

# SPLIT EXTENSIONS, SEMIDIRECT PRODUCT, AND HOLOMORPH OF CATEGORICAL GROUPS

STEFANO KASANGIAN, GIUSEPPE METERE, AND ENRICO M. VITALE

**ABSTRACT.** Working in the context of categorical groups, we show that the semidirect product provides a biequivalence between actions and points. From this biequivalence, we deduce a 2-dimensional classification of split extensions of categorical groups, as well as the universal property of the holomorph of a categorical group. We also discuss the link between the holomorph and inner autoequivalences.

## 1. Introduction

A categorical group is a monoidal groupoid in which the objects are invertible, up to isomorphism, with respect to the tensor product. Using categorical groups, several well-known notions and results which classically hold for groups can be raised to a categorical level. For example, the correspondence between actions of a group  $H$  on a group  $G$  and split extensions of  $G$  by  $H$  has a categorical version established by Garzón and Inassaridze [15]. More precisely, following [7] they define a split extension as a diagram of categorical groups and monoidal functors

$$\mathbb{G} \xrightarrow{\mathbf{F}} \mathbb{A} \begin{array}{c} \xleftarrow{\mathbf{G}} \\ \xrightarrow{\mathbf{S}} \end{array} \mathbb{H}$$

together with a monoidal natural transformation  $\varphi: \mathbf{GF} \Rightarrow 0$  such that  $\mathbf{G}$  is a Grothendieck dense fibration, the factorization of  $\mathbf{F}$  through the kernel of  $\mathbf{G}$  is an equivalence and  $\mathbf{GS} = 1_{\mathbb{H}}$ . Then, they prove that equivalence classes of split extensions bijectively correspond to equivalence classes of actions of  $\mathbb{H}$  on  $\mathbb{G}$  (where an action of  $\mathbb{H}$  on  $\mathbb{G}$  is defined as a monoidal functor  $\mathbb{H} \rightarrow \mathcal{E}q(\mathbb{G})$ ,  $\mathcal{E}q(\mathbb{G})$  being the categorical group of monoidal autoequivalences of  $\mathbb{G}$ ).

The quoted paper [15] is one of a series of papers devoted to extensions and derivations in the context of categorical groups, see [6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 23]; these papers contain a number of examples showing that categorical groups provide a unifying framework for studying classical homological algebra. Despite of its interest, the paper [15] suffers of a mixture of 1-dimensional and 2-dimensional arguments, and the aim of this note is to put some of the results by Garzón and Inassaridze into their proper 2-categorical context. More precisely, the aim of this note is to improve the results by Garzón and Inassaridze observing that:

- The assumption that  $\mathbf{G}: \mathbb{A} \rightarrow \mathbb{H}$  is a Grothendieck fibration can be avoided and, consequently, the equation  $\mathbf{GS} = 1_{\mathbb{H}}$  can be replaced by an invertible 2-cell. The

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notion of morphism of extensions can also be weakened, and (split) extensions can be organized in a 2-category (in fact, a 2-groupoid).

- The bijection between classes of split extensions and actions is induced by a biequivalence between the 2-category of split extensions and a locally discrete 2-category of actions. (Let us point out here that in [23], Theorem 2, Rousseau establishes a complete classification of not necessarily split extensions of categorical groups. Rousseau’s classification is based on the notion of monoidal fibration of bitorsors and is quite far from that we propose here for split extensions.)
- The biequivalence between split extensions and actions is just the restriction of a more fundamental biequivalence between the 2-category of actions and a suitable 2-category whose objects are “points”, *i.e.* diagrams of categorical groups of the form

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{1_{\mathbb{H}}} & \mathbb{H} \\ & \searrow \mathbf{B} & \nearrow \mathbf{A} \\ & \mathbb{A} & \end{array}$$

Moreover, we introduce the holomorph of a categorical group  $\mathbb{G}$  as the semidirect product

$$\mathcal{H}ol\mathbb{G} = \mathbb{G} \rtimes \mathcal{E}q(\mathbb{G})$$

(where the action  $\mathcal{E}q(\mathbb{G}) \times \mathbb{G} \rightarrow \mathbb{G}$  is the evaluation) and we deduce the universal property of  $\mathcal{H}ol\mathbb{G}$  directly from the biequivalence between split extensions and actions. Finally, we explain in what sense the holomorph is the universal solution to the problem of converting autoequivalences into inner autoequivalences.

The paper is organized as follows. In Section 2, for the reader convenience, we recall the classification of split extensions of groups following the modern approach due to Bourn and Janelidze [5, 19]. We also discuss various universal properties of the holomorph; the first of them has been studied in the more general context of semi-abelian categories by Borceux, Janelidze and Kelly in [3, 4]. In Section 3 we recall basic definitions and properties about categorical groups, and we fix notations. The content of Section 4 is the biequivalence between points and actions, and its restriction to split extensions. The final section is devoted to the holomorph of a categorical group.

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## 2. The case of groups

### 2.1. THE EQUIVALENCE BETWEEN ACTIONS AND POINTS.

Let us first describe the category **Act** of actions in the category **Grp** of groups. An object is given by a group  $G$ , a group  $H$  and an action of  $H$  on  $G$ , that is, a map

$$H \times G \rightarrow G \quad (x, a) \mapsto {}^x a$$

such that  $x \cdot y a = x(y a)$ ,  $1 a = a$  and  $x(a + b) = x a + x b$  (we use multiplicative notation for  $H$  and additive notation for  $G$ ). Given two actions  $H \times G \rightarrow G$  and  $H' \times G' \rightarrow G'$ , a morphism is a pair of equivariant group homomorphisms  $g: G \rightarrow G'$  and  $h: H \rightarrow H'$ , that is  $g(x a) = h(x) g(a)$ .

We pass now to the category **Pt** of points in the category of groups. An object is a diagram of groups and homomorphisms

$$A \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} H$$

such that  $\alpha\beta = 1_H$ . An arrow from  $(A, H, \alpha, \beta)$  to  $(A', H', \alpha', \beta')$  is a pair of group homomorphisms  $f: A \rightarrow A'$  and  $h: H \rightarrow H'$  such that  $f\beta = \beta'h$  and  $h\alpha = \alpha'f$ .

There is a functor

$$\mathcal{K}: \mathbf{Pt} \rightarrow \mathbf{Act}$$

which sends a point  $(A, H, \alpha, \beta)$  to the action of  $H$  on  $\mathcal{K}er\alpha$ , the kernel of  $\alpha$ , given by

$$H \times \mathcal{K}er\alpha \rightarrow \mathcal{K}er\alpha \quad (x, a) \mapsto \beta(x) + a - \beta(x)$$

This functor is an equivalence, with quasi-inverse given by semidirect product

$$\rtimes: \mathbf{Act} \rightarrow \mathbf{Pt}$$

Indeed, if  $H \times G \rightarrow G$  is an action, the semidirect product  $G \rtimes H$  is the cartesian product  $G \times H$  equipped with a group structure given by  $(a, x) + (b, y) = (a + x b, x \cdot y)$ ,  $0 = (0, 1)$ ,  $-(a, x) = (-x^{-1} a, x^{-1})$ . Moreover, we get a point

$$G \rtimes H \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} H$$

with  $p(a, x) = x$  and  $i(x) = (0, x)$ .

## 2.2. THE CLASSIFICATION OF SPLIT EXTENSIONS.

There are two forgetful functors:

$$\begin{array}{ll} \mathcal{U}_*: \mathbf{Pt} \rightarrow \mathbf{Grp} & (A, H, \alpha, \beta) \mapsto H \\ \mathcal{O}: \mathbf{Act} \rightarrow \mathbf{Grp} & H \times G \rightarrow G \mapsto H \end{array}$$

making the diagram

$$\begin{array}{ccc} \mathbf{Pt} & \xrightarrow{\mathcal{K}} & \mathbf{Act} \\ & \searrow \mathcal{U}_* & \swarrow \mathcal{O} \\ & \mathbf{Grp} & \end{array}$$

commutative. If we fix a group  $H$  and consider the fibers over  $H$  of these forgetful functors, the equivalence  $\mathcal{K}$  restricts to an equivalence  $\mathcal{K}': \mathbf{Pt}(H) \rightarrow \mathbf{Act}(H)$ .

Now we can define two other functors:

$$\begin{aligned} \mathcal{N}: \mathbf{Pt}(H) &\rightarrow \mathbf{Grp} & (A, \alpha, \beta) &\mapsto \mathcal{Ker}\alpha \\ \mathcal{U}: \mathbf{Act}(H) &\rightarrow \mathbf{Grp} & H \times G \rightarrow G &\mapsto G \end{aligned}$$

Once again, the diagram

$$\begin{array}{ccc} \mathbf{Pt}(H) & \xrightarrow{\mathcal{K}'} & \mathbf{Act}(H) \\ & \searrow \mathcal{N} & \swarrow \mathcal{U} \\ & \mathbf{Grp} & \end{array}$$

commutes and, if we fix a group  $G$ , the equivalence  $\mathcal{K}'$  restricts to an equivalence

$$\mathcal{K}'': \mathbf{Ext}_{split}(H, G) \rightarrow \mathbf{Act}(H, G)$$

on the fibers. Explicitly,  $\mathbf{Act}(H, G)$  is the set (= discrete category) of actions of  $H$  on  $G$ , and  $\mathbf{Ext}_{split}(H, G)$  is the groupoid whose objects are split extensions of  $G$  by  $H$ , that is, points  $(A, H, \alpha, \beta)$  such that  $G$  is the kernel of  $\alpha$ , and arrows are homomorphisms  $f: A \rightarrow A'$  such that  $\alpha'f = \alpha$ ,  $f\beta = \beta'$ , and inducing the identity on  $G$ .

