d, "x'_{n-1}"'] & L_{n-1}(X_{n-1}) \otimes A'_n \arrow[d, "L_{n-1}(p_{n-2}) \otimes 1"] \\
L_{n-1}(X'_{n-1}) \otimes A'_n & L_{n-1}(L_{n-2}(P_{n-2}) \otimes X'_{n-1}) \otimes A'_n \arrow[d, "\simeq"] \\
I \otimes L_{n-1}(X'_{n-1}) \otimes A'_n \arrow[u, "\simeq"'] & L_{n-1}(L_{n-2}(P_{n-2})) \otimes L_{n-1}(X'_{n-1}) \otimes A'_n \arrow[l, bend left, "\alpha_{n-2}(P_{n-2}) \otimes 1 \otimes 1"]
\end{tikzcd}$$

Remark 3.2 From the previous description, it is evident that
$$\pi_0(H^n(\mathbb{A}_\bullet)) \simeq \pi_1(H^{n+1}(\mathbb{A}_\bullet))$$
This will be useful in Section 5 to make some proofs shorter.

Let us look now at the functoriality of H^n. A morphism $F_\bullet\colon \mathbb{A}_\bullet \to \mathbb{B}_\bullet$ of complexes in SCG is pictured in the following diagram

$$
\begin{tikzcd}
\dots\mathbb{A}_{n-1} \arrow[r, "L_{n-1}"] \arrow[d, "F_{n-1}"'] \arrow[rr, bend left=50, "0"] \arrow[dr, phantom, "\Downarrow\lambda_{n-1}"] & \mathbb{A}_n \arrow[r, "L_n"] \arrow[d, "F_n"] \arrow[u, phantom, "\Uparrow\alpha_{n-1}"] \arrow[dr, phantom, "\Downarrow\lambda_n"] & \mathbb{A}_{n+1}\dots \arrow[d, "F_{n+1}"] \\
\dots\mathbb{B}_{n-1} \arrow[r, "M_{n-1}"'] \arrow[rr, bend right=50, "0"'] & \mathbb{B}_n \arrow[r, "M_n"'] \arrow[d, phantom, "\Downarrow\beta_{n-1}"] & \mathbb{B}_{n+1}\dots
\end{tikzcd}
$$

where the family of 2-cells $\{\lambda_n\colon L_n \cdot F_{n+1} \Rightarrow F_n \cdot M_n\}_{n\geq 0}$ makes commutative the following diagram

$$
\begin{tikzcd}
L_{n-1}\cdot L_n\cdot F_{n+1} \arrow[r, Rightarrow, "L_{n-1}\cdot\lambda_n"] \arrow[d, Rightarrow, "\alpha_{n-1}\cdot F_{n+1}"'] & L_{n-1}\cdot F_n\cdot M_n \arrow[r, Rightarrow, "\lambda_{n-1}\cdot M_n"] & F_{n-1}\cdot M_{n-1}\cdot M_n \arrow[d, Rightarrow, "F_{n-1}\cdot\beta_{n-1}"] \\
0\cdot F_{n+1} \arrow[r, Rightarrow, "can"'] & 0 & F_{n-1}\cdot 0 \arrow[l, Rightarrow, "can"]
\end{tikzcd}
$$

Such a morphism induces, for each n, a morphism of symmetric categorical groups $H^n(F_\bullet)\colon H^n(\mathbb{A}_\bullet) \to H^n(\mathbb{B}_\bullet)$. Its existence follows from the universal property of the relative kernels and cokernels involved. It can be described explicitly: given an object $(A_n \in \mathbb{A}_n, a_n\colon L_n(A_n) \to I)$ in $H^n(\mathbb{A}_\bullet)$, we have

$$H^n(F_\bullet)(A_n, a_n) = (F_n(A_n) \in \mathbb{B}_n, \lambda_n^{-1}(A_n) \cdot F_{n+1}(a_n)\colon$$

$$M_n(F_n(A_n)) \to F_{n+1}(L_n(A_n)) \to F_{n+1}(I) \simeq I)$$

The fact that $(F_n(A_n), \lambda_n^{-1}(A_n) \cdot F_{n+1}(a_n))$ is an object of the relative kernel $Ker(M_n, \beta_n)$ depends on the condition on the family $\{\lambda_n\}$. Given an arrow

$$[X_{n-1} \in \mathbb{A}_{n-1}, x_{n-1}\colon A_n \to L_{n-1}(X_{n-1}) \otimes A'_n]\colon (A_n, a_n) \to (A'_n, a'_n)$$

in $H^n(\mathbb{A}_\bullet)$, we have

$$H^n(F_\bullet)[X_{n-1}, x_{n-1}] = [F_{n-1}(X_{n-1}) \in \mathbb{B}_{n-1}, F_n(x_{n-1}) \cdot (\lambda_{n-1}(X_{n-1}) \otimes 1)\colon$$

$$F_n(A_n) \to F_n(L_{n-1}(X_{n-1}) \otimes A'_n) \simeq$$

$$\simeq F_n(L_{n-1}(X_{n-1})) \otimes F_n(A'_n) \to M_{n-1}(F_{n-1}(X_{n-1})) \otimes F_n(A'_n)]$$

4 The long cohomology sequence

Recall that an extension of symmetric categorical groups is a diagram in SCG

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow\varphi"] & \mathbb{C} \\
& \mathbb{B} \arrow[ur, "G"'] &
\end{tikzcd}$$

which is 2-exact, F is faithful and G is essentially surjective (see [1, 12]). Equivalently, an extension is a diagram in SCG of the form

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[rr, bend left=50, "0"] & \mathbb{A} \arrow[r, "F"] \arrow[u, phantom, "\Uparrow can"] \arrow[rr, bend right=50, "0"'] & \mathbb{B} \arrow[r, "G"] \arrow[d, phantom, "\Downarrow\varphi"] \arrow[rr, bend left=50, "0"] & \mathbb{C} \arrow[r, "0"] \arrow[u, phantom, "\Uparrow can"] & 0
\end{tikzcd}$$

which is relative 2-exact in $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$. (Indeed, 2-exactness and relative 2-exactness are equivalent conditions in $\mathbb{B}$, because $Ker(G, can) \simeq KerG$ and can-full means full. Now, if the factorization of G through $CokerF$ is full and faithful, the relative 2-exactness in $\mathbb{A}$ of $(0, can, F, \varphi, G)$ is equivalent to the relative 2-exactness in $\mathbb{A}$ of $(0, can, F, \pi_F, P_F)$, that is, by Example 2.4, to the faithfulness of F. The argument for the essential surjectivity of G is dual.)

A morphism of extensions is pictured in the following diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "L"'] \arrow[rr, bend left=50, "0"] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathbb{B} \arrow[r, "G"] \arrow[d, "M"'] \arrow[u, phantom, "\Uparrow\varphi"] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathbb{C} \arrow[d, "N"] \\
\mathbb{A}' \arrow[r, "F'"'] \arrow[rr, bend right=50, "0"'] & \mathbb{B}' \arrow[r, "G'"'] \arrow[d, phantom, "\Downarrow\varphi'"] & \mathbb{C}' \\
& &
\end{tikzcd}$$

where the 2-cells make commutative the following diagram

$$\begin{tikzcd}
F \cdot G \cdot N \arrow[r, Rightarrow, "F \cdot \mu"] \arrow[d, Rightarrow, "\varphi \cdot N"'] & F \cdot M \cdot G' \arrow[r, Rightarrow, "\lambda \cdot G'"] & L \cdot F' \cdot G' \arrow[d, Rightarrow, "L \cdot \varphi'"] \\
0 \cdot N \arrow[r, Rightarrow, "can"'] & 0 & L \cdot 0 \arrow[l, Rightarrow, "can"']
\end{tikzcd}$$

Definition 4.1 An extension of complexes in SCG is a diagram

$$\begin{tikzcd} \mathbb{A}_\bullet \arrow[rr, "0"] \arrow[dr, "F_\bullet"'] & \arrow[d, phantom, "\Uparrow \varphi_\bullet"] & \mathbb{C}_\bullet \\ & \mathbb{B}_\bullet \arrow[ur, "G_\bullet"'] & \end{tikzcd}$$

where

$$\mathbb{A}_\bullet \xrightarrow{F_\bullet} \mathbb{B}_\bullet \xrightarrow{G_\bullet} \mathbb{C}_\bullet$$

are morphisms of complexes, and $\varphi_\bullet = \{\varphi_n \colon F_n \cdot G_n \Rightarrow 0\}_{n\geq 0}$ is a family of 2-cells such that, for each n,

$$\begin{tikzcd} \mathbb{A}_n \arrow[rr, "0"] \arrow[dr, "F_n"'] & \arrow[d, phantom, "\Uparrow \varphi_n"] & \mathbb{C}_n \\ & \mathbb{B}_n \arrow[ur, "G_n"'] & \end{tikzcd}$$

is an extension of symmetric categorical groups and

$$\begin{tikzcd} \mathbb{A}_n \arrow[rr, bend left, "0"] \arrow[r, "F_n"] \arrow[d, "L_n"'] \arrow[dr, phantom, "\overset{\lambda_n}{\Rightarrow}"] & \mathbb{B}_n \arrow[u, phantom, "\Uparrow \varphi_n"] \arrow[r, "G_n"] \arrow[d, "M_n"] \arrow[dr, phantom, "\overset{\mu_n}{\Rightarrow}"] & \mathbb{C}_n \arrow[d, "N_n"] \\ \mathbb{A}_{n+1} \arrow[r, "F_{n+1}"'] \arrow[rr, bend right, "0"'] & \mathbb{B}_{n+1} \arrow[d, phantom, "\Downarrow \varphi_{n+1}"] \arrow[r, "G_{n+1}"'] & \mathbb{C}_{n+1} \end{tikzcd}$$

is a morphism of extensions.