### 2.3. FIRST UNIVERSAL PROPERTY OF THE HOLOMORPH.

If we denote by  $\mathbf{EXT}_{split}(H, G)$  the set of isomorphism classes of split extensions, the equivalence  $\mathcal{K}'': \mathbf{Ext}_{split}(H, G) \rightarrow \mathbf{Act}(H, G)$  induces a bijection

$$\mathbf{EXT}_{split}(H, G) \simeq \mathbf{Act}(H, G)$$

Moreover,  $\mathbf{Act}(H, G)$  is also in bijection with the set of group homomorphisms from  $H$  to the group of automorphisms of  $G$

$$\mathbf{Act}(H, G) \simeq \mathbf{Hom}(H, \mathcal{A}utG)$$

Putting together these bijections and letting the group  $H$  vary, we get a natural isomorphism of functors:

$$\theta: \mathbf{EXT}_{split}(-, G) \Longrightarrow \mathbf{Hom}(-, \mathcal{A}utG) : \mathbf{Grp}^{op} \rightarrow \mathbf{Set}$$

where  $\mathbf{EXT}_{split}(-, G)$  is defined on a morphism  $h: H \rightarrow H'$  by the following pullback

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \alpha \downarrow & & \downarrow \alpha' \\ H & \xrightarrow{h} & H' \end{array}$$

which is a split-pullback, in the sense that as section  $\beta$  of  $\alpha$  one takes the one induced by the section  $\beta'$  of  $\alpha'$ .

It is a general fact, coming from Yoneda's lemma, that a functor  $F: \mathcal{C}^{op} \rightarrow \mathbf{Set}$  is representable, that is, there is an object  $R \in \mathcal{C}$  and a natural isomorphism  $\theta: F \xrightarrow{\cong} \mathcal{C}(-, R)$ , if and only if there is an element  $a \in F(R)$ , given by  $\theta_R^{-1}(1_R)$ , satisfying the following universal property: for all  $H \in \mathcal{C}$  and for all  $x \in F(H)$ , there is a unique  $h: H \rightarrow R$  such that  $x = F(h)(a)$ .

If we apply this general fact to  $\theta: \text{EXT}_{\text{split}}(-, G) \xrightarrow{\cong} \mathbf{Hom}(-, \mathcal{A}utG)$ , we get as element  $\theta^{-1}(1_{\mathcal{A}utG})$  the (isomorphism class of) the holomorph  $\mathcal{H}ol(G) = G \rtimes \mathcal{A}utG$  of  $G$

$$G \xrightarrow{in_1} G \rtimes \mathcal{A}utG \begin{array}{c} \xleftarrow{pr} \\ \xrightarrow{in_2} \end{array} \mathcal{A}utG$$

Accordingly, its universal property is: for any group  $H$  and for any split extension  $G \longrightarrow A \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} H$ , there is a unique homomorphism  $h: H \rightarrow \mathcal{A}utG$  such that  $A \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} H$  is the split-pullback of  $G \rtimes \mathcal{A}utG \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{A}utG$  along  $h$ .

#### 2.4. SECOND UNIVERSAL PROPERTY OF THE HOLOMORPH.

Since inner automorphisms are easier to handle than general automorphisms, a natural question is if a group  $G$  can be embedded into a group  $G'$  in such a way that each automorphism of  $G$  is the restriction of an inner automorphism of  $G'$ . This is the way the holomorph is classically introduced [22]: the canonical injections

$$G \xrightarrow{in_1} \mathcal{H}ol(G) = G \rtimes \mathcal{A}ut(G) \xleftarrow{in_2} \mathcal{A}ut(G)$$

are such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{in_1} & \mathcal{H}ol(G) \\ f \downarrow & & \downarrow \mathcal{I}_{in_2(f)} \\ G & \xrightarrow{in_1} & \mathcal{H}ol(G) \end{array}$$

commutes for any  $f \in \mathcal{A}ut(G)$ , where  $\mathcal{I}_{in_2(f)} \in \mathcal{A}ut(\mathcal{H}ol(G))$  is the inner automorphism associated with the element  $in_2(f) \in \mathcal{H}ol(G)$ . In fact, more is true: the holomorph is the universal solution for such a problem. Given a group  $A$  and two morphisms

$$G \xrightarrow{g} A \xleftarrow{h} \mathcal{A}ut(G)$$

such that, for every  $f \in \mathcal{A}ut(G)$ , the diagram

$$\begin{array}{ccc} G & \xrightarrow{g} & A \\ f \downarrow & & \downarrow \mathcal{I}_{h(f)} \\ G & \xrightarrow{g} & A \end{array} \tag{1}$$

commutes, there is a unique morphism  $\gamma: \mathcal{H}ol(G) \rightarrow A$  such that  $\gamma \cdot in_1 = g, \gamma \cdot in_2 = h$  (just put  $\gamma(x, f) = g(x)h(f)$  for  $x \in G, f \in \mathcal{A}ut(G)$ ).

This second universal property of the holomorph is a special case of a more general fact: the functor

$$\times: \mathbf{Act} \rightarrow \mathbf{Grp}$$

has a right adjoint

$$\mathcal{C}: \mathbf{Grp} \rightarrow \mathbf{Act}$$

which associates to a group  $A$  the conjugation  $A \times A \rightarrow A : (x, y) \mapsto xyx^{-1}$ . Indeed, equivariant pairs  $(g: G \rightarrow A, h: H \rightarrow A)$  bijectively correspond to group homomorphisms  $\gamma: G \times H \rightarrow A$ : given  $(g, h)$ , put  $\gamma(a, x) = g(a)h(x)$ ; conversely, precompose  $\gamma$  with the canonical injections in the semidirect product. Now, if one specializes this situation taking  $\mathcal{A}ut(G)$  as group  $H$  and the evaluation  $\mathcal{A}ut(G) \times G \rightarrow G$  as action, then the condition on  $(g, h)$  to be equivariant precisely says that, for all  $f \in \mathcal{A}ut(G)$ , diagram (1) commutes. (Note that, via the equivalence  $\mathbf{Act} \simeq \mathbf{Pt}$ , the adjunction  $\times \dashv \mathcal{C}$  corresponds to the adjunction  $\mathcal{U}^* \dashv \Delta: \mathbf{Pt} \rightarrow \mathbf{Grp}$ , where  $\mathcal{U}^*(A, H, \alpha, \beta) = A$  and  $\Delta(A)$  is the diagonal point.)

## 2.5. HOLOMORPH AND THE MAL'TSEV OPERATION.

The holomorph of a group  $G$  can be described also via the canonical Mal'tsev operation. For the sake of completeness, we recall this description here, but we leave the generalization to categorical groups as an exercise for the reader. Recall that the canonical Mal'tsev operation on  $G$  is the ternary operation defined by

$$m: G \times G \times G \rightarrow G \quad m(x, y, z) = x - y + z$$

Let us call  $\mathcal{M}al(G)$  the subgroup of the group of transformations on  $G$  given by those bijections  $t: G \rightarrow G$  such that  $t(x - y + z) = t(x) - t(y) + t(z)$ . There is an equivariant pair

$$g: G \rightarrow \mathcal{M}al(G) \quad g_x(y) = x + y \quad ; \quad h: \mathcal{A}ut(G) \rightarrow \mathcal{M}al(G) \quad h = \text{inclusion}$$

By the second universal property of the holomorph, from the pair  $(g, h)$  we get a morphism

$$\gamma: \mathcal{H}ol(G) \rightarrow \mathcal{M}al(G) \quad \gamma(x, f) = g_x \cdot f$$

The morphism  $\gamma$  is an isomorphism: if we put  $t = g_x \cdot f$ , we can obtain  $x$  and  $f$  using the formulae  $x = t(0)$  and  $f = g_{t(0)}^{-1} \cdot t$ .

## 3. Preliminaries on categorical groups

### 3.1. THE 2-CATEGORY OF CATEGORICAL GROUPS.

Basic facts on categorical groups can be found in [1, 7, 14, 20, 21, 24, 26]. Let us recall that a categorical group  $\mathbb{G} = (\mathbb{G}, \otimes, I, \dots)$  is a monoidal groupoid in which the objects are invertible, up to isomorphism, with respect to the tensor product. This means that, for each object  $A$  in  $\mathbb{G}$ , there is an object  $A^*$  and a morphism  $\epsilon_A: A^* \otimes A \rightarrow I$ . One can then

choose a morphism  $\eta_A: I \rightarrow A \otimes A^*$  so that  $(A^*, A, \eta_A, \epsilon_A)$  is a duality. Extending this to arrows, one gets an op-monoidal equivalence  $\mathbb{G} \rightarrow \mathbb{G}^{op}$ . Categorical groups are the objects of a 2-category  $\mathbf{CG}$ , homomorphisms of categorical groups are just monoidal functors, and 2-cells are monoidal natural transformations (which always are natural isomorphisms). A point of notation: if  $\mathbf{F}: \mathbb{G} \rightarrow \mathbb{H}$  is a monoidal functor, its monoidal structure is denoted by  $\mu^{\mathbf{F}} = \mu_{X,Y}^{\mathbf{F}}: F(X \otimes Y) \rightarrow F(X) \otimes F(Y)$  (a canonical arrow  $\mu_0 = \mu_0^{\mathbf{F}}: F(I) \rightarrow I$  can be deduced).

### 3.2. THE 2-CATEGORY OF SPLIT EXTENSIONS.

The kernel  $(\mathcal{N}(\mathbf{F}), \mathbf{e}_{\mathbf{F}}, \lambda^{\mathbf{F}})$  of an homomorphism  $\mathbf{F}: \mathbb{G} \rightarrow \mathbb{H}$  (see [21]) is the universal solution, in the sense of a bilimit, to the problem of finding a categorical group  $\mathbb{K}$ , an homomorphism  $\mathbf{G}$  and a 2-cell  $\phi$  as in the following diagram, where 0 is the functor sending each arrow of  $\mathbb{K}$  on the identity of the unit object of  $\mathbb{H}$ ,

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \phi & \curvearrowleft & \\ \mathbb{K} & \xrightarrow{\mathbf{G}} & \mathbb{G} & \xrightarrow{\mathbf{F}} & \mathbb{H} \end{array}$$

An explicit description of the kernel is provided by the comma-category whose objects are pairs  $(A \in \mathbb{G}, a: F(A) \rightarrow I)$ , and whose arrows are arrows  $f: A \rightarrow B$  in  $\mathbb{G}$  such that  $a = bF(f)$ . The monoidal structure of  $\mathcal{N}(\mathbf{F})$  is defined from those of  $\mathbb{G}$  and  $\mathbf{F}$  in the only sensible way. The functor  $\mathbf{e}_{\mathbf{F}}: \mathcal{N}(\mathbf{F}) \rightarrow \mathbb{G}$  is the canonical forgetful functor, and the 2-cell  $\lambda^{\mathbf{F}}: \mathbf{F}\mathbf{e}_{\mathbf{F}} \Rightarrow 0$  is  $\lambda_{(A,a)}^{\mathbf{F}} = a$ .

Using kernels, we can define 2-exactness and extensions [21, 23].

**Definition 3.1** *Consider the following diagram in  $\mathbf{CG}$*

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \eta & \curvearrowleft & \\ \mathbb{A} & \xrightarrow{\mathbf{F}} & \mathbb{B} & \xrightarrow{\mathbf{G}} & \mathbb{C} \end{array}$$

1. *The triple  $(\mathbf{F}, \eta, \mathbf{G})$  is 2-exact if the factorization of  $\mathbf{F}$  through the kernel of  $\mathbf{G}$  is a full and essentially surjective functor.*
2. *The 2-exact triple  $(\mathbf{F}, \eta, \mathbf{G})$  is an extension of categorical groups if  $\mathbf{F}$  is faithful and  $\mathbf{G}$  is essentially surjective on objects.*

Observe that if  $(\mathbf{F}, \eta, \mathbf{G})$  is an extension, the factorization of  $\mathbf{F}$  through the kernel of  $\mathbf{G}$  is an equivalence.

Extensions of a categorical group  $\mathbb{C}$  by a categorical group  $\mathbb{A}$  are the objects of the 2-category  $\mathbf{Ext}(\mathbb{C}, \mathbb{A})$ . If  $(\mathbf{F}, \eta, \mathbf{G})$  and  $(\mathbf{F}', \eta', \mathbf{G}')$  are two such extensions, an arrow

between them is a triple  $(\sigma, \mathbf{E}, \tau)$  in  $\mathbf{CG}$  as in the following diagram

$$\begin{array}{ccccc}
 & & \mathbb{B} & & \\
 & \mathbf{F} & \nearrow & & \mathbf{G} \\
 \mathbb{A} & & & & \mathbb{C} \\
 & \mathbf{F}' & \searrow & & \mathbf{G}' \\
 & & \mathbb{B}' & & 
 \end{array}$$

$\sigma$  (double arrow from  $\mathbb{A}$  to  $\mathbb{B}$ )  
 $\tau$  (double arrow from  $\mathbb{B}'$  to  $\mathbb{C}$ )  
 $\mathbf{E}$  (vertical arrow from  $\mathbb{B}$  to  $\mathbb{B}'$ )

with  $\sigma$  and  $\tau$  compatible with  $\eta$  and  $\eta'$ . Finally, if  $(\sigma, \mathbf{E}, \tau)$  and  $(\sigma', \mathbf{E}', \tau')$  are arrows as above, a 2-cell  $\theta$  between them is a 2-cell  $\theta: \mathbf{E} \Longrightarrow \mathbf{E}'$  in  $\mathbf{CG}$  compatible with  $\sigma, \sigma'$  and with  $\tau, \tau'$ . In analogy with the case of groups, the 2-category  $\mathbf{Ext}(\mathbb{C}, \mathbb{A})$  is a 2-groupoid [23, 6]: 1-cells are equivalences and 2-cells are isomorphisms.

For our purposes, we are interested into a sub-2-category of  $\mathbf{Ext}(\mathbb{C}, \mathbb{A})$ , namely that of split extensions. A split extension is an extension for which the epi part admits a (not necessarily strict) section in  $\mathbf{CG}$ . More precisely, a split extension is a 5-tuple  $(\mathbf{F}, \eta, \mathbf{G}, \mathbf{S}, \varsigma)$ , where  $(\mathbf{F}, \eta, \mathbf{G})$  is a categorical group extension,  $\mathbf{S}: \mathbb{C} \rightarrow \mathbb{B}$  is a homomorphism, and  $\varsigma: \mathbf{G}\mathbf{S} \Longrightarrow 1_{\mathbb{C}}$  is a 2-cell in  $\mathbf{CG}$ . An arrow between two split extensions is a 4-tuple  $(\sigma, \mathbf{E}, \tau, s): (\mathbf{F}, \eta, \mathbf{G}, \mathbf{S}, \varsigma) \rightarrow (\mathbf{F}', \eta', \mathbf{G}', \mathbf{S}', \varsigma')$ , where  $(\sigma, \mathbf{E}, \tau)$  is an arrow in  $\mathbf{Ext}(\mathbb{C}, \mathbb{A})$ , and  $s: \mathbf{E}\mathbf{S} \Longrightarrow \mathbf{S}'$  is a 2-cell of  $\mathbf{CG}$  making commutative the following diagram

$$\begin{array}{ccc}
 \mathbf{G}'\mathbf{E}\mathbf{S} & \xrightarrow{\mathbf{G}'s} & \mathbf{G}'\mathbf{S}' \\
 \tau\mathbf{S} \Downarrow & & \Downarrow \varsigma' \\
 \mathbf{G}\mathbf{S} & \xrightarrow{\varsigma} & 1_{\mathbb{C}}
 \end{array}$$

A 2-cell  $\theta: (\sigma, \mathbf{E}, \tau, s) \rightarrow (\sigma', \mathbf{E}', \tau', s')$  is a 2-cell of  $\mathbf{Ext}(\mathbb{C}, \mathbb{A})$  compatible with  $s, s'$ . The 2-category of split categorical group extensions of  $\mathbb{C}$  by  $\mathbb{A}$  will be denoted by  $\mathbf{Ext}_{split}(\mathbb{C}, \mathbb{A})$ .

### 3.3. ACTIONS AND SEMIDIRECT PRODUCT.

Let  $\mathbb{H}$  be a categorical group. Recall from [15] that a  $\mathbb{H}$ -categorical group is a categorical group  $\mathbb{G}$  equipped with a homomorphism

$$\Sigma: \mathbb{H} \rightarrow \mathcal{E}q(\mathbb{G})$$

or, equivalently, equipped with an action, that is a functor

$$ac: \mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}, \quad (X, A) \mapsto {}^X A = \Sigma_X(A)$$

and three natural isomorphisms

$$\psi = \psi_{X,A,B}: {}^X(A \otimes B) \rightarrow {}^X A \otimes {}^X B, \quad \phi = \phi_{X,Y,A}: {}^{X \otimes Y} A \rightarrow {}^X ({}^Y A), \quad \phi_0 = \phi_{0,A}: {}^I A \rightarrow A$$

satisfying suitable coherence conditions. Note that a categorical group always acts on itself by conjugation [7, 23]

$$\mathcal{I}: \mathbb{G} \rightarrow \mathcal{E}q(\mathbb{G})$$



where  $\mathcal{I}_X$  is the monoidal auto-equivalence  $A \mapsto (X \otimes A) \otimes X^*$ .

Let  $\mathbb{G}$  be an  $\mathbb{H}$ -categorical group. The semidirect product  $\mathbb{G} \rtimes \mathbb{H}$  has the product  $\mathbb{G} \times \mathbb{H}$  as underlying groupoid, the tensor product is  $(A, X) \otimes (B, Y) = (A \otimes^X B, X \otimes Y)$ , and constraints and inverses are constructed using  $\psi, \phi, \phi_0$  and those of  $\mathbb{G}$  and  $\mathbb{H}$ . Semidirect product provides the typical example of split extension. Indeed

$$(\mathbf{Q}, =, \mathbf{P}, \mathbf{S}, =) : \quad \mathbb{G} \xrightarrow{\mathbf{Q}} \mathbb{G} \rtimes \mathbb{H} \xleftarrow[\mathbf{S}]{\mathbf{P}} \mathbb{H}$$

is a split extension, where  $\mathbf{Q}, \mathbf{S}, \mathbf{P}$  are the canonical injections and projections.

#### 4. Actions and points

In this section, we introduce the 2-category  $\mathbf{Pt}(\mathbf{CG})$  of points in  $\mathbf{CG}$  and the 2-category  $\mathbf{Act}(\mathbf{CG})$  of categorical group actions, and we prove that they are biequivalent.

##### 4.1. THE GENERAL SETTING.

The 2-category  $\mathbf{Pt}(\mathbf{CG})$  consists of the following data: objects are 5-tuples  $(\mathbb{A}, \mathbb{H}, \mathbf{A}, \mathbf{B}, \alpha)$  in  $\mathbf{CG}$  as below:

$$\mathbb{A} \xrightleftharpoons[\mathbf{B}]{\mathbf{A}} \mathbb{H}, \quad \alpha : \mathbf{A}\mathbf{B} \Longrightarrow \mathbf{1}_{\mathbb{H}}$$

Arrows are 4-ples  $(\mathbf{F}, \mathbf{H}, \varphi, \chi) : (\mathbb{A}, \mathbb{H}, \mathbf{A}, \mathbf{B}, \alpha) \rightarrow (\mathbb{A}', \mathbb{H}', \mathbf{A}', \mathbf{B}', \alpha')$  in  $\mathbf{CG}$  as below:

$$\mathbf{F} : \mathbb{A} \rightarrow \mathbb{A}', \quad \mathbf{H} : \mathbb{H} \rightarrow \mathbb{H}', \quad \varphi : \mathbf{F}\mathbf{B} \Longrightarrow \mathbf{B}'\mathbf{H}, \quad \chi : \mathbf{A}'\mathbf{F} \Longrightarrow \mathbf{H}\mathbf{A}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{A}'\mathbf{F}\mathbf{B} & \xrightarrow{\chi\mathbf{B}} & \mathbf{H}\mathbf{A}\mathbf{B} \\ \mathbf{A}'\varphi \Downarrow & & \Downarrow \mathbf{H}\alpha \\ \mathbf{A}'\mathbf{B}'\mathbf{H} & \xrightarrow{\alpha'\mathbf{H}} & \mathbf{H} \end{array}$$