Theorem 4.2 *Let*

$$\begin{tikzcd} \mathbb{A}_\bullet \arrow[rr, "0"] \arrow[dr, "F_\bullet"'] & \arrow[d, phantom, "\Uparrow \varphi_\bullet"] & \mathbb{C}_\bullet \\ & \mathbb{B}_\bullet \arrow[ur, "G_\bullet"'] & \end{tikzcd}$$

be an extension of complexes of symmetric categorical groups. For each n*, there is a morphism* Δ_n *and three 2-cells* $H^n(\varphi_\bullet), \Sigma_n$ *and* Ψ_n *making the following* *long sequence 2-exact in each point*

$$
\begin{tikzcd}
H^n(\mathbb{A}_\bullet) \arrow[r, "H^n(F_\bullet)"] \arrow[rr, bend left=50, "0"] \arrow[r, phantom, shift left=6ex, "\Uparrow H^n(\varphi_\bullet)"] & H^n(\mathbb{B}_\bullet) \arrow[r, "H^n(G_\bullet)"] \arrow[rr, bend right=50, "0"'] & H^n(\mathbb{C}_\bullet) \arrow[r, "\Delta_n"] \arrow[rr, bend left=40, "0"] \arrow[r, phantom, shift right=6ex, "\Downarrow \Sigma_n"] \arrow[r, phantom, shift left=6ex, "\Uparrow \Psi_n"] & H^{n+1}(\mathbb{A}_\bullet) \arrow[r, "H^{n+1}(F_\bullet)"] & H^{n+1}(\mathbb{B}_\bullet)
\end{tikzcd}
$$

Proof. We give the construction of the morphisms and 2-cells involved in the statement. As far as 2-exactness is concerned, we concentrate on the 2-exactness in $H^n(\mathbb{C}_\bullet)$, which is the most delicate part of the proof. In fact, we give a first construction of Δ_n and Σ_n. We use these constructions to show that the factorization of $H^n(G_\bullet)$ through the kernel of Δ_n is essentially surjective. Then we give a second construction of $H^n(\mathbb{C}_\bullet), \Delta_n$ and Σ_n, and we use them to show that the factorization of $H^n(G_\bullet)$ is full.

Construction of $H^n(\varphi_\bullet)$: given an object $(A_n \in \mathbb{A}_n, a_n \colon L_n(A_n) \to I)$ in $H^n(\mathbb{A}_\bullet)$, if we apply $H^n(F_\bullet)$ and $H^n(G_\bullet)$ we obtain the following object of $H^n(\mathbb{C}_\bullet)$:

$$(G_n(F_n(A_n)) \in \mathbb{C}_n, \mu_n^{-1}(F_n(A_n)) \cdot G_{n+1}(\lambda_n^{-1}(A_n)) \cdot G_{n+1}(F_{n+1}(a_n)) \colon$$
$$N_n(G_n(F_n(A_n))) \to G_{n+1}(M_n(F_n(A_n))) \to$$
$$\to G_{n+1}(F_{n+1}(L_n(A_n))) \to G_{n+1}(F_{n+1}(I)) \simeq I)$$

Such an object is naturally isomorphic to $(I \in \mathbb{C}_n, N_n(I) \simeq I)$, which is the unit object in $H^n(\mathbb{C}_\bullet)$, via the morphism

$$H^n(\varphi_\bullet) = [I \in \mathbb{C}_{n-1}, \varphi_n(A_n) \colon G_n(F_n(A_n)) \to I \simeq N_{n-1}(I) \otimes I]\,.$$

First construction of Δ_n : let $(C_n \in \mathbb{C}_n, c_n \colon N_n(C_n) \to I)$ be an object in $KerN_n$; since $G_n \colon \mathbb{B}_n \to \mathbb{C}_n$ is essentially surjective, there are $B_n \in \mathbb{B}_n$ and $i \colon G_n(B_n) \to C_n$. Since

$$(M_n(B_n), \mu_n(B_n) \cdot N_n(i) \cdot c_n \colon G_{n+1}(M_n(B_n)) \to N_n(G_n(B_n)) \to N_n(C_n) \to I)$$

is an object of $KerG_{n+1}$ and the factorization of $F_{n+1} \colon \mathbb{A}_{n+1} \to \mathbb{B}_{n+1}$ through $KerG_{n+1}$ is an equivalence, there are $A_{n+1} \in \mathbb{A}_{n+1}$ and $j \colon F_{n+1}(A_{n+1}) \to M_n(B_n)$ such that $G_{n+1}(j) \cdot \mu_n(B_n) \cdot N_n(i) \cdot c_n = \varphi_{n+1}(A_{n+1})$. Now we need an arrow $a_{n+1} \colon L_{n+1}(A_{n+1}) \to I$. Since the factorization F'_{n+2} of F_{n+2} through $KerG_{n+2}$ is an equivalence, it is enough to find an arrow $F'_{n+2}(L_{n+1}(A_{n+1})) \to F'_{n+2}(I)$. This is given by

$$\lambda_{n+1}(A_{n+1}) \cdot M_{n+1}(j) \cdot \beta_n(B_n) \colon F_{n+2}(L_{n+1}(A_{n+1})) \to I \simeq F_{n+2}(I)$$

Finally, we put $\Delta_n(C_n, c_n) = (A_{n+1}, a_{n+1})$. This is an object of $H^{n+1}(\mathbb{A}_\bullet)$: the condition $L_{n+2}(a_{n+1}) = \alpha_{n+1}(A_{n+1})$ can be checked applying the faithful

functor F_{n+3}.
Consider now an arrow

$$[Z_{n-1} \in \mathbb{C}_{n-1}, z_{n-1} \colon C_n \to N_{n-1}(Z_{n-1}) \otimes C'_n] \colon (C_n, c_n) \to (C'_n, c'_n)$$

in $H^n(\mathbb{C}_\bullet)$. We look for an arrow

$$[X_n \in \mathbb{A}_n, x_n \colon A_{n+1} \to L_n(X_n) \otimes A'_{n+1}] \colon (A_{n+1}, a_{n+1}) \to (A'_{n+1}, a'_{n+1})$$

in $H^{n+1}(\mathbb{A}_\bullet)$. Since $G_{n-1} \colon \mathbb{B}_{n-1} \to \mathbb{C}_{n-1}$ is essentially surjective, there are $Y_{n-1} \in \mathbb{B}_{n-1}$ and $l \colon G_{n-1}(Y_{n-1}) \to Z_{n-1}$. We get the following arrow in $\mathbb{C}_n$

$$i \cdot z_{n-1} \cdot (N_{n-1}(l^{-1}) \otimes 1) \cdot (\mu_{n-1}^{-1}(Y_{n-1}) \otimes 1) \cdot (1 \otimes (i')^{-1}) \colon$$

$$G_n(B_n) \to G_n(M_{n-1}(Y_{n-1})) \otimes G_n(B_n) \simeq G_n(M_{n-1}(Y_{n-1}) \otimes B_n)$$

Since the factorization of G_n through $CokerF_n$ is an equivalence, we get the corresponding arrow in $CokerF_n$

$$[X_n \in \mathbb{A}_n, s \colon B_n \to F_n(X_n) \otimes M_{n-1}(Y_{n-1}) \otimes B'_n] \colon B_n \to M_{n-1}(Y_{n-1}) \otimes B'_n$$

This allows us to construct an arrow

$$F'_{n+1}(A_{n+1}) \to F'_{n+1}(L_n(X_n) \otimes A'_{n+1})$$

in $KerG_{n+1}$ in the following way

$$\begin{array}{c}
F_{n+1}(A_{n+1}) \\
\Big\downarrow {\scriptstyle j} \\
M_n(B_n) \\
\Big\downarrow {\scriptstyle M_n(s)} \\
M_n(F_n(X_n) \otimes M_{n-1}(Y_{n-1}) \otimes B'_n) \\
\Big\downarrow {\scriptstyle \simeq} \\
M_n(F_n(X_n)) \otimes M_n(M_{n-1}(Y_{n-1})) \otimes M_n(B'_n) \\
\Big\downarrow {\scriptstyle 1 \otimes \beta_{n-1}(Y_{n-1}) \otimes 1} \\
M_n(F_n(X_n)) \otimes M_n(B'_n) \\
\Big\downarrow {\scriptstyle \lambda_n^{-1}(X_n) \otimes (j')^{-1}} \\
F_{n+1}(L_n(X_n)) \otimes F_{n+1}(A'_{n+1}) \simeq F_{n+1}(L_n(X_n) \otimes A'_{n+1})
\end{array}$$

Since F'_{n+1} is an equivalence, we get a uniquely determined arrow $x_n \colon A_{n+1} \to L_n(X_n) \otimes A'_{n+1}$. Finally, we put $\Delta_n[Z_{n-1}, z_{n-1}] = [X_n, x_n]$: the condition to be an arrow in $H^{n+1}(\mathbb{A}_\bullet)$ can be checked applying the faithful functor F_{n+2}.