Finally, a 2-cell  $(\delta, \epsilon) : (\mathbf{F}, \mathbf{H}, \varphi, \chi) \Longrightarrow (\mathbf{F}', \mathbf{H}', \varphi', \chi')$  is a pair of 2-cells  $\delta : \mathbf{F} \Longrightarrow \mathbf{F}'$  and  $\epsilon : \mathbf{H} \Longrightarrow \mathbf{H}'$  in  $\mathbf{CG}$  such that the following diagrams commute

$$\begin{array}{ccc} \mathbf{F}\mathbf{B} & \xrightarrow{\delta\mathbf{B}} & \mathbf{F}'\mathbf{B} & \quad & \mathbf{A}'\mathbf{F} & \xrightarrow{\mathbf{A}'\delta} & \mathbf{A}'\mathbf{F}' \\ \varphi \Downarrow & & \Downarrow \varphi' & & \chi \Downarrow & & \Downarrow \chi' \\ \mathbf{B}'\mathbf{H} & \xrightarrow[\mathbf{B}'\epsilon]{} & \mathbf{B}'\mathbf{H}' & & \mathbf{H}\mathbf{A} & \xrightarrow[\epsilon\mathbf{A}]{} & \mathbf{H}'\mathbf{A} \end{array}$$

Extending the definition of the 2-category of  $\mathbb{H}$ -categorical groups, given in [11], in order to let  $\mathbb{H}$  change, we get the 2-category  $\mathbf{Act}(\mathbf{CG})$  of categorical groups actions. It consists of the following data: objects are categorical group actions; a morphism between the  $\mathbb{H}$ -categorical group  $\mathbb{G}$  and the  $\mathbb{H}'$ -categorical group  $\mathbb{G}'$  is a pair of homomorphisms

$$\mathbf{R} : \mathbb{H} \rightarrow \mathbb{H}' \quad \mathbf{T} : \mathbb{G} \rightarrow \mathbb{G}'$$

in  $\mathbf{CG}$ , equivariant with respect to the actions, that is, equipped with a natural transformation  $\rho$

$$\begin{array}{ccc} \mathbb{H} \times \mathbb{G} & \xrightarrow{ac} & \mathbb{G} \\ \mathbf{R} \times \mathbf{T} \downarrow & \swarrow \rho & \downarrow \mathbf{T} \\ \mathbb{H}' \times \mathbb{G}' & \xrightarrow{ac'} & \mathbb{G}' \end{array}$$

making commutative the following diagrams:

$$\begin{array}{ccccc} T(X \otimes Y A) & \xrightarrow{\rho} & R(X \otimes Y) T A & \xrightarrow{\mu^{\mathbf{R}} T A} & R(X) \otimes R(Y) T A \\ T(\phi) \downarrow & & & & \downarrow \phi' \\ T(X(Y A)) & \xrightarrow{\rho} & R^X T(Y A) & \xrightarrow{\rho} & R^X(R^Y A) \end{array}$$

$$\begin{array}{ccc} T(I A) & \xrightarrow{T\phi_0} & T A \\ \rho \downarrow & & \uparrow \phi'_0 \\ R^I T A & \xrightarrow{\mu_0^{\mathbf{R}} T A} & I T A \end{array}$$

$$\begin{array}{ccccc} T(X(A \otimes B)) & \xrightarrow{T\psi} & T(X A \otimes^X B) & \xrightarrow{\mu^{\mathbf{T}}} & T(X A) \otimes T(X B) \\ \rho \downarrow & & & & \downarrow \rho \otimes \rho \\ R^X T(A \otimes B) & \xrightarrow{R^X \mu^{\mathbf{T}}} & R^X(T A \otimes T B) & \xrightarrow{\psi'} & R^X T A \otimes R^X T B \end{array}$$

Finally, a 2-cell  $(\mathbf{R}, \mathbf{T}, \rho) \Rightarrow (\mathbf{R}', \mathbf{T}', \rho')$  is a pair of 2-cells  $\alpha: \mathbf{R} \Rightarrow \mathbf{R}'$  and  $\beta: \mathbf{T} \Rightarrow \mathbf{T}'$  in  $\mathbf{CG}$  such that the following diagram commutes:

$$\begin{array}{ccc} T(X A) & \xrightarrow{\beta_{X A}} & T'(X A) \\ \rho_{X, A} \downarrow & & \downarrow \rho'_{X, A} \\ R^X T A & \xrightarrow{\alpha_X \beta_A} & R^{X'} T' A \end{array}$$

#### 4.2. THE MAIN BIEQUIVALENCE.

The semi-direct product described in Section 3.3 is part of a 2-functor

$$\times: \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{Pt}(\mathbf{CG})$$

which may be viewed as follows:

$$\begin{array}{ccc} \begin{array}{ccc} \mathbb{H} \times \mathbb{G} & \xrightarrow{ac} & \mathbb{G} \\ \mathbf{R} \times \mathbf{T} \downarrow & \swarrow \rho & \downarrow \mathbf{T} \\ \mathbb{H}' \times \mathbb{G}' & \xrightarrow{ac'} & \mathbb{G}' \end{array} & \mapsto & \begin{array}{ccc} \mathbb{G} \times \mathbb{H} & \xrightleftharpoons[\mathbf{S}]{\mathbf{P}} & \mathbb{H} \\ \mathbf{T} \times \mathbf{R} \downarrow & (\varphi, \chi) & \downarrow \mathbf{R} \\ \mathbb{G}' \times \mathbb{H}' & \xrightleftharpoons[\mathbf{S}']{\mathbf{P}'} & \mathbb{H}' \end{array} \end{array}$$

More precisely, given an action  $ac: \mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}$ , its image is the point

$$\mathbb{G} \rtimes \mathbb{H} \begin{array}{c} \xrightarrow{\mathbf{P}} \\ \xleftarrow{\mathbf{S}} \end{array} \mathbb{H}, \quad \mathbf{PS} = 1_{\mathbb{H}}$$

Given a 1-cell  $(\mathbf{R}, \mathbf{T}, \rho)$  from  $\mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}$  to  $\mathbb{H}' \times \mathbb{G}' \rightarrow \mathbb{G}'$  in  $\mathbf{Act}(\mathbf{CG})$ , we get a 1-cell  $(\mathbf{T} \rtimes \mathbf{R}, \mathbf{R}, \varphi, \chi)$  in  $\mathbf{Pt}(\mathbf{CG})$ , with

$$\begin{array}{ccc} T \rtimes R((A, X) \otimes (B, Y)) & \xrightarrow{\mu_{(A, X), (B, Y)}^{\mathbf{T} \rtimes \mathbf{R}}} & T \rtimes R((A, X)) \otimes T \rtimes R((B, Y)) \\ \parallel & & \parallel \\ (T(A^X B), R(XY)) & \xrightarrow{(\mu^1, \mu^2)} & (TA^{RX}TB, RXRY) \end{array}$$

(where  $\mu^1 = (TA \otimes (\rho_{X, B})) \circ \mu_{A, X, B}^{\mathbf{T}}$  and  $\mu^2 = \mu_{X, Y}^{\mathbf{R}}$ ,  $\varphi_X = (\mu_0^{\mathbf{T}}, id): (TI, RX) \rightarrow (I, RX)$ , and  $\chi_{(A, X)} = id: RX \rightarrow RX$ . Finally, the image of a 2-cell  $(\alpha, \beta)$  of  $\mathbf{Act}(\mathbf{CG})$ , is the 2-cell  $((\beta, \alpha), \alpha)$  of  $\mathbf{Pt}(\mathbf{CG})$ .

**Proposition 4.1** *The 2-functor*

$$\rtimes: \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{Pt}(\mathbf{CG})$$

*is a biequivalence of 2-categories.*

*Proof.* We split the proof in two lemmas: recall that a 2-functor  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$  is a biequivalence if it induces equivalences on hom-categories, and if it is biessentially surjective, that is, for every object  $Y$  of  $\mathcal{B}$  there exists an object  $X$  of  $\mathcal{A}$  and an equivalence  $\mathcal{F}(X) \rightarrow Y$ . ■

**Lemma 4.2** *Consider two objects  $ac: \mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}$  and  $ac': \mathbb{H}' \times \mathbb{G}' \rightarrow \mathbb{G}'$  in  $\mathbf{Act}(\mathbf{CG})$ . The induced functor on hom-categories*

$$\Gamma = \rtimes_{ac, ac'}: \mathbf{Hom}_{\mathbf{Act}(\mathbf{CG})} \left( \begin{array}{ccc} \mathbb{H} \times \mathbb{G} & & \mathbb{H}' \times \mathbb{G}' \\ ac \downarrow & , & \downarrow ac' \\ \mathbb{G} & & \mathbb{G}' \end{array} \right) \rightarrow \mathbf{Hom}_{\mathbf{Pt}(\mathbf{CG})} \left( \begin{array}{ccc} \mathbb{G} \rtimes \mathbb{H} & & \mathbb{G}' \rtimes \mathbb{H}' \\ \mathbf{S} \updownarrow \mathbf{P} & , & \mathbf{S}' \updownarrow \mathbf{P}' \\ \mathbb{H} & & \mathbb{H}' \end{array} \right)$$

*is an equivalence.*

*Proof.*  $\Gamma$  is clearly faithful. As far as fullness is concerned, consider  $(\mathbf{R}, \mathbf{T}, \rho)$  and  $(\mathbf{R}', \mathbf{T}', \rho')$  in  $\mathbf{Hom}_{\mathbf{Act}(\mathbf{CG})}(ac, ac')$ , and  $(\delta, \epsilon): \Gamma(\mathbf{R}, \mathbf{T}, \rho) \Longrightarrow \Gamma(\mathbf{R}', \mathbf{T}', \rho')$ . Observe that  $\delta$  has two components, say

$$\delta_{A, X} = (\delta_{A, X}^{(1)}, \delta_{A, X}^{(2)}): (TA, RX) \rightarrow (T'A, R'X)$$

So we get  $\alpha: \mathbf{R} \Longrightarrow \mathbf{R}'$  by  $\alpha_X = \delta_{I,X}^{(2)}$ , and  $\beta: \mathbf{T} \Longrightarrow \mathbf{T}'$  by  $\beta_A = \delta_{A,I}^{(1)}$ . Using that  $\delta$  is monoidal, one checks that  $(\alpha, \beta): (\mathbf{R}, \mathbf{T}, \rho) \Longrightarrow (\mathbf{R}', \mathbf{T}', \rho')$  is a 2-cell in  $\mathbf{Act}(\mathbf{CG})$ . Moreover, using that  $(\delta, \epsilon)$  is a 2-cell in  $\mathbf{Pt}(\mathbf{CG})$ , one checks that  $\delta_{A,X}^{(2)} = \epsilon_X$  and  $\delta_{A,X}^{(1)} = \delta_{(A,I)}^{(1)}$ , so that  $\Gamma(\alpha, \beta) = (\delta, \epsilon)$ .