First construction of Σ_n : let $(B_n \in \mathbb{B}_n, b_n \colon M_n(B_n) \to I)$ be an object of $H^n(\mathbb{B}_\bullet)$; we put

$$\Sigma_n(B_n, b_n) = [I \in \mathbb{A}_n, \sigma(B_n, b_n) \colon A_{n+1} \to L_n(I)]$$

where $(A_{n+1}, a_{n+1}) = \Delta_n(H^n(G_\bullet)(B_n, b_n))$ and $\sigma(B_n, b_n)$ corresponds to the arrow

$$j \cdot b_n \colon F'_{n+1}(A_{n+1}) = F_{n+1}(A_{n+1}) \to I \simeq F_{n+1}(L_n(I)) = F'_{n+1}(L_n(I))$$

of $KerG_{n+1}$ via the equivalence $F'_{n+1} \colon \mathbb{A}_{n+1} \to KerG_{n+1}$. Indeed, the fact that $[I, \sigma(B_n, b_n)]$ is an arrow in $H^{n+1}(\mathbb{A}_\bullet)$ can be checked applying the faithful functor F_{n+2}.

2-exactness in $H^n(\mathbb{C}_\bullet)$: let us call Γ the factorization of $H^n(G_\bullet)$ through $Ker\Delta_n$. We are going to prove that Γ is essentially surjective. Let

$$\langle\, (C_n \in \mathbb{C}_n, c_n \colon N_n(C_n) \to I),$$

$$[\overline{C}_n \in \mathbb{A}_n, \overline{c}_n \colon A_{n+1} \to L_n(\overline{C}_n)] \colon (A_{n+1}, a_{n+1}) = \Delta_n(C_n, c_n) \to I \,\rangle$$

be an object of $Ker\Delta_n$. Using the notations introduced in the first construction of Δ_n, we construct the following object of $H^n(\mathbb{B}_\bullet)$:

$$(F_n(\overline{C}^*_n) \otimes B_n, \tau = (\lambda_n^{-1}(\overline{C}^*_n) \otimes 1) \cdot (F_{n+1}(\overline{c}^*_n) \otimes 1) \cdot ((j^{-1})^* \otimes 1) \colon$$

$$M_n(F_n(\overline{C}^*_n) \otimes B_n) \simeq M_n(F_n(\overline{C}^*_n)) \otimes M_n(B_n) \to M_n(B_n)^* \otimes M_n(B_n) \simeq I)$$

and the needed isomorphism

$$\Gamma(F_n(\overline{C}^*_n) \otimes B_n, \tau) \to \langle\, (C_n, c_n), [\overline{C}_n, \overline{c}_n] \,\rangle$$

is given by

$$[I \in \mathbb{C}_{n-1}, \varphi_n(\overline{C}^*_n) \otimes i \colon G_n(F_n(\overline{C}^*_n) \otimes B_n) \simeq G_n(F_n(\overline{C}^*_n)) \otimes G_n(B_n) \to C_n]$$

Second description of $H^n(\mathbb{C}_\bullet)$: since (F_n, φ_n, G_n) is an extension, $\mathbb{C}_n$ is equivalent to the cokernel of F_n, and we get the following description of $H^n(\mathbb{C}_\bullet)$. An object is a pair

$$(B_n \in \mathbb{B}_n, [A_{n+1} \in \mathbb{A}_{n+1}, a_{n+1} \colon M_n(B_n) \to F_{n+1}(A_{n+1})]\,),$$

where $[A_{n+1}, a_{n+1}] \colon M_n(B_n) \to I$ is an arrow in $CokerF_{n+1}$, such that there exists $t_{n+2} \colon L_{n+1}(A_{n+1}) \to I$ making commutative the following diagram

$$\begin{tikzcd}
M_{n+1}(M_n(B_n)) \arrow[r, "M_{n+1}(a_{n+1})"] \arrow[d, "\beta_n(B_n)"'] & M_{n+1}(F_{n+1}(A_{n+1})) \arrow[d, "\lambda^{-1}_{n+1}(A_{n+1})"] \\
I \simeq F_{n+2}(I) & F_{n+2}(L_{n+1}(A_{n+1})) \arrow[l, "F_{n+2}(t_{n+2})"]
\end{tikzcd}$$

(note that such an arrow t_{n+2} is necessarily unique because F_{n+2} is faithful). An arrow $(B_n, [A_{n+1}, a_{n+1}]) \to (B'_n, [A'_{n+1}, a'_{n+1}])$ is a class of pairs

$$(B_{n-1} \in \mathbb{B}_{n-1}, [A_n \in \mathbb{A}_n, a_n \colon B_n \to F_n(A_n) \otimes M_{n-1}(B_{n-1}) \otimes B'_n]\,)\,,$$

where $[A_n, a_n] \colon B_n \to M_{n-1}(B_{n-1}) \otimes B'_n$ is an arrow in $CokerF_n$, such that there exists $\overline{a}_n \colon A_{n+1} \to L_n(A_n) \otimes A'_{n+1}$ making commutative the following diagram

$$\begin{tikzcd}
M_n(B_n) \arrow[d, "a_{n+1}"'] \arrow[dr, "M_n(a_n)"] & \\
F_{n+1}(A_{n+1}) \arrow[d, "F_{n+1}(\overline{a}_n)"'] & M_n(F_n(A_n) \otimes M_{n-1}(B_{n-1}) \otimes B'_n) \arrow[d, "\simeq"] \\
F_{n+1}(L_n(A_n) \otimes A'_{n+1}) \arrow[d, "\simeq"'] & M_n(F_n(A_n)) \otimes M_n(M_{n-1}(B_{n-1})) \otimes M_n(B'_n) \arrow[d, "\lambda_n^{-1}(A_n) \otimes \beta_{n-1}(B_{n-1}) \otimes 1"] \\
F_{n+1}(L_n(A_n)) \otimes F_{n+1}(A'_{n+1}) & F_{n+1}(L_n(A_n)) \otimes M_n(B'_n) \arrow[l, "1 \otimes a'_{n+1}"]
\end{tikzcd}$$

(once again the arrow $\overline{a}_n$ is necessarily unique because F_{n+1} is faithful). Finally, two parallel pairs $(B_{n-1}, [A_n, a_n])$ and $(B'_{n-1}, [A'_n, a'_n])$ are identified if there are $B_{n-2} \in \mathbb{B}_{n-2}, A_{n-1} \in \mathbb{A}_{n-1}, a_{n-1} \colon B_{n-1} \to F_{n-1}(A_{n-1}) \otimes M_{n-2}(B_{n-2}) \otimes B'_{n-1}$ and $\overline{a}_{n-1} \colon A'_n \to A_n \otimes L_{n-1}(A_{n-1})$ such that the following compositions are equal

$$\begin{tikzcd}
B_n \arrow[d, "a_n"] \\
F_n(A_n) \otimes M_{n-1}(B_{n-1}) \otimes B'_n \arrow[d, "1 \otimes M_{n-1}(a_{n-1}) \otimes 1"] \\
F_n(A_n) \otimes M_{n-1}(F_{n-1}(A_{n-1}) \otimes M_{n-2}(B_{n-2}) \otimes B'_{n-1}) \otimes B'_n \arrow[d, "1 \otimes \lambda_{n-1}^{-1}(A_{n-1}) \otimes \beta_{n-2}(B_{n-2}) \otimes 1"] \\
F_n(A_n) \otimes F_n(L_{n-1}(A_{n-1})) \otimes M_{n-1}(B'_{n-1}) \otimes B'_n \arrow[d, "\simeq"] \\
F_n(A_n \otimes L_{n-1}(A_{n-1})) \otimes M_{n-1}(B'_{n-1}) \otimes B'_n
\end{tikzcd}$$

$$\begin{tikzcd}
B_n \arrow[d, "a'_n"'] \\
F_n(A'_n) \otimes M_{n-1}(B'_{n-1}) \otimes B'_n \arrow[d, "F_n(\overline{a}_{n-1}) \otimes 1 \otimes 1"'] \\
F_n(A_n \otimes L_{n-1}(A_{n-1})) \otimes M_{n-1}(B'_{n-1}) \otimes B'_n
\end{tikzcd}$$

Second construction of Δ_n : using the second description of $H^n(\mathbb{C}_\bullet)$, we can define the functor

$$\Delta_n \colon H^n(\mathbb{C}_\bullet) \to H^{n+1}(\mathbb{A}_\bullet)$$

on objects by

$$\Delta_n(B_n, [A_{n+1}, a_{n+1}]) = (A_{n+1}, t_{n+2} \colon L_{n+1}(A_{n+1}) \to I)$$

and on arrows by

$$\Delta_n[B_{n-1}, [A_n, a_n]] = [A_n, \overline{a}_n \colon A_{n+1} \to L_n(A_n) \otimes A'_{n+1}] \, .$$

Second description of $H^n(G_\bullet)$: we have to adapt the description of the functor

$$H^n(G_\bullet) \colon H^n(\mathbb{B}_\bullet) \to H^n(\mathbb{C}_\bullet)$$

to the second description of $H^n(\mathbb{C}_\bullet)$. The image of an object

$$(B_n \in \mathbb{B}_n, b_n \colon M_n(B_n) \to I)$$

of $H^n(\mathbb{B}_\bullet)$ is the object

$$(B_n \in \mathbb{B}_n, [I \in \mathbb{A}_{n+1}, b_n \colon M_n(B_n) \to I \simeq F_{n+1}(I)] \,)$$

(as arrow t_{n+2} one takes the canonical isomorphism $L_{n+1}(I) \simeq I$) and the image of an arrow

$$[Y_{n-1} \in \mathbb{B}_{n-1}, y_{n-1} \colon B_n \to M_{n-1}(Y_{n-1}) \otimes B'_n] \colon (B_n, b_n) \to (B'_n, b'_n)$$

of $H^n(\mathbb{B}_\bullet)$ is the arrow

$$[Y_{n-1}, [I \in \mathbb{A}_n, y_{n-1} \colon B_n \to M_{n-1}(Y_{n-1}) \otimes B'_n \simeq F_n(I) \otimes M_{n-1}(Y_{n-1}) \otimes B'_n]] \, .$$

Second construction of Σ_n : using the second description of the functors Δ_n and $H^n(G_\bullet)$, the 2-cell Σ_n is the identity 2-cell.

2-exactness in $H^n(\mathbb{C}_\bullet)$: we are going to prove that $\Gamma \colon H^n(\mathbb{B}_\bullet) \to Ker\Delta_n$ is full. For this, observe that an object in $Ker\Delta_n$ is a pair

$$\langle (B_n, [A_{n+1}, a_{n+1}]) \in H^n(\mathbb{C}_\bullet), [X_n, x_n] \colon (A_{n+1}, t_{n+2}) \to I \in H^{n+1}(\mathbb{A}_\bullet) \rangle$$

and an arrow in $Ker\Delta_n$ is an arrow $[B_{n-1}, [A_n, a_n]]$ in $H^n(\mathbb{C}_\bullet)$ (with its $\overline{a}_n$) such that there are $P_{n-1} \in \mathbb{A}_{n-1}$ and $p_{n-1} \colon A_n \otimes X'_n \to L_{n-1}(P_{n-1}) \otimes X_n$ making commutative a certain diagram. Consider now two objects $(B_n, b_n), (B'_n, b'_n)$ in $H^n(\mathbb{B}_\bullet)$ and an arrow

$$[B_{n-1}, [A_n, a_n]]\colon \Gamma(B_n, b_n) \to \Gamma(B'_n, b'_n)$$

in $Ker\Delta_n$. We put $Y_{n-1} = F_{n-1}(P_{n-1}) \otimes B_{n-1}$ and we define y_{n-1} by the following composition

$$\begin{tikzcd}
B_n \arrow[d, "a_n"] \\
F_n(A_n) \otimes M_{n-1}(B_{n-1}) \otimes B'_n \arrow[d, "F_n(p_{n-1})\otimes 1 \otimes 1"] \\
F_n(L_{n-1}(P_{n-1})) \otimes M_{n-1}(B_{n-1}) \otimes B'_n \arrow[d, "\lambda_{n-1}(P_{n-1})\otimes 1\otimes 1"] \\
M_{n-1}(F_{n-1}(P_{n-1})) \otimes M_{n-1}(B_{n-1}) \otimes B'_n \arrow[d, "\simeq"] \\
M_{n-1}(F_{n-1}(P_{n-1}) \otimes B_{n-1}) \otimes B'_n
\end{tikzcd}$$

Then $[Y_{n-1}, y_{n-1}]\colon (B_n, b_n) \to (B'_n, b'_n)$ is an arrow in $H^n(\mathbb{B}_\bullet)$. Finally, to check that $\Gamma[Y_{n-1}, y_{n-1}] = [B_{n-1}, [A_n, a_n]]$, we put $B_{n-2} = I, A_{n-1} = P_{n-1}, a_{n-1} = 1$ and $\overline{a}_{n-1} = p_{n-1}$.

Construction of Ψ_n : given an object

$$(B_n \in \mathbb{B}_n, [A_{n+1} \in \mathbb{A}_{n+1}, a_{n+1}\colon M_n(B_n) \to F_{n+1}(A_{n+1})]\,)$$

in $H^n(\mathbb{C}_\bullet)$, if we apply Δ_n and $H^{n+1}(F_\bullet)$ we obtain the following object of $H^{n+1}(\mathbb{B}_\bullet)$:

$$(F_{n+1}(A_{n+1}) \in \mathbb{A}_{n+2}, \lambda^{-1}_{n+1}(A_{n+1}) \cdot F_{n+2}(t_{n+2})\colon$$

$$M_{n+1}(F_{n+1}(A_{n+1})) \to F_{n+2}(L_{n+1}(A_{n+1})) \to F_{n+2}(I) \simeq I\,)$$

Such an object is naturally isomorphic to $(I \in \mathbb{B}_{n+1}, M_{n+1}(I) \simeq I)$, which is the unit object in $H^{n+1}(\mathbb{B}_\bullet)$, via the morphism

$$\Psi_n(B_n, [A_{n+1}, a_{n+1}]) = [B_n \in \mathbb{B}_n, a^{-1}_{n+1}\colon F_{n+1}(A_{n+1}) \to M_n(B_n)]\,.$$

$\square$

Remark 4.3 At this point, the reader probably wonders why we define the cohomology categorical groups of a complex using the relative kernels and relative cokernels, instead of the usual kernels and cokernels. The reason is the construction of the functor Δ_n involved in the previous theorem: such a functor does not exist if we define cohomology using the usual kernel and cokernel. To make clear the problem, imagine to define $H^n(\mathbb{C}_\bullet)$ using the usual kernel and cokernel, so that an object in $H^n(\mathbb{C}_\bullet)$ is just an object of $KerN_n$. Now, given an object

$$(C_n \in \mathbb{C}_n, c_n \colon N_n(C_n) \to I)$$

in $KerN_n$, we look for an object

$$\Delta_n(C_n, c_n) = (A_{n+1}, a_{n+1} \colon L_{n+1}(A_{n+1}))$$

in $KerL_{n+1}$. Since $G_n \colon \mathbb{B}_n \to \mathbb{C}_n$ is essentially surjective, there are an object $B_n \in \mathbb{B}_n$ and an arrow $b_n \colon G_n(B_n) \to C_n$, so that

$$(M_n(B_n), \mu_n(B_n) \cdot N_n(b_n) \cdot c_n \colon G_{n+1}(M_n(B_n)) \to I)$$

is an object in $KerG_{n+1}$. Since $(F_{n+1}, \varphi_{n+1}, G_{n+1})$ is 2-exact, there are an object $A_{n+1} \in \mathbb{A}_{n+1}$ and an arrow

$$x_{n+1} \colon (F_{n+1}(A_{n+1}), \varphi_{n+1}(A_{n+1})) \to (M_n(B_n), \mu_n(B_n) \cdot N_n(b_n) \cdot c_n)$$

in $KerG_{n+1}$. It remains to find an arrow

$$a_{n+1} \colon L_{n+1}(A_{n+1}) \to I$$

in $\mathbb{A}_{n+2}$. Since $(F_{n+2}, \varphi_{n+2}, G_{n+2})$ is 2-exact, it is enough to find an arrow

$$\tau \colon (F_{n+2}(L_{n+1}(A_{n+1})), \varphi_{n+2}(L_{n+1}(A_{n+1}))) \to (F_{n+2}(I), \varphi_{n+2}(I))$$

in $KerG_{n+2}$. We could take as τ the following composition

$$\lambda_{n+1}(A_{n+1}) \cdot M_{n+1}(x_{n+1}) \cdot \beta_n(B_n) \colon$$

$$F_{n+2}(L_{n+1}(A_{n+1})) \to M_{n+1}(F_{n+1}(A_{n+1})) \to M_{n+1}(M_n(B_n)) \to I \simeq F_{n+2}(I)\,.$$

Now, to check that τ is an arrow in $KerG_{n+2}$ amounts to check the commutativity of the following diagram

$$\begin{tikzcd}
N_{n+1}(N_n(C_n)) \arrow[rr, "N_{n+1}(c_n)"] \arrow[dr, "\gamma_n(C_n)"'] & & N_{n+1}(I) \\
& I \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

which precisely means that (C_n, c_n) is indeed an object of the relative kernel $Ker(N_n, \gamma_n)$.

5 Examples and applications

5.1 Complexes of abelian groups

First of all, let us point out that, when the complex of symmetric categorical groups is in fact a complex of abelian groups, then we get the usual cohomology groups applying π_0 and π_1 to the cohomology categorical groups. More precisely, consider a complex of abelian groups

$$A_\bullet = \; A_0 \xrightarrow{l_0} A_1 \xrightarrow{l_1} A_2 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} A_n \xrightarrow{l_n} A_{n+1} \xrightarrow{l_{n+1}} \cdots$$

with cohomology groups $H^n(A_\bullet) = Ker(l_n)/Im(l_{n-1})$. We can construct two complexes of symmetric categorical groups:

$$A_\bullet[0] = \; A_0[0] \xrightarrow{l_0[0]} A_1[0] \xrightarrow{l_1[0]} A_2[0] \; \ldots$$

$$A_\bullet[1] = \; A_0[1] \xrightarrow{l_0[1]} A_1[1] \xrightarrow{l_1[1]} A_2[1] \; \ldots$$

Proposition 5.1 *With the previous notations, we have*

1. $\pi_0(H^n(A_\bullet[0])) = H^n(A_\bullet) = \pi_1(H^{n+1}(A_\bullet[0]))$
2. $\pi_0(H^n(A_\bullet[1])) = H^{n+1}(A_\bullet) = \pi_1(H^{n+1}(A_\bullet[1]))$

Proof. We check only part 1 because the proof of part 2 is similar. If we specialize the description of $H^n(\mathbb{A}_\bullet)$ given in Section 3 to the case of $\mathbb{A}_\bullet = A_\bullet[0]$, we have that the objects are the elements of $Ker(l_n)$, and a pre-morphism $a_n \to a'_n$ is an element $x_{n-1} \in A_{n-1}$ such that $a_n = l_{n-1}(x_{n-1}) + a'_n$. It is now clear that $\pi_0(H^n(A_\bullet[0])) = H^n(A_\bullet)$. $\square$

5.2 Takeuchi-Ulbrich cohomology

Consider a complex of symmetric categorical groups

$$\begin{tikzcd} \mathbb{A}_\bullet = \; \mathbb{A}_0 \arrow[r, "L_0"] \arrow[rr, bend right, "0"'] & \mathbb{A}_1 \arrow[d, phantom, "\overset{\alpha_0}{\Downarrow}"] \arrow[r, "L_1"] & \mathbb{A}_2 \; \ldots \\ & {} & \end{tikzcd}$$

Each object $X_{n-1} \in \mathbb{A}_{n-1}$ gives rise to an object $(L_{n-1}(X_{n-1}), \alpha_{n-1}(X_{n-1})) \in Ker(L_n, \alpha_n)$. The isomorphism classes of these objects constitute a subgroup of the group of connected components $\pi_0(Ker(L_n, \alpha_n))$. From [15, 16], we recall the following definition.