We prove now that  $\Gamma$  is essentially surjective. Consider

$$(F, H, \varphi, \chi) \in \mathbf{Hom}_{\mathbf{Pt}(\mathbf{CG})} \left( \begin{array}{ccc} \mathbb{G} \times \mathbb{H} & & \mathbb{G}' \times \mathbb{H}' \\ \mathbb{S} \uparrow \downarrow \mathbb{P} & , & \mathbb{S}' \uparrow \downarrow \mathbb{P}' \\ \mathbb{H} & & \mathbb{H}' \end{array} \right)$$

We look for  $(\mathbf{R}, \mathbf{T}, \rho) \in \mathbf{Hom}_{\mathbf{Act}(\mathbf{CG})}(ac, ac')$  and a 2-cell  $(\delta, \epsilon): \Gamma(\mathbf{R}, \mathbf{T}, \rho) \Longrightarrow (\mathbf{F}, \mathbf{H}, \varphi, \chi)$  in  $\mathbf{Pt}(\mathbf{CG})$ . One can choose  $\mathbf{R} = \mathbf{H}: \mathbb{H} \rightarrow \mathbb{H}'$  and  $\mathbf{T} = F_1 Q: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{H} \rightarrow \mathbb{G}'$ , where  $F_1, F_2$  are the components of  $\mathbf{F}: \mathbb{G} \times \mathbb{H} \rightarrow \mathbb{G}' \times \mathbb{H}'$ . The monoidal structure of  $\mathbf{T}$  is given by the first component of the following chain of natural isomorphisms:

$$\begin{aligned} (F_1(A, I) \otimes F_1(B, I), I) &\simeq (F_1(A, I), I) \otimes (F_1(B, I), I) \simeq \\ &\simeq (F_1(A, I), F_2(A, I)) \otimes (F_1(B, I), F_2(B, I)) = F(A, I) \otimes F(B, I) \simeq \\ &\simeq F((A, I) \otimes (B, I)) \simeq F(A \otimes B, I) = (F_1(A \otimes B, I), F_2(A \otimes B, I)) \end{aligned}$$

where the second isomorphism is provided by  $\chi$ , and the other isomorphisms are canonical. Similarly, the 2-cell  $\rho$  is the first component of the following chain of natural isomorphisms:

$$\begin{aligned} (F_1({}^X A, I), F_2({}^X A, I)) &= F({}^X A, I) \simeq F((I, X) \otimes (A, I) \otimes (I, X^*)) \simeq \\ &\simeq F(I, X) \otimes F(A, I) \otimes F(I, X^*) = \\ &= (F_1(I, X), F_2(I, X)) \otimes (F_1(A, I), F_2(A, I)) \otimes (F_1(I, X^*), F_2(I, X^*)) \simeq \\ &\simeq (I, H(X)) \otimes (F_1(A, I), I) \otimes (I, H(X^*)) \simeq ({}^{H(X)} F_1(A, I), I) \end{aligned}$$

where the third isomorphism is provided by  $\varphi$  on the first component of the first and of the third object, by  $\chi$  on the second component of the first and of the third object, and by  $\chi$  again on the second object, and the other isomorphisms are canonical. It remains to construct the 2-cell  $(\delta, \epsilon): \epsilon$  is the identity, and  $\delta$  is given by the following chain of natural isomorphisms:

$$\begin{aligned} (F_1(A, I), H(X)) &\simeq (F_1(A, I), F_2(I, X)) \simeq (F_1(A, I), I) \otimes (I, F_2(I, X)) \simeq \\ &\simeq (F_1(A, I), F_2(A, I)) \otimes (F_1(I, X), F_2(I, X)) = F(A, I) \otimes F(I, X) \simeq \\ &\simeq F((A, I) \otimes (I, X)) \simeq F(A, X) \end{aligned}$$

where the first isomorphism is provided by  $\chi$ , the third isomorphism is provided by  $\chi$  on the first object and by  $\varphi$  on the second object, and the other isomorphisms are canonical. We leave to the reader the (quite long) verification that  $(\mathbf{R}, \mathbf{T}, \rho)$  is a 1-cell in  $\mathbf{Act}(\mathbf{CG})$  and  $(\delta, \epsilon)$  is a 2-cell in  $\mathbf{Pt}(\mathbf{CG})$ .  $\blacksquare$

**Lemma 4.3** *The 2-functor*

$$\times: \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{Pt}(\mathbf{CG})$$

is biessentially surjective.

Proof. Consider an object

$$\mathbb{A} \begin{array}{c} \xrightarrow{\mathbf{A}} \\ \xleftarrow{\mathbf{B}} \end{array} \mathbb{H}, \quad \alpha: \mathbf{AB} \Longrightarrow \mathbf{1}_{\mathbb{H}}$$

in  $\mathbf{Pt}(\mathbf{CG})$ . There is an action  $\mathbb{H} \times \mathcal{N}(\mathbf{A}) \rightarrow \mathcal{N}(\mathbf{A})$  which sends a pair  $(X, (A, a))$  on the object  $\mathbf{B}(X) \otimes A \otimes \mathbf{B}(X)^*$  with structural map

$$\mathbf{A}(\mathbf{B}(X) \otimes A \otimes \mathbf{B}(X)^*) \simeq \mathbf{A}(\mathbf{B}(X)) \otimes \mathbf{A}(A) \otimes \mathbf{A}(\mathbf{B}(X)^*) \xrightarrow{1 \otimes a \otimes 1} \mathbf{A}(\mathbf{B}(X)) \otimes \mathbf{A}(\mathbf{B}(X)^*) \simeq I$$

We look for homomorphisms  $\mathbf{F}$  and  $\mathbf{H}$  as in the following diagram

$$\begin{array}{ccc} \mathcal{N}(\mathbf{A}) \times \mathbb{H} & \begin{array}{c} \xleftarrow{\mathbf{P}} \\ \xrightarrow{\mathbf{S}} \end{array} & \mathbb{H} \\ \mathbf{F} \downarrow & & \downarrow \mathbf{H} \\ \mathbb{A} & \begin{array}{c} \xleftarrow{\mathbf{A}} \\ \xrightarrow{\mathbf{B}} \end{array} & \mathbb{H} \end{array}$$

together with 2-cells  $\varphi: \mathbf{FS} \Longrightarrow \mathbf{BH}$  and  $\chi: \mathbf{AF} \Longrightarrow \mathbf{HP}$ . We take as  $\mathbf{H}$  the identity functor, and we define  $\mathbf{F}(A, a, X) = A \otimes \mathbf{B}(X)$ . The monoidal structure of  $\mathbf{F}$  comes from that of  $\mathbf{B}$  in the following way:

$$\mathbf{F}((A, a, X) \otimes (B, b, Y)) = A \otimes \mathbf{B}(X) \otimes B \otimes \mathbf{B}(X)^* \otimes \mathbf{B}(X \otimes Y) \simeq$$

$$\simeq A \otimes \mathbf{B}(X) \otimes B \otimes \mathbf{B}(X)^* \otimes \mathbf{B}(X) \otimes \mathbf{B}(Y) \simeq A \otimes \mathbf{B}(X) \otimes B \otimes \mathbf{B}(Y) = \mathbf{F}(A, a, X) \otimes \mathbf{F}(B, b, Y)$$

Finally, we put

$$\varphi_X: \mathbf{F}(\mathbf{S}(X)) = I \otimes \mathbf{B}(X) \simeq \mathbf{B}(X)$$

$$\chi_{(A,a,X)}: \mathbf{A}(\mathbf{F}(A, a, X)) \simeq \mathbf{A}(A) \otimes \mathbf{A}(\mathbf{B}(X)) \xrightarrow{a \otimes \alpha_X} I \otimes X \simeq X = \mathbf{P}(A, a, X)$$

Since the kernel of  $\mathbf{P}$  is equivalent to the kernel of  $\mathbf{A}$ , the short five lemma for categorical groups [23] implies that  $(\mathbf{F}, \mathbf{1}_{\mathbb{H}}, \varphi, \chi)$  is an equivalence in  $\mathbf{Pt}(\mathbf{CG})$ .  $\blacksquare$

From the proof of the previous lemma, we can extract a 2-functor

$$\mathcal{K}: \mathbf{Pt}(\mathbf{CG}) \rightarrow \mathbf{Act}(\mathbf{CG})$$

quasi-inverse of the 2-functor  $\times: \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{Pt}(\mathbf{CG})$ . The 2-functor  $\mathcal{K}$  acts as in the following diagram:

$$\begin{array}{ccc} \begin{array}{ccc} \mathbb{A} & \xrightleftharpoons[\mathbf{B}]{\mathbf{A}} & \mathbb{H} \\ \mathbf{F} \downarrow & (\varphi, \chi) & \downarrow \mathbf{H} \\ \mathbb{A}' & \xrightleftharpoons[\mathbf{B}']{\mathbf{A}'} & \mathbb{H}' \end{array} & \mapsto & \begin{array}{ccc} \mathbb{H} \times \mathcal{N}(\mathbf{A}) & \xrightarrow{ac} & \mathcal{N}(\mathbf{A}) \\ \mathbf{H} \times \mathbf{F}_I \downarrow & \swarrow \rho & \downarrow \mathbf{F}_I \\ \mathbb{H}' \times \mathcal{N}(\mathbf{A}') & \xrightarrow{ac'} & \mathcal{N}(\mathbf{A}') \end{array} \end{array}$$

where  $\mathbf{F}_I$  is the restriction of  $\mathbf{F}$  to the kernel, that is,  $F_I(A, a) = (F(A), H(a) \cdot \chi_A)$ , and the component of  $\rho$  at  $(X, A, a)$  is given by the composition

$$\begin{array}{c} F(B(X) \otimes A \otimes B(X)^*) \\ \downarrow \mu^F \\ F(B(X)) \otimes F(A) \otimes F(B(X))^* \\ \downarrow \varphi_X \otimes 1 \otimes \varphi_X^* \\ B'(H(X)) \otimes F(A) \otimes B'(H(X))^* \end{array}$$

Finally, for a 2-cell  $(\delta, \epsilon)$  in  $\mathbf{Pt}(\mathbf{CG})$ ,  $\mathcal{K}(\delta, \epsilon) = (\epsilon, \bar{\delta})$ , where  $\bar{\delta}$  is the restriction of  $\delta$  to  $\mathbf{F}_I$ .