Definition 5.2 With the previous notations, the n-th Takeuchi-Ulbrich cohomology group of the complex $\mathbb{A}_\bullet$ is the quotient group

$$H^n_U(\mathbb{A}_\bullet) = \pi_0(Ker(L_n, \alpha_n))/\langle\, [L_{n-1}(X_{n-1}), \alpha_{n-1}(X_{n-1})]\,\rangle_{X_{n-1}\in\mathbb{A}_{n-1}}$$

Proposition 5.3 *With the previous notations, we have group isomorphisms*

$$\pi_0(H^n(\mathbb{A}_\bullet)) \simeq H^n_U(\mathbb{A}_\bullet) \simeq \pi_1(H^{n+1}(\mathbb{A}_\bullet))$$

Proof. Explicitly, $\pi_0(Ker(L_n, \alpha_n))/\langle\, [L_{n-1}(X_{n-1}), \alpha_{n-1}(X_{n-1})]\,\rangle_{X_{n-1}\in\mathbb{A}_{n-1}}$ is the group of equivalence classes of pairs $(A_n \in \mathbb{A}_n, a_n \colon L_n(A_n) \to I)$ such that $L_{n+1}(a_n) = \alpha_n(A_n)$. Two pairs (A_n, a_n) and (A'_n, a'_n) are equivalent if there is $X_{n-1} \in \mathbb{A}_{n-1}$ such that (A_n, a_n) and $(L_{n-1}(X_{n-1}), \alpha_{n-1}(X_{n-1})) \otimes (A'_n, a'_n)$ are isomorphic in $Ker(L_n, \alpha_n)$. This amounts to ask that there is $x_{n-1} \colon A_n \to L_{n-1}(X_{n-1}) \otimes A'_n$ making commutative the following diagram

$$\begin{tikzcd}
L_n(A_n) \arrow[r, "L_n(x_{n-1})"] \arrow[d, "a_n"'] & L_n(L_{n-1}(X_{n-1}) \otimes A'_n) \arrow[d, "\simeq"] \\
I \simeq I \otimes I & L_n(L_{n-1}(X_{n-1})) \otimes L_n(A'_n) \arrow[l, "\alpha_{n-1}(X_{n-1})\otimes a'_n"]
\end{tikzcd}$$

If we look now at the description of $H^n(\mathbb{A}_\bullet)$ given in Section 3, it is clear that the previous description corresponds to $\pi_0(H^n(\mathbb{A}_\bullet))$. □

Since the functor

$$\pi_0 \colon \mathrm{SCG} \to \text{Abelian Groups}$$

sends 2-exact sequences into exact sequences (and π_1 also, see [19]), from Theorem 4.2 and Proposition 5.3 we get the following corollary.

Corollary 5.4 *Let*

$$\begin{tikzcd}
\mathbb{A}_\bullet \arrow[rr, "0"] \arrow[dr, "F_\bullet"'] & \Uparrow \varphi_\bullet & \mathbb{C}_\bullet \\
& \mathbb{B}_\bullet \arrow[ur, "G_\bullet"'] &
\end{tikzcd}$$

be an extension of complexes of symmetric categorical groups. There is a long exact sequence of abelian groups

$$\cdots \longrightarrow H^n_U(\mathbb{A}_\bullet) \longrightarrow H^n_U(\mathbb{B}_\bullet) \longrightarrow H^n_U(\mathbb{C}_\bullet) \longrightarrow H^{n+1}_U(\mathbb{A}_\bullet) \longrightarrow \cdots$$

5.3 Ulbrich exact sequence

If $\mathbb{B}$ is a symmetric categorical group, we can construct a canonical extension

$$\begin{tikzcd}
\pi_1(\mathbb{B})[1] \arrow[rr, "0"] \arrow[dr] & = & \pi_0(\mathbb{B})[0] \\
& \mathbb{B} \arrow[ur] &
\end{tikzcd}$$

where $\pi_1(\mathbb{B})[1] \to \mathbb{B}$ is just the inclusion, and $\mathbb{B} \to \pi_0(\mathbb{B})[0]$ sends an object on its isomorphism class (see [1]). Starting from a complex $\mathbb{B}_\bullet$ of symmetric categorical groups and repeating the previous construction at each degree, we obtain an extension of complexes

$$\pi_1(\mathbb{B}_\bullet)[1] \longrightarrow \mathbb{B}_\bullet \longrightarrow \pi_0(\mathbb{B}_\bullet)[0]$$

and we can apply Theorem 4.2. Using Proposition 5.1 and Proposition 5.3 to calculate π_0 of the 2-exact sequence of cohomology categorical groups, we get the following corollary, which is the main general result contained in [17].

Corollary 5.5 *Let $\mathbb{B}_\bullet$ be a complex of symmetric categorical groups. There is a long exact sequence of abelian groups*

$$\ldots H^{n+1}(\pi_1(\mathbb{B}_\bullet)) \longrightarrow H^n_U(\mathbb{B}_\bullet) \longrightarrow H^n(\pi_0(\mathbb{B}_\bullet)) \longrightarrow H^{n+2}(\pi_1(\mathbb{B}_\bullet)) \ldots$$

5.4 Hattori-Villamayor-Zelinsky exact sequence

If $\mathbb{C}$ is any (symmetric) monoidal category, the Picard categorical group $\mathbb{P}ic(\mathbb{C})$ is the (symmetric) categorical group of invertible objects and isomorphisms in $\mathbb{C}$. In particular, if R is a commutative ring with unit, $\mathbb{P}ic(R)$ is by definition $\mathbb{P}ic(R$-*mod*). It follows that $\pi_0(\mathbb{P}ic(R))$ is the usual Picard group of R, and $\pi_1(\mathbb{P}ic(R))$ is the group of units of R. Moreover, each ring homomorphism $f\colon R \to S$ induces a monoidal functor R-*mod* $\to$ S-*mod* and then a morphism of symmetric categorical groups (denoted with the same name) $f\colon \mathbb{P}ic(R) \to \mathbb{P}ic(S)$.

Starting from the ring homomorphism $f\colon R \to S$, we can construct the n-th tensor power

$$S^{\otimes n} = S \otimes_R S \otimes_R \ldots \otimes_R S.$$

Moreover, for each n, we have $n+1$ face homomorphisms

$$f_i\colon S^{\otimes n} \longrightarrow S^{\otimes n+1}$$

determined by $f_i(s_1 \otimes \ldots \otimes s_n) = s_1 \otimes \ldots \otimes s_i \otimes 1 \otimes s_{i+1} \otimes \ldots \otimes s_n$. The induced morphisms of symmetric categorical groups

$$f_i\colon \mathbb{P}ic(S^{\otimes n}) \longrightarrow \mathbb{P}ic(S^{\otimes n+1})$$

can be pasted together to obtain a complex $\mathbb{P}ic(S^{\otimes \bullet})$:

$$\ldots\ \mathbb{P}ic(S^{\otimes n-1}) \xrightarrow{L_{n-1}} \mathbb{P}ic(S^{\otimes n}) \xrightarrow{L_n} \mathbb{P}ic(S^{\otimes n+1})\ \ldots$$

where L_n is a kind of alternating tensor product:

$$L_n(X) = f_1(X)^* \otimes f_2(X) \otimes f_3(X)^* \otimes \ldots .$$

If we apply Corollary 5.5 to the complex $\mathbb{P}ic(S^{\otimes\bullet})$, we obtain the Hattori-Villamayor-Zelinsky sequence [9, 18], that is the $\mathcal{U}$-*Pic*-exact sequence associated with the ring homomorphism $f\colon R \to S$ (notations of Theorem 4.14 in [18], but ours $H^n(S/R,\mathcal{U})$ and $H^n(S/R, Pic)$ are their H^{n-1})

$$\ldots H^{n+1}(S/R,\mathcal{U}) \to H^n_U(S/R) \to H^n(S/R, Pic) \to H^{n+2}(S/R,\mathcal{U}) \ldots$$

(see also Theorem 6.1.3 in [2]).

5.5 Takeuchi exact sequence

If $\mathbb{C}$ is a symmetric monoidal category with stable coequalizers, a new symmetric monoidal category $Bim(\mathbb{C})$ can be obtained by taking as objects $\mathbb{C}$-monoids and as arrows isomorphism classes of bimodules. The Brauer categorical group of $\mathbb{C}$ is by definition

$$\mathbb{B}r(\mathbb{C}) = \mathbb{P}ic(Bim(\mathbb{C}))$$

(see [19]). If R is a commutative ring with unit, we put $\mathbb{B}r(R) = \mathbb{B}r(R\text{-}mod)$. One has that $\pi_0(\mathbb{B}r(R))$ is the usual Brauer group of R and $\pi_1(\mathbb{B}r(R))$ is the Picard group of R. Once again, a ring homomorphism $f\colon R \to S$ induces a morphism of symmetric categorical groups $\mathbb{B}r(R) \to \mathbb{B}r(S)$. Working in the same way as in the previous subsection, we get a complex of symmetric categorical groups $\mathbb{B}r(S^{\otimes\bullet})$:

$$\ldots\ \mathbb{B}r(S^{\otimes n-1}) \xrightarrow{L_{n-1}} \mathbb{B}r(S^{\otimes n}) \xrightarrow{L_n} \mathbb{B}r(S^{\otimes n+1})\ \ldots$$

If we apply Corollary 5.5 to the complex $\mathbb{B}r(S^{\otimes\bullet})$, we obtain the Takeuchi sequence [13], that is the *Picard-Brauer* exact sequence associated with the ring homomorphism $f\colon R \to S$ (notations of Theorem 6.4.2 in [2])

$$\ldots H^{n+1}(S/R, Pic) \to H^n_U(S/R, \underline{\underline{A}}) \to H^n(S/R, Br) \to H^{n+2}(S/R, Pic) \ldots$$

5.6 Simplicial cohomology, I

Given a simplicial set $X_\bullet$ with degeneracies

$$\delta_i\colon X_{n+1} \to X_n, \quad i = 0, \ldots, n+1$$

and a symmetric categorical group $\mathbb{A}$, following [11, 4] we can construct a cosimplicial complex $\mathbb{A}^{X_\bullet}$ of symmetric categorical groups and strict homomorphisms:

- $\mathbb{A}^{X_n}$ is the symmetric categorical group of functors from the discrete groupoid X_n to $\mathbb{A}$, under pointwise tensor product;
- the codegeneracies are given by composition with the degeneracies

$$d_i = - \cdot \delta_i\colon \mathbb{A}^{X_n} \to \mathbb{A}^{X_{n+1}}; \quad i = 0, \ldots, n+1\,.$$

Now, by taking alternating tensor product we get a complex of symmetric categorical groups $\mathcal{C}(\mathbb{A}^{X_\bullet})$:

$$\ldots\ \mathbb{A}^{X_{n-1}} \xrightarrow{L_{n-1}} \mathbb{A}^{X_n} \xrightarrow{L_n} \mathbb{A}^{X_{n+1}}\ \ldots$$

with $L_n(H) = d_0(H) \otimes d_1(H)^* \otimes d_2(H) \otimes \ldots$
The cohomology categorical groups of this complex are denoted by $H^n(X_\bullet, \mathbb{A})$.

Since a discrete groupoid X is "projective" with respect to essentially surjective functors, any extension

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow \varphi"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd}$$

in SCG gives rise to a new extension

$$\begin{tikzcd} \mathbb{A}^X \arrow[rr, "0"] \arrow[dr, "-\cdot F"'] & \arrow[d, phantom, "\Uparrow -\cdot\varphi"] & \mathbb{C}^X \\ & \mathbb{B}^X \arrow[ur, "-\cdot G"'] & \end{tikzcd}$$

By Theorem 4.2, we get the following corollary.

Corollary 5.6 *Let*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow \varphi"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd}$$

be an extension of symmetric categorical groups, and fix a simplicial set $X_\bullet$. There is a long 2-exact sequence of symmetric categorical groups

$$\ldots \to H^n(X_\bullet, \mathbb{A}) \to H^n(X_\bullet, \mathbb{B}) \to H^n(X_\bullet, \mathbb{C}) \to H^{n+1}(X_\bullet, \mathbb{A}) \to \ldots$$

Applying the functor $\pi_0 \colon \mathrm{SCG} \to$ Abelian Groups to the previous 2-exact sequence, we get the long exact sequence of abelian groups obtained in [4], Proposition 2.4.

5.7 Simplicial cohomology, II

Let $\mathbb{D}$ be a category. As simplicial set $X_\bullet$, we can take the nerve $Ner(\mathbb{D})$ of $\mathbb{D}$.

Proposition 5.7 *Let $\mathbb{D}$ be a category and $\mathbb{A}$ a symmetric categorical group. There is an equivalence of symmetric categorical groups*

$$Hom_{Cat}(\mathbb{D}, \mathbb{A}) \simeq H^0(Ner(\mathbb{D}), \mathbb{A}) .$$

Proof. Indeed, an object of $H^0(Ner(\mathbb{D}), \mathbb{A})$ is a pair (A_0, a_0), where A_0 is a map from the objects of $\mathbb{D}$ to those of $\mathbb{A}$, and a_0 associates to any arrow $f \colon X \to Y$ in $\mathbb{D}$ an arrow $a_0(f) \colon A_0(X) \otimes A_0(Y)^* \to I$. To such an arrow canonically corresponds an arrow $\tilde{a}_0(f) \colon A_0(X) \to A_0(Y)$, and the condition $L_1(a_0) = \alpha_0(A_0)$ gives that the pair $(A_0, \tilde{a}_0)$ is a functor from $\mathbb{D}$ to $\mathbb{A}$. (In fact, the condition $L_1(a_0) = \alpha_0(A_0)$ means that $\tilde{a}_0$ preserves the composition. This implies that it preserves also the identity arrows, because $\mathbb{A}$ is a groupoid.) $\square$

If $\mathbb{D}$ is a category and $\mathbb{A}$ a categorical group, the groupoid $Tors(\mathbb{D}, \mathbb{A})$ of $\mathbb{D}$-torsors under $\mathbb{A}$ has been studied in [5]. A $\mathbb{D}$-torsor under $\mathbb{A}$ is a Grothendieck cofibration $p \colon \mathbb{E} \to \mathbb{D}$ such that, for any $X \in \mathbb{D}$, the fibre category $\mathbb{E}_X$ is equivalent to $\mathbb{A}$ via a given action of $\mathbb{A}$ on $\mathbb{E}$. The arrows in $Tors(\mathbb{D}, \mathbb{A})$ are the $\mathbb{A}$-equivariant $\mathbb{D}$-functors. This groupoid is a 2-groupoid adding as 2-cells the $\mathbb{A}$-equivariant $\mathbb{D}$-homotopies. If $\mathbb{A}$ is symmetric, the next proposition provides the classifying groupoid of $Tors(\mathbb{D}, \mathbb{A})$ with a structure of symmetric categorical group.

Proposition 5.8 *Let $\mathbb{D}$ be a category and $\mathbb{A}$ a symmetric categorical group. There is an equivalence of groupoids*

$$cl(Tors(\mathbb{D}, \mathbb{A})) \simeq H^1(Ner(\mathbb{D}), \mathbb{A}) \,.$$

Proof. We limit our proof to the construction of the morphism

$$H^1(Ner(\mathbb{D}), \mathbb{A}) \to cl(Tors(\mathbb{D}, \mathbb{A}))$$

The objects of $H^1(Ner(\mathbb{D}), \mathbb{A})$ are the systems $(A_f, t_{g,f})$ consisting of

- for any morphism $f \colon X \to Y$ of $\mathbb{D}$, an object $A_f \in \mathbb{A}$, and

- for any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{D}$, a morphism $t_{g,f} \colon A_f \otimes A_g \to A_{fg}$ in $\mathbb{A}$

which satisfy a cocycle condition. So, an object of $H^1(Ner(\mathbb{D}), \mathbb{A})$ can be identified with a 2-cocycle in $\mathbb{D}$ with coefficients in $\mathbb{A}$ (see [5]). Thus any object (A, t) of $H^1(Ner(\mathbb{D}), \mathbb{A})$ defines a pseudo-functor and, following the Grothendieck construction, has canonically associated a cofibration $P \colon \mathbb{E}_{(A,t)} \to \mathbb{D}$. In Theorem 4.9 in [5] it is proved that $\mathbb{E}_{(A,t)}$ is in fact a $\mathbb{D}$-torsor under $\mathbb{A}$.
A pre-arrow $\varphi \colon (A, t) \to (A', t')$ in $H^1(Ner(\mathbb{D}), \mathbb{A})$ is a system $\varphi = (\varphi_X, \varphi_f)$ consisting of

- for any object $X \in \mathbb{D}$, an object $\varphi_X \in \mathbb{A}$, and

- for any morphism $f \colon X \to Y$ in $\mathbb{D}$, a morphism $\varphi_f \colon A_f \otimes \varphi_Y \to \varphi_X \otimes A'_f$ in $\mathbb{A}$

which makes certain diagrams commutative. A pre-arrow $\varphi \colon (A, t) \to (A', t')$ defines an $\mathbb{A}$-equivariant $\mathbb{D}$-functor $\mathbb{E}_\varphi \colon \mathbb{E}_{(A,t)} \to \mathbb{E}_{(A',t')}$ which sends an object $(B \in \mathbb{A}, X \in \mathbb{D}) \in \mathbb{E}_{(A,t)}$ to $(B \otimes \varphi_X, X)$.
Two pre-arrows $\varphi, \varphi' \colon (A, t) \to (A', t')$ of $H^1(Ner(\mathbb{D}), \mathbb{A})$ are identified if there is a collection of morphisms $\nu = \{\nu_X \colon \varphi_X \to \varphi'_X \mid X \in \mathbb{D}\}$ making a certain diagram commutative. It is easy to get an homotopy $\mathbb{E}_\nu \colon \mathbb{E}_\varphi \to \mathbb{E}_{\varphi'}$ from such a collection ν. $\square$