#### 4.3. INDUCED BIEQUIVALENCES.

There are forgetful 2-functors  $\mathcal{U}_*: \mathbf{Pt}(\mathbf{CG}) \rightarrow \mathbf{CG}$  and  $\mathcal{O}: \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{CG}$  defined as in the following diagrams

$$\begin{array}{ccc} \mathcal{U}_*: \mathbf{Pt}(\mathbf{CG}) & \longrightarrow & \mathbf{CG} \\ \begin{array}{ccc} \mathbb{A} & \xrightleftharpoons[\mathbf{B}]{\mathbf{A}} & \mathbb{H} \\ \mathbf{F} \downarrow \delta \downarrow \mathbf{F}' & & \downarrow \mathbf{H} \downarrow \mathbf{H}' \\ \mathbb{A}' & \xrightleftharpoons[\mathbf{B}']{\mathbf{A}'} & \mathbb{H}' \end{array} & \mapsto & \begin{array}{ccc} \mathbb{H} & & \\ \downarrow & \epsilon & \downarrow \\ \mathbb{H}' & & \end{array} \end{array}$$
  

$$\begin{array}{ccc} \mathcal{O}: \mathbf{Act}(\mathbf{CG}) & \longrightarrow & \mathbf{CG} \\ \begin{array}{ccc} \mathbb{H} \times \mathbb{G} & \xrightarrow{ac} & \mathbb{G} \\ \mathbf{R} \times \mathbf{T} \downarrow (\alpha, \beta) \downarrow \mathbf{R}' \times \mathbf{T}' & \swarrow \rho' \swarrow \rho & \downarrow \mathbf{T} \downarrow \mathbf{T}' \\ \mathbb{H}' \times \mathbb{G}' & \xrightarrow{ac'} & \mathbb{G}' \end{array} & \mapsto & \begin{array}{ccc} \mathbb{H} & & \\ \downarrow & \alpha & \downarrow \\ \mathbb{H}' & & \end{array} \end{array}$$

If  $\mathbb{H}$  is a fixed categorical group, the strict 2-fibres  $\mathbf{Pt}(\mathbb{H})$  and  $\mathbf{Act}(\mathbb{H})$  of  $\mathcal{U}_*$  and  $\mathcal{O}$  on  $\mathbb{H}$  can be described in the following way:

- Objects, 1-cells and 2-cells in  $\mathbf{Pt}(\mathbb{H})$  are of the form

$$\delta: (\mathbf{F}, \varphi, \chi) \rightrightarrows (\mathbf{F}', \varphi', \chi'): (\mathbb{A}, \mathbf{A}, \mathbf{B}, \alpha) \rightrightarrows (\mathbb{A}', \mathbf{A}', \mathbf{B}', \alpha')$$

where  $(\mathbb{A}, \mathbb{H}, \mathbf{A}, \mathbf{B}, \alpha)$ ,  $(\mathbb{A}', \mathbb{H}, \mathbf{A}', \mathbf{B}', \alpha')$ ,  $(\mathbf{F}, 1_{\mathbb{H}}, \varphi, \chi)$ ,  $(\mathbf{F}', 1_{\mathbb{H}}, \varphi', \chi')$ ,  $(\delta, 1_{\mathbb{H}})$  are objects, 1-cells and 2-cells in  $\mathbf{Pt}(\mathbf{CG})$ .

- Objects, 1-cells and 2-cells in  $\mathbf{Act}(\mathbb{H})$  are of the form

$$\beta: (\mathbf{T}, \rho) \rightrightarrows (\mathbf{T}', \rho'): (\mathbb{G}, ac) \rightrightarrows (\mathbb{G}', ac')$$

where  $(\mathbb{H}, \mathbb{G}, ac)$ ,  $(\mathbb{H}, \mathbb{G}', ac')$ ,  $(1_{\mathbb{H}}, \mathbf{T}, \rho)$ ,  $(1_{\mathbb{H}}, \mathbf{T}', \rho')$ ,  $(1_{\mathbb{H}}, \beta)$  are objects, 1-cells and 2-cells in  $\mathbf{Act}(\mathbf{CG})$ .

Since the diagram

$$\begin{array}{ccc} \mathbf{Act}(\mathbf{CG}) & \xleftarrow{\kappa} & \mathbf{Pt}(\mathbf{CG}) \\ & \searrow \mathcal{O} & \swarrow \mathcal{U}_* \\ & \mathbf{CG} & \end{array}$$

commutes, from Proposition 4.1 we get the following result.

**Corollary 4.4** *The biequivalence*

$$\mathcal{K}: \mathbf{Pt}(\mathbf{CG}) \rightarrow \mathbf{Act}(\mathbf{CG})$$

*restricts to a biequivalence*

$$\mathcal{K}': \mathbf{Pt}(\mathbb{H}) \rightarrow \mathbf{Act}(\mathbb{H}).$$

Another forgetful 2-functor can be defined now:

$$\begin{array}{ccc} \mathcal{U}: \mathbf{Act}(\mathbb{H}) & \longrightarrow & \mathbf{CG} \\ \begin{array}{ccc} \mathbb{H} \times \mathbb{G} & \xrightarrow{ac} & \mathbb{G} \\ \downarrow \scriptstyle (1, \beta) & \swarrow \scriptstyle \rho' / \rho & \downarrow \scriptstyle \beta \\ 1_{\mathbb{H}} \times \mathbf{T} & \xrightarrow{\quad} & \mathbf{T} \\ \downarrow & & \downarrow \\ \mathbb{H} \times \mathbb{G}' & \xrightarrow{ac'} & \mathbb{G}' \end{array} & \mapsto & \begin{array}{ccc} \mathbb{G} & & \\ \downarrow \scriptstyle \beta & & \\ \mathbf{T} & \xrightarrow{\quad} & \mathbf{T}' \\ \downarrow & & \downarrow \\ \mathbb{G}' & & \end{array} \end{array}$$

Composing  $\mathcal{K}'$  and  $\mathcal{U}$ , we get a 2-functor

$$\begin{array}{ccc} \mathcal{N}: \mathbf{Pt}(\mathbb{H}) & \longrightarrow & \mathbf{CG} \\ \begin{array}{ccc} \mathbb{A} & \xrightleftharpoons[\mathbf{B}]{\mathbf{A}} & \mathbb{H} \\ \downarrow \scriptstyle \delta & & \downarrow \scriptstyle 1_{\mathbb{H}} \\ \mathbf{F} & \xrightarrow{\quad} & \mathbf{F}' \\ \downarrow & & \downarrow \\ \mathbb{A}' & \xrightleftharpoons[\mathbf{B}']{\mathbf{A}'} & \mathbb{H} \end{array} & \mapsto & \begin{array}{ccc} \mathcal{N}(\mathbb{A}) & & \\ \downarrow \scriptstyle \bar{\delta} & & \\ \mathbf{F}_{\mathbb{I}} & \xrightarrow{\quad} & \mathbf{F}'_{\mathbb{I}} \\ \downarrow & & \downarrow \\ \mathcal{N}(\mathbb{A}') & & \end{array} \end{array}$$

where  $\bar{\delta}: \mathbf{F}_1 \Longrightarrow \mathbf{F}_1'$  are the restriction of  $\delta: \mathbf{F} \Longrightarrow \mathbf{F}'$  to the kernel. Fix now a categorical group  $\mathbb{G}$  and let us look at the 2-fibres of  $\mathcal{U}$  and  $\mathcal{N}$  on  $\mathbb{G}$ :

- The strict 2-fibre of  $\mathcal{U}$  on  $\mathbb{G}$  is the groupoid (= locally discrete 2-groupoid) whose objects are actions  $ac: \mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}$  of  $\mathbb{H}$  on  $\mathbb{G}$ , and whose arrows are natural transformations  $\rho: ac \Longrightarrow ac'$  such that  $(1_{\mathbb{H}}, 1_{\mathbb{G}}, \rho)$  is a 1-cell in  $\mathbf{Act}(\mathbf{CG})$ .
- The homotopy 2-fibre of  $\mathcal{N}$  on  $\mathbb{G}$  is precisely the 2-groupoid of split extensions  $\mathbf{Ext}_{split}(\mathbb{H}, \mathbb{G})$  described in Section 3.2.

**Corollary 4.5** *The biequivalence*

$$\mathcal{K}': \mathbf{Pt}(\mathbb{H}) \rightarrow \mathbf{Act}(\mathbb{H})$$

*restricts to a biequivalence*

$$\mathcal{K}'': \mathbf{Ext}_{split}(\mathbb{H}, \mathbb{G}) \rightarrow \mathbf{Act}(\mathbb{H}, \mathbb{G}).$$

Proof. Since  $\mathcal{N} = \mathcal{U} \cdot \mathcal{K}'$ , the biequivalence  $\mathcal{K}'$  restricts to a biequivalence between the strict 2-fibres on  $\mathbb{G}$ . Moreover, the 2-functor  $\mathcal{N}$  is a fibration of 2-groupoids, so that the strict fibre and the homotopy fibre are biequivalent [13]. ■

## 5. Universal properties of the holomorph of a categorical group

### 5.1. BIREPRESENTABILITY.

To generalize the first universal property of the holomorph to our context, we need the following birepresentability criterion, established in [25].

**Proposition 5.1** *Let  $\mathcal{C}$  be a bicategory, consider the 2-category  $\mathbf{Cat}$  of small categories, and a morphism*

$$\mathcal{F}: \mathcal{C}^{op} \rightarrow \mathbf{Cat}$$

*i.e.  $\mathcal{F}$  preserves composition and identities up to natural coherent invertible 2-cells (super-script “op” refers to notation in [25], meaning that  $\mathcal{F}$  inverts 1-cells but does not invert 2-cells). The following conditions are equivalent:*

1. *there exists an object  $R$  of  $\mathcal{C}$ , and a natural equivalence  $\theta: \mathcal{F} \Longrightarrow \mathcal{C}(-, R)$ , i.e. an equivalence  $\theta$  in  $\mathbf{Hom}(\mathcal{C}^{op}, \mathbf{Cat})$*
2. *there exist an object  $R$  of  $\mathcal{C}$  and an object  $a \in \mathcal{F}(R)$  such that*
  - *for every object  $H$  of  $\mathcal{C}$ , and every object  $x \in \mathcal{F}(H)$  there exist an arrow  $h: H \rightarrow R$  in  $\mathcal{C}$ , and an isomorphism  $u: x \rightarrow \mathcal{F}(h)(a)$  in  $\mathcal{F}(H)$*
  - *given two arrows  $h': H \rightarrow R$ ,  $\alpha: \mathcal{F}(h)(a) \rightarrow \mathcal{F}(h')(a)$  there exists a unique 2-cell  $\sigma: h \Longrightarrow h'$  such that the component at  $a$  of the natural transformation  $\mathcal{F}(\sigma)$  is  $\alpha$ .*

The aim of the present section is to show that split extensions of categorical groups are represented by the holomorph split extension. In order to present the result, a few preliminary notions are needed.



## 5.2. PULLBACKS OF CATEGORICAL GROUPS.

Given two homomorphisms of categorical groups  $\mathbf{F}: \mathbb{B} \rightarrow \mathbb{A}$  and  $\mathbf{G}: \mathbb{C} \rightarrow \mathbb{A}$ , what we call here their pullback is a universal (in the sense of bilimits) 4-tuple  $(\mathbb{P}, \mathbf{F}', \mathbf{G}', \varpi)$ , represented by the following diagram in  $\mathbf{CG}$

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\mathbf{F}'} & \mathbb{B} \\ \mathbf{G}' \downarrow & \swarrow \varpi & \downarrow \mathbf{F} \\ \mathbb{C} & \xrightarrow{\mathbf{G}} & \mathbb{A} \end{array}$$

This means that, for any other diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\mathbf{H}} & \mathbb{B} \\ \mathbf{K} \downarrow & \swarrow \varphi & \downarrow \mathbf{F} \\ \mathbb{C} & \xrightarrow{\mathbf{G}} & \mathbb{A} \end{array}$$

there are an homomorphism  $\mathbf{D}: \mathbb{D} \rightarrow \mathbb{P}$ , and two 2-cells  $\varphi_H: \mathbf{H} \Rightarrow \mathbf{F}'\mathbf{D}$  and  $\varphi_K: \mathbf{K} \Rightarrow \mathbf{G}'\mathbf{D}$  such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{FH} & \xrightarrow{\mathbf{F}\varphi_H} & \mathbf{FF}'\mathbf{D} \\ \varphi \downarrow & & \downarrow \varpi\mathbf{D} \\ \mathbf{GK} & \xrightarrow{\mathbf{G}\varphi_K} & \mathbf{GG}'\mathbf{D} \end{array}$$

Moreover, for any other triple  $(\mathbf{D}', \varphi'_H, \varphi'_K)$  satisfying the same property, there is a unique 2-cell  $\psi: \mathbf{D} \Rightarrow \mathbf{D}'$  such that the following diagrams commute

$$\begin{array}{ccc} \mathbf{F}'\mathbf{D} & \xrightarrow{\mathbf{F}'\psi} & \mathbf{F}'\mathbf{D}' \\ \swarrow \varphi_H & & \searrow \varphi'_H \\ & \mathbf{H} & \\ \swarrow \varphi_K & & \searrow \varphi'_K \\ \mathbf{G}'\mathbf{D} & \xrightarrow{\mathbf{G}'\psi} & \mathbf{G}'\mathbf{D}' \end{array}$$

In [6], an explicit description of the pullback is given:  $\mathbb{P}$  is the comma-groupoid whose objects are triples  $(B \in \mathbb{B}, u: FB \rightarrow GC, C \in \mathbb{C})$ ; an arrow in  $\mathbb{P}$  is a pair  $f_B: B \rightarrow B', f_C: C \rightarrow C'$  such that the following square commutes

$$\begin{array}{ccc} FB & \xrightarrow{u} & GC \\ Ff_B \downarrow & & \downarrow Gf_C \\ FB & \xrightarrow{u'} & GC \end{array}$$

Tensor products of  $\mathbb{B}$  and  $\mathbb{C}$  extend naturally to  $\mathbb{P}$ , so that  $\mathbb{P}$  is indeed a categorical group. Homomorphism  $\mathbf{F}'$  forgets the  $\mathbb{C}$  side,  $\mathbf{G}'$  forgets the  $\mathbb{B}$  side, and the 2-cell  $\varpi$  is given by  $\varpi_{(B,u,C)} = u$ .

Many properties of pullbacks can be revisited in dimension 2 just using the universal property of the bilimit. The relevant one for our purpose is that, given an homomorphism  $\mathbf{F}: \mathbb{K} \rightarrow \mathbb{H}$  of categorical groups and a point

$$\mathbb{A} \begin{array}{c} \xrightarrow{\mathbf{A}} \\ \xleftarrow{\mathbf{B}} \end{array} \mathbb{H}, \quad \alpha: \mathbf{A}\mathbf{B} \Longrightarrow \mathbf{1}_{\mathbb{H}}$$

over  $\mathbb{H}$ , pulling back along  $\mathbf{F}$  produces a point over  $\mathbb{K}$ . Once an explicit description of the pullback is chosen, this produces a 2-functor

$$\mathbf{F}^*: \mathbf{Pt}(\mathbb{H}) \rightarrow \mathbf{Pt}(\mathbb{K})$$

Moreover, such a 2-functor induces a 2-functor

$$\mathbf{Ext}_{split}(\mathbf{F}, \mathbb{G}): \mathbf{Ext}_{split}(\mathbb{H}, \mathbb{G}) \rightarrow \mathbf{Ext}_{split}(\mathbb{K}, \mathbb{G})$$

### 5.3. THE HOLOMORPH OF A CATEGORICAL GROUP.

Let  $\mathbb{G}$  be a categorical group. We call holomorph of  $\mathbb{G}$ , and we denote it by  $\mathcal{H}ol(\mathbb{G})$ , the semidirect product  $\mathbb{G} \rtimes \mathcal{E}q(\mathbb{G})$ , with action given by evaluation. In other words, the holomorph of  $\mathbb{G}$  fits into a split extension of categorical groups

$$\mathcal{H} = ( \mathbb{G} \longrightarrow \mathcal{H}ol(\mathbb{G}) \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \mathcal{E}q(\mathbb{G}) )$$

that is, an object of  $\mathbf{Ext}_{split}(\mathcal{E}q(\mathbb{G}), \mathbb{G})$

Recall from [2] that the classifying category of a bicategory  $\mathcal{C}$  has the same objects of  $\mathcal{C}$ , and arrows are 2-isomorphism classes of 1-cells of  $\mathcal{C}$ . Let us denote by  $\mathbf{EXT}_{split}(\mathbb{H}, \mathbb{G})$  the classifying category of  $\mathbf{Ext}_{split}(\mathbb{H}, \mathbb{G})$  (in fact,  $\mathbf{EXT}_{split}(\mathbb{H}, \mathbb{G})$  a groupoid, since it is the classifying category of a 2-groupoid). The biequivalence

$$\mathcal{K}'' : \mathbf{Ext}_{split}(\mathbb{H}, \mathbb{G}) \rightarrow \mathbf{Act}(\mathbb{H}, \mathbb{G}).$$

of Corollary 4.5 induces an equivalence of groupoids

$$\mathbf{EXT}_{split}(\mathbb{H}, \mathbb{G}) \rightarrow \mathbf{Act}(\mathbb{H}, \mathbb{G}).$$

Moreover, the previous discussion on pullbacks of categorical groups allows us to convert this equivalence into a natural equivalence

$$\mathbf{EXT}_{split}(-, \mathbb{G}) \Longrightarrow \mathbf{Act}(-, \mathbb{G}).$$

Finally, let us observe that the correspondence between action of  $\mathbb{H}$  on  $\mathbb{G}$  and homomorphisms  $\mathbb{H} \rightarrow \mathcal{E}q(\mathbb{G})$ , already quoted in Section 3.3, is indeed part of an equivalence

$$\mathbf{Act}(\mathbb{H}, \mathbb{G}) \rightarrow \mathbf{Hom}_{\mathbf{CG}}(\mathbb{H}, \mathcal{E}q(\mathbb{G})).$$

Once again, this equivalence can be converted into a natural equivalence

$$\mathbf{Act}(-, \mathbb{G}) \rightarrow \mathbf{Hom}_{\mathbf{CG}}(-, \mathcal{E}q(\mathbb{G})).$$

Putting together the previous natural equivalences and applying the birepresentability criterion of Section 5.1 to the 2-functor

$$\mathbf{EXT}_{split}(-, \mathbb{G}) : \mathbf{CG}^{op} \rightarrow \mathbf{Cat}$$

we get the following universal property of the holomorph.

**Proposition 5.2** *Consider two categorical groups  $\mathbb{G}$  and  $\mathbb{H}$ .*

1. *For any split extension*

$$\mathcal{E} = ( \mathbb{G} \longrightarrow \mathbb{A} \begin{array}{c} \xrightarrow{\mathbf{A}} \\ \xleftarrow{\mathbf{B}} \end{array} \mathbb{H} , \alpha : \mathbf{AB} \Longrightarrow \mathbf{1}_{\mathbb{H}} )$$

*there is a 1-cell  $\Sigma : \mathbb{H} \rightarrow \mathcal{E}q(\mathbb{G})$  in  $\mathbf{CG}$  and an isomorphism between the given split extension and the split extension obtained pulling back the holomorph split extension*

$$\mathcal{H} = ( \mathbb{G} \longrightarrow \mathcal{H}ol(\mathbb{G}) \rightleftarrows \mathcal{E}q(\mathbb{G}) )$$

*along  $\Sigma$ , say  $u : \mathcal{E} \rightarrow \mathbf{EXT}_{split}(\Sigma, \mathbb{G})(\mathcal{H})$ .*

2. *Moreover, for any other 1-cell  $\Sigma' : \mathbb{H} \rightarrow \mathcal{E}q(\mathbb{G})$  in  $\mathbf{CG}$  and for any other isomorphism  $u' : \mathcal{E} \rightarrow \mathbf{EXT}_{split}(\Sigma', \mathbb{G})(\mathcal{H})$ , there is a unique 2-cell  $\sigma : \Sigma \Longrightarrow \Sigma'$  in  $\mathbf{CG}$  such that  $\mathbf{EXT}_{split}(\sigma, \mathbb{G})(\mathcal{H}) \cdot u = u'$ .*

**Remark 5.3** In spite of the fact that  $\mathbf{CG}$  is a 2-category, in order to apply the 2-dimensional representability criterion of Section 5.1, it is necessary to pass through a quotient that identifies isomorphic 1-cells. Indeed, representability of  $\mathbf{Ext}_{split}(-, \mathbb{G})$  lives in a 3-categorical context, when categorical groups are seen as bicategories with only one object. Nevertheless, we decided to avoid 3-categorical complications because the 2-dimensional representability recaptures the universality of the holomorph construction we are interested into. The reason is that the wider flexibility available in 3-categories trivializes because of local discreteness of categorical group actions.