Corollary 5.9 *Let*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow\varphi"] & \mathbb{C} \\ & \mathbb{B} \arrow[ur, "G"'] & \end{tikzcd}$$

be an extension of symmetric categorical groups, and fix a category $\mathbb{D}$. There is a 2-exact sequence of symmetric categorical groups

$$\begin{tikzcd} Hom_{Cat}(\mathbb{D},\mathbb{A}) \arrow[r] & Hom_{Cat}(\mathbb{D},\mathbb{B}) \arrow[r] & Hom_{Cat}(\mathbb{D},\mathbb{C}) \arrow[dll] \\ cl(Tors(\mathbb{D},\mathbb{A})) \arrow[r] & cl(Tors(\mathbb{D},\mathbb{B})) \arrow[r] & cl(Tors(\mathbb{D},\mathbb{C})) \end{tikzcd}$$

If $\mathbb{D} = D[1]$ for D a group and $\mathbb{A} = A[1]$ for A an abelian group, then $\pi_0(cl(Tors(\mathbb{D},\mathbb{A}))) = Ext_{cen}(D,A)$, the group of equivalence classes of central extensions of D by A (Example 3.9 in [5]). So, applying the functor π_0 to the previous 2-exact sequence, we get an exact sequence of abelian groups involving the groups of central extensions.

5.8 Simplicial cohomology, III

In [3], the nerve $Ner_2(\mathbb{D})$ of a categorical group $\mathbb{D}$ has been introduced. Let us recall that $Ner_2(\mathbb{D})$ is the 3-coskeleton of the following truncated simplicial set:

- $Ner_2(\mathbb{D})_0 = \{0\}$,
- $Ner_2(\mathbb{D})_1 = Obj(\mathbb{D})$,
- $Ner_2(\mathbb{D})_2 = \{(x, D_0, D_1, D_2) \in Mor(\mathbb{D}) \times Obj(\mathbb{D})^3 \mid x\colon D_0 \otimes D_2 \to D_1\}$,
- $Ner_2(\mathbb{D})_3$ is the set of commutative diagrams in $\mathbb{D}$ of the form

$$\begin{tikzcd} D_{00} \otimes D_{03} \otimes D_{23} \arrow[r, "1\otimes x_3"] \arrow[d, "x_0\otimes 1"'] & D_{00} \otimes D_{13} \arrow[d, "x_1"] \\ D_{02} \otimes D_{23} \arrow[r, "x_2"'] & D_{11} \end{tikzcd}$$

Proposition 5.10 *Let $\mathbb{D}$ be a categorical group and $\mathbb{A}$ a symmetric categorical group. There is an equivalence of symmetric categorical groups*

$$Hom_{CG}(\mathbb{D},\mathbb{A}) \simeq H^1(Ner_2(\mathbb{D}),\mathbb{A})\,.$$

Proof. Let us restrict ourselves to the description of objects. An object of $H^1(Ner_2(\mathbb{D}),\mathbb{A})$ is a system (A_D, a_x) consisting of

- for any object $D \in \mathbb{D}$, an object $A_D \in \mathbb{A}$, and

- for any morphism $x\colon D_0 \otimes D_2 \to D_1$ in $\mathbb{D}$, a morphism $a_x\colon A_{D_0} \otimes A_{D_2} \to A_{D_1}$ in $\mathbb{A}$ such that, for all $(x_0, x_1, x_2, x_3) \in Ner_2(\mathbb{D})_3$, the following diagram commutes

$$\begin{tikzcd}
A_{D_{00}} \otimes A_{D_{03}} \otimes A_{D_{23}} \arrow[r, "1\otimes a_{x_3}"] \arrow[d, "a_{x_0}\otimes 1"'] & A_{D_{00}} \otimes A_{D_{13}} \arrow[d, "a_{x_1}"] \\
A_{D_{02}} \otimes A_{D_{23}} \arrow[r, "a_{x_2}"'] & A_{D_{11}}
\end{tikzcd}$$

Thus, we have a monoidal functor $A\colon \mathbb{D} \to \mathbb{A}$ defined by $A(D) = A_D$, with canonical morphisms given by $a_{1_{D_0\otimes D_2}}\colon A_{D_0} \otimes A_{D_2} \to A_{D_0\otimes D_2}$. $\square$

Finally, if the categorical group $\mathbb{D}$ is symmetric, it is possible to refine again its nerve to take into account the symmetric structure. We refere to [3] for a detailed description of the nerve $Ner_3(\mathbb{D})$ of a symmetric categorical group $\mathbb{D}$.

Proposition 5.11 *Let $\mathbb{D}$ and $\mathbb{A}$ be symmetric categorical groups. There is an equivalence of symmetric categorical groups*

$$Hom_{SCG}(\mathbb{D}, \mathbb{A}) \simeq H^2(Ner_3(\mathbb{D}), \mathbb{A})\,.$$

6 The kernel-cokernel lemma

In this section, we obtain the kernel-cokernel (or "snake" lemma) for symmetric categorical groups as a particular case of the long cohomology sequence of Theorem 4.2. We will then apply the lemma to get a low-dimensional cohomology sequence involving derivations of categorical groups.

6.1 The kernel-cokernel lemma for symmetric categorical groups

We start with two general lemmas on symmetric categorical groups.

Lemma 6.1 *Consider the following diagram in* SCG

$$\begin{tikzcd}
KerF \arrow[r, "e_F"] \arrow[d, "L"'] \arrow[rr, bend left, "0"] & \mathbb{A} \arrow[r, "F"] \arrow[d, "M"'] \arrow[u, phantom, "\Uparrow\epsilon_F"] & \mathbb{B} \arrow[d, "N"] \\
KerG \arrow[r, "e_G"'] \arrow[rr, bend right, "0"'] \arrow[ur, phantom, "\Rightarrow\psi"] & \mathbb{C} \arrow[r, "G"'] \arrow[d, phantom, "\Downarrow\epsilon_G"] \arrow[ur, phantom, "\Rightarrow\varphi"] & \mathbb{D}
\end{tikzcd}$$

where L and ψ are induced by the universal property of $KerG$ (so that $\psi, \varphi, \epsilon_F$ and ϵ_G are compatible).

1. *If N is full and faithful, then the left-hand square is a bi-pullback;*

2. *If, moreover, M is full (faithful) (essentially surjective), then L is full (faithful) (essentially surjective).*

Proof. 1. From [10], Proposition 5.2, recall that N is full and faithful iff for all $\mathbb{G} \in$ SCG, the functor $Hom_{SCG}(\mathbb{G}, N)$ is full and faithful. Using this fact, the proof is a (long) argument on bi-limits which holds in any 2-category.
2. It follows from the first part, using the stability under bi-pullback of the involved classes of morphisms (see Proposition 5.2 in [1]). $\square$

Lemma 6.2 *Consider the following diagram in* SCG

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\overset{\pi_F}{\Uparrow}"] \arrow[d, "L"'] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & \mathbb{B} \arrow[r, "P_F"] \arrow[d, "M"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & CokerF \arrow[d, "N"] \\
\mathbb{C} \arrow[r, "G"'] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\overset{\pi_G}{\Downarrow}"] & \mathbb{D} \arrow[r, "P_G"'] & CokerG
\end{tikzcd}$$

where N and φ are induced by the universal property of $CokerF$ (so that ψ, φ, π_F and π_G are compatible).

1. *If L is full and essentially surjective, then the right-hand square is a bi-pushout;*

2. *If, moreover, M is full (faithful) (essentially surjective), then N is full (faithful) (essentially surjective).*

Proof. Dual of the previous one: by Proposition 5.3 in [10], L is full and essentially surjective iff for all $\mathbb{G} \in$ SCG, the functor $Hom_{SCG}(N, \mathbb{G})$ is full and faithful; the stability under bi-pushout is established in [1], Proposition 5.1. $\square$

Fix now the following diagram in SCG

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "L"'] \arrow[rr, bend left=50, "0"] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathbb{B} \arrow[r, "G"] \arrow[d, "M"] \arrow[u, phantom, "\overset{\varphi}{\Uparrow}"] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathbb{C} \arrow[d, "N"] \\
\mathbb{A}' \arrow[r, "F'"'] \arrow[rr, bend right=50, "0"'] & \mathbb{B}' \arrow[r, "G'"'] \arrow[d, phantom, "\overset{\varphi'}{\Downarrow}"] & \mathbb{C}'
\end{tikzcd} \tag{5}$$

where (F, φ, G) and (F', φ', G') are 2-exact sequences, G is essentially surjective and F' is faithful. We assume also that $\varphi, \varphi', \lambda$ and μ are compatible (as at the beginning of Section 4).