#### 5.4. HOLOMORPH AND INNER AUTOEQUIVALENCES.

Composing the biequivalence  $\times : \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{Pt}(\mathbf{CG})$  of Section 4.2 with the forgetful 2-functor  $\mathcal{U}^* : \mathbf{Pt}(\mathbf{CG}) \rightarrow \mathbf{CG}$ ,  $\mathcal{U}^*(\mathbb{A}, \mathbb{H}, \mathbf{A}, \mathbf{B}) = \mathbb{A}$ , we get a 2-functor

$$\times : \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{CG}$$

Moreover, the conjugation of a categorical group described in Section 3.3 plainly extends to a 2-functor

$$\mathcal{C} : \mathbf{CG} \rightarrow \mathbf{Act}(\mathbf{CG})$$

**Proposition 5.4** *The 2-functor  $\mathcal{C} : \mathbf{CG} \rightarrow \mathbf{Act}(\mathbf{CG})$  is a right biadjoint to the 2-functor  $\times : \mathbf{Act}(\mathbf{CG}) \rightarrow \mathbf{CG}$ .*

Proof. We refer to [25] for the definition of biadjunction, and we limit ourselves to the description of the natural equivalence between hom-categories

$$\mathbf{Hom}_{\mathbf{Act}(\mathbf{CG})}(ac: \mathbb{H} \times \mathbb{G} \rightarrow \mathbb{G}, \mathcal{C}(\mathbb{A})) \simeq \mathbf{Hom}_{\mathbf{CG}}(\mathbb{G} \rtimes \mathbb{H}, \mathbb{A})$$

Given a morphism of actions

$$\begin{array}{ccc} \mathbb{H} \times \mathbb{G} & \xrightarrow{ac} & \mathbb{G} \\ \mathbf{R} \times \mathbf{T} \downarrow & \nearrow \rho & \downarrow \mathbf{T} \\ \mathbb{A} \times \mathbb{A} & \xrightarrow{conj} & \mathbb{A} \end{array}$$

we define  $\mathbf{\Gamma}: \mathbb{G} \rtimes \mathbb{H} \rightarrow \mathbb{A}$  by  $\mathbf{\Gamma}(A, X) = \mathbf{T}(A) \otimes \mathbf{R}(X)$ . The monoidal structure of  $\mathbf{\Gamma}$  is given by

$$\begin{array}{ccc} \mathbf{T}(A \otimes {}^X B) \otimes \mathbf{R}(X \otimes Y) & \xrightarrow{\simeq} & \mathbf{T}A \otimes \mathbf{T}^X B \otimes \mathbf{R}X \otimes \mathbf{R}Y \\ & & \downarrow 1 \otimes \rho_{X,B} \otimes 1 \otimes 1 \\ \mathbf{T}A \otimes \mathbf{R}X \otimes \mathbf{T}B \otimes \mathbf{R}Y & \xrightarrow{\simeq} & \mathbf{T}A \otimes \mathbf{R}X \otimes \mathbf{T}B \otimes \mathbf{R}X^* \otimes \mathbf{R}X \otimes \mathbf{R}Y \end{array}$$

Given a 2-cell  $(\alpha: \mathbf{R} \Rightarrow \mathbf{R}', \beta: \mathbf{T} \Rightarrow \mathbf{T}')$  in  $\mathbf{Act}(\mathbf{CG})$ , we get a 2-cell  $\mathbf{\Gamma} \Rightarrow \mathbf{\Gamma}'$  in  $\mathbf{CG}$  by  $\beta_A \otimes \alpha_X: \mathbf{T}A \otimes \mathbf{R}X \rightarrow \mathbf{T}'A \otimes \mathbf{R}'X$ . Conversely, given a morphism  $\mathbf{\Gamma}: \mathbb{G} \rtimes \mathbb{H} \rightarrow \mathbb{A}$ , we get  $\mathbf{R}: \mathbb{H} \rightarrow \mathbb{A}$  and  $\mathbf{T}: \mathbb{G} \rightarrow \mathbb{A}$  precomposing  $\mathbf{\Gamma}$  with the canonical injections of  $\mathbb{H}$  and  $\mathbb{G}$  into the semidirect product. Finally, consider  $(X, B) \in \mathbb{H} \times \mathbb{G}$ : we obtain the natural transformation  $\rho$  tensoring with  $\mathbf{R}X^*$  the following composition, which is constructed using the monoidal structure of  $\mathbf{\Gamma}$

$$\begin{array}{ccc} \mathbf{T}^X B \otimes \mathbf{R}X = \mathbf{\Gamma}({}^X B, I) \otimes \mathbf{\Gamma}(I, X) & \longrightarrow & \mathbf{\Gamma}(({}^X B, I) \otimes (I, X)) \\ & & \downarrow \simeq \\ \mathbf{R}X \otimes \mathbf{T}B = \mathbf{\Gamma}(I, X) \otimes \mathbf{\Gamma}(B, I) & \longleftarrow & \mathbf{\Gamma}((I, X) \otimes (B, I)) \end{array}$$

■

Observe now that the canonical morphisms

$$\mathbb{G} \xrightarrow{in_1} \mathcal{H}ol(\mathbb{G}) \xleftarrow{in_2} \mathcal{E}q(\mathbb{G})$$

are equipped with a natural family of 2-cells

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{in_1} & \mathcal{H}ol(\mathbb{G}) \\ \mathbf{F} \downarrow & \nearrow \varphi_{\mathbf{F}} & \downarrow \mathcal{I}_{in_2(\mathbf{F})} \\ \mathbb{G} & \xrightarrow{in_1} & \mathcal{H}ol(\mathbb{G}) \end{array}$$

where  $\mathbf{F}$  varies in  $\mathcal{E}q(\mathbb{G})$ . If, in the equivalence described in the proof of Proposition 5.4, we consider as action the evaluation  $\mathcal{E}q(\mathbb{G}) \times \mathbb{G} \rightarrow \mathbb{G}$ , we obtain the second universal property of the holomorph: for any triple

$$\mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{A} \xleftarrow{\mathbf{R}} \mathcal{E}q(\mathbb{G})$$

in  $\mathbf{CG}$  equipped with a natural family of 2-cells

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\mathbf{T}} & \mathbb{A} \\ \mathbf{F} \downarrow & \rho_{\mathbf{F}} \swarrow & \downarrow \mathcal{I}_{\mathbf{R}(\mathbf{F})} \\ \mathbb{G} & \xrightarrow{\mathbf{T}} & \mathbb{A} \end{array}$$

there is a triple  $(\alpha, \Gamma, \beta)$  in  $\mathbf{CG}$

$$\begin{array}{ccccc} & & \mathcal{H}ol(\mathbb{G}) & & \\ & in_1 \nearrow & \downarrow \Gamma & \nwarrow in_2 & \\ \mathbb{G} & & & & \mathcal{E}q(\mathbb{G}) \\ & \searrow \mathbf{T} & & \swarrow \mathbf{R} & \\ & & \mathbb{A} & & \end{array}$$

$\alpha \Downarrow$        $\beta \Downarrow$

which renders commutative the following diagram of 2-cells, where the unlabelled 2-cell is determined by the monoidal structure of  $\Gamma$

$$\begin{array}{ccccc} \mathcal{I}_{\Gamma(in_2(\mathbf{F}))} \Gamma in_1 & \xrightarrow{\mathcal{I}_{\beta_{\mathbf{F}}} \Gamma in_1} & \mathcal{I}_{\mathbf{R}(\mathbf{F})} \Gamma in_1 & \xrightarrow{\mathcal{I}_{\mathbf{R}(\mathbf{F})} \alpha} & \mathcal{I}_{\mathbf{R}(\mathbf{F})} \mathbf{T} \\ \uparrow \parallel & & & & \downarrow \rho_{\mathbf{F}} \\ \Gamma \mathcal{I}_{in_2(\mathbf{F})} in_1 & \xrightarrow{\Gamma \varphi_{\mathbf{F}}} & \Gamma in_1 \mathbf{F} & \xrightarrow{\alpha_{\mathbf{F}}} & \mathbf{T} \mathbf{F} \end{array}$$

Moreover, if  $(\alpha', \Gamma', \beta')$  is another such a triple, then there is a unique 2-cell  $\omega: \Gamma \Rightarrow \Gamma'$  in  $\mathbf{CG}$  making commutative the following diagrams

$$\begin{array}{ccc} \Gamma in_1 & \xrightarrow{\omega in_1} & \Gamma' in_1 \\ \searrow \alpha & & \swarrow \alpha' \\ & \mathbf{T} & \end{array} \quad \begin{array}{ccc} \Gamma in_2 & \xrightarrow{\omega in_2} & \Gamma' in_2 \\ \searrow \beta & & \swarrow \beta' \\ & \mathbf{R} & \end{array}$$

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*Dipartimento di Matematica  
 Università di Milano  
 Via Saldini 50  
 I 20133 Milano, Italia*

*Département de Mathématique Pure et Appliquée  
 Université catholique de Louvain  
 Chemin du Cyclotron 2  
 B 1348 Louvain-la-Neuve, Belgique*

Email:

`Stefano.Kasangian@mat.unimi.it`, `metere@mat.unimi.it`, `vitale@math.ucl.ac.be`