Proposition 6.3 (The kernel-cokernel lemma) *There are a morphism and two 2-cells in* SCG

$$\Delta \colon KerN \to CokerL \qquad \Sigma \colon \overline{G} \cdot \Delta \Rightarrow 0 \qquad \Psi \colon \Delta \cdot \overline{F'} \Rightarrow 0$$

making the following sequence 2-exact in each point

$$\begin{tikzcd}
 & KerM \arrow[rr, "0"] \arrow[dr, "\overline{G}"] \arrow[d, phantom, "\overset{\overline{\varphi}}{\Downarrow}"] & & CokerL \arrow[rr, "0"] \arrow[dr, "\overline{F'}"] \arrow[d, phantom, "\overset{\Psi}{\Downarrow}"] & & CokerN \\
KerL \arrow[ur, "\overline{F}"] \arrow[rr, "0"'] & & KerN \arrow[ur, "\Delta"] \arrow[rr, "0"'] \arrow[u, phantom, "\overset{\Sigma}{\Uparrow}"] & & CokerM \arrow[ur, "\overline{G'}"'] \arrow[u, phantom, "\overset{\overline{\varphi'}}{\Uparrow}"] &
\end{tikzcd}$$

Proof. Consider the factorization of F as a full and essentially surjective functor F_1 followed by a faithful functor F_2 (Proposition 2.1 in [10]). Consider also the factorization of G' as an essentially surjective functor G_1 followed by a full and faithful functor G_2 (Proposition 2.3 in [10]

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F_1"] \arrow[d, "L"'] & \mathbb{I} \arrow[r, "F_2"] \arrow[dl, dotted, "H"] \arrow[dr, phantom, "\overset{\alpha''}{\Rightarrow}"] & \mathbb{B} \arrow[rr, "G"] \arrow[d, "M"] & & \mathbb{C} \arrow[d, "N"] \arrow[dl, dotted, "K"'] \\
\mathbb{A}' \arrow[rr, "F'"'] \arrow[u, phantom, "\overset{\alpha'}{\Leftarrow}"] & & \mathbb{B}' \arrow[r, "G_1"'] \arrow[ur, phantom, "\overset{\beta'}{\Leftarrow}"] & \mathbb{I}' \arrow[r, "G_2"'] \arrow[ur, phantom, "\overset{\beta''}{\Rightarrow}"] & \mathbb{C}'
\end{tikzcd}$$

Since F_1 is orthogonal to F' (Proposition 4.3 in [10]) and G is orthogonal to G_2 (Proposition 4.6 in [10]), we get the fill-in H, α', α'' and K, β', β'' as in the previous diagram. Moreover, since F_1 is full and essentially surjective, there is a unique 2-cell $\psi \colon F_2 \cdot G \Rightarrow 0$ such that $F_1 \cdot \psi = \varphi$; since G_2 is full and faithful, there is a unique 2-cell $\psi' \colon F' \cdot G_1 \Rightarrow 0$ such that $\psi' \cdot G_2 = \varphi'$. In this way, we have constructed a new diagram in SCG

$$
\begin{tikzcd}
\mathbb{I} \arrow[rr, bend left, "0"] \arrow[r, "F_2"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\alpha''}{\Rightarrow}"] & \mathbb{B} \arrow[u, phantom, "\overset{\psi}{\Uparrow}"] \arrow[r, "G"] \arrow[d, "M"] \arrow[dr, phantom, "\overset{(\beta')^{-1}}{\Rightarrow}"] & \mathbb{C} \arrow[d, "K"] \\
\mathbb{A}' \arrow[rr, bend right, "0"'] \arrow[r, "F'"'] & \mathbb{B}' \arrow[d, phantom, "\overset{\psi'}{\Downarrow}"] \arrow[r, "G_1"'] & \mathbb{I}'
\end{tikzcd}
\tag{6}
$$

Composing with F_1 and G_2, we can check the compatibility of the 2-cells in (6) using that of the 2-cells in (5). Moreover, (F_2, ψ, G) is 2-exact (and then it is an extension) because, by Lemma 6.2, the cokernel of F_2 is equivalent to the cokernel of F. Analogously, (F', ψ', G_1) is 2-exact because, by Lemma 6.1, the kernel of G_1 is equivalent to the kernel of G.
Now, adding zero-morphisms and canonical 2-cells, we can turn the morphism of extensions (6) into an extension of complexes. The only non trivial cohomology categorical groups of these complexes are the (usual) kernels and cokernels of H, M and K. Therefore, Theorem 4.2 gives us the following 2-exact sequence

$$
\begin{tikzcd}
KerH \arrow[r] & KerM \arrow[r] & KerK \arrow[dll] \\
CokerH \arrow[r] & CokerM \arrow[r] & CokerK
\end{tikzcd}
$$

Observe now that, by Lemma 6.1, $KerK$ and $KerN$ are equivalent, and, by Lemma 6.2, $CokerH$ and $CokerL$ are equivalent. Moreover, by Lemma 6.1 again, the comparison $KerL \to KerH$ is full and essentially surjective, so that the 2-exactness of $KerH \to KerM \to KerK$ implies the 2-exactness of $KerL \to KerM \to KerK$. In the same way, by Lemma 6.2 the comparison $CokerK \to CokerN$ is full and faithful, so that $CokerH \to CokerM \to CokerN$ is 2-exact. Finally, we have proved the 2-exactness of

$$
\begin{tikzcd}
KerL \arrow[r] & KerM \arrow[r] & KerN \arrow[dll] \\
CokerL \arrow[r] & CokerM \arrow[r] & CokerN\,.
\end{tikzcd}
$$

□

6.2 Derivations of categorical groups

To end, we explain how the low-dimensional cohomology sequence obtained in [7], Theorem 6.2, is a special case of the 2-exact sequence of Proposition 6.3. For detailed definitions about derivations of categorical groups, we refer to [7, 8].

Fix a categorical group $\mathbb{G}$ and a symmetric $\mathbb{G}$-module $\mathbb{B}$ with action

$$\cdot\colon \mathbb{G} \times \mathbb{B} \to \mathbb{B}\,.$$

A *derivation* is a functor $D\colon \mathbb{G} \to \mathbb{B}$ together with a natural and coherent family of isomorphisms

$$\delta_{X,Y}\colon D(X) \otimes X \cdot D(Y) \to D(X \otimes Y)\,.$$

Derivations and their morphisms give rise to a groupoid $Der(\mathbb{G},\mathbb{B})$, which is a symmetric categorical group under pointwise tensor product. (Observe that, in general, if the $\mathbb{G}$-module $\mathbb{B}$ is only braided, the categorical group $Der(\mathbb{G},\mathbb{B})$ is no longer braided.) This construction plainly extends to a 2-functor from the 2-category of symmetric $\mathbb{G}$-modules and equivariant morphisms to SCG. Moreover, for any symmetric $\mathbb{G}$-module $\mathbb{B}$, there is an "inner derivation" morphism

$$\mathcal{I}\colon \mathbb{B} \to Der(\mathbb{G},\mathbb{B}) \qquad \mathcal{I}(B)\colon \mathbb{G} \to \mathbb{B} \qquad \mathcal{I}(B)(X) = X \cdot B \otimes B^*$$

whose kernel and cokernel are denoted by $\mathcal{H}^0(\mathbb{G},\mathbb{B})$ and $\mathcal{H}^1(\mathbb{G},\mathbb{B})$ and called the low-dimensional cohomology categorical groups of $\mathbb{G}$ with coefficients in $\mathbb{B}$. Now, if $F\colon \mathbb{A} \to \mathbb{B}$ is an equivariant morphism of symmetric $\mathbb{G}$-modules, its equivariant structure induces a 2-cell in SCG

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "\mathcal{I}"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathbb{B} \arrow[d, "\mathcal{I}"] \\
Der(\mathbb{G},\mathbb{A}) \arrow[r, "-\cdot F"'] & Der(\mathbb{G},\mathbb{B})
\end{tikzcd}$$

Finally, if

$$\begin{tikzcd}
& \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\varphi"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C}
\end{tikzcd}$$

is an extension of symmetric $\mathbb{G}$-modules, by Proposition 3.4 in [8] we get a diagram in SCG

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[rr, bend left, "0"] \arrow[d, "\mathcal{I}"'] \arrow[dr, phantom, "\Rightarrow\lambda"] & \mathbb{B} \arrow[r, "G"] \arrow[u, phantom, "\Uparrow\varphi"] \arrow[d, "\mathcal{I}"] \arrow[dr, phantom, "\Rightarrow\mu"] & \mathbb{C} \arrow[d, "\mathcal{I}"] \\
Der(\mathbb{G},\mathbb{A}) \arrow[r, "-\cdot F"'] \arrow[rr, bend right, "0"'] & Der(\mathbb{G},\mathbb{B}) \arrow[r, "-\cdot G"'] \arrow[d, phantom, "\Downarrow -\cdot\varphi"] & Der(\mathbb{G},\mathbb{C}) \\
& {} &
\end{tikzcd}$$

with $(- \cdot F, - \cdot \varphi, - \cdot G)$ 2-exact and $- \cdot F$ faithful. Since it is straightforward to check the compatibility of λ, μ, φ and $- \cdot \varphi$, as a corollary of Proposition 6.3 we get the 2-exact cohomology sequence

$$\mathcal{H}^0(\mathbb{G}, \mathbb{A}) \to \mathcal{H}^0(\mathbb{G}, \mathbb{B}) \to \mathcal{H}^0(\mathbb{G}, \mathbb{C}) \to \mathcal{H}^1(\mathbb{G}, \mathbb{A}) \to \mathcal{H}^1(\mathbb{G}, \mathbb{B}) \to \mathcal{H}^1(\mathbb{G}, \mathbb{C}) \quad (7)$$

If $\mathbb{G}$ is a discrete categorical group, and $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ are discrete $\mathbb{G}$-modules, then applying π_0 to the previous sequence we recover the familiar exact sequence of low-dimensional cohomology groups. Several other particular cases of interest are discussed in [7]. The non symmetric analogue of the 2-exact sequence (7) is studied in [6].

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