THE SNAIL LEMMA

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Abstract. The classical snake lemma produces a six terms exact sequence starting from a commutative square with one of the edge being a regular epimorphism. We establish a new diagram lemma, that we call snail lemma, removing such a condition. We also show that the snail lemma subsumes the snake lemma and we give an interpretation of the snail lemma in terms of strong homotopy kernels. Our results hold in any pointed regular protomodular category.

Contents

1. Introduction

One of the basic diagram lemmas in homological algebra is the snake lemma (also called kernel-cokernel lemma). In an abelian category, it can be stated in the following way : from the commutative diagram

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ \mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \tag{1}$$

and under the assumption that f is an epimorphism, it is possible to get an exact sequence

$$\mathrm{Ker}(\mathrm{K}(\alpha)) \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow \mathrm{Cok}(\mathrm{K}(\alpha)) \longrightarrow \mathrm{Cok}(\alpha) \longrightarrow \mathrm{Cok}(\beta)$$

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If we replace "epimorphism" with "regular epimorphism" and if α, β and $\mathrm{K}(\alpha)$ are proper morphisms (Definition 2.2) the snake lemma holds also in several important non abelian categories, like groups, crossed modules, Lie algebras, not necessarily unitary rings.
Despite its very clear formulation, the snake lemma is somehow asymmetric because of the hypothesis on the morphism f. The aim of this paper is to study what happens if we remove such condition. What we prove is that we can get a six terms exact sequence (the snail sequence) starting from any commutative diagram like

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \tag{2}$$

(with α and β proper if we work in a non abelian context) ; moreover, the snail sequence coincides with the usual snake sequence whenever f is a regular epimorphism. In order to get the snail sequence, we have to replace the kernels appearing in diagram (1) with a different (very simple) construction, which in fact is a kind of 2-dimensional kernel of diagram (2).

More in detail, the layout of this paper is as follows : in Section 2 we recall the snake lemma in its general form established by D. Bourn in the context of pointed regular protomodular categories (= homological categories, in the terminology of [3]), as well as some other results from [2]. A general reference for protomodular categories is [3], a more concise introduction can be found in [4] ; abelian categories as well as groups and all the other examples quoted above are pointed regular protomodular categories. Pointed regular protomodular categories are also the framework of Section 3, which is completely devoted to state and prove the snail lemma, and of Section 4, where we show that the snail lemma subsumes the snake lemma. It is worthwile to note that, in order to compare the snail sequence and the snake sequence, we need an intermediate result (Lemma 4.1) which is very close to the characterization of subtractive categories established in [8]. The precise relation between the snail lemma and the axiomatic for subtractive categories will be explaineded in [9]. In Section 5 we give a precise explication of the 2-categorical meaning of the construction involved in the snail lemma (in contrast with the construction involved in the snake lemma, which is a categorical kernel but does not satisfy any 2-dimensional universal property). Finally, in Section 6 we specialize the situation to abelian categories : in this case, by duality we easily get a longer version of the snail sequence.

We use diagrammatic notation for composition : $A \xrightarrow{f} B \xrightarrow{g} C$ is written $f \cdot g$.

2. The snake lemma

Let $\mathcal{A}$ be a pointed, regular and protomodular category.

Notation 2.1 Here is the notation we use for kernels, cokernels, and the induced morphisms.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "c_f"] \arrow[d, "\beta"] & \mathrm{Cok}(f) \arrow[d, "\mathrm{C}(\beta)"] \\
\mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0 \arrow[r, "c_{f_0}"'] & \mathrm{Cok}(f_0)
\end{tikzcd}$$

To start, we recall the snake lemma as proved by D. Bourn in [2].

Definition 2.2 A morphism $\alpha\colon A \to A_0$ in $\mathcal{A}$ is *proper* if it admits a cokernel c_α and the factorization $\overline{\alpha}$ of α along the kernel of c_α is a regular epimorphism

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\overline{\alpha}"'] & & A_0 \arrow[r, "c_\alpha"] & \mathrm{Cok}(\alpha) \\
& \mathrm{Ker}(c_\alpha) \arrow[ur, "k_{c_\alpha}"'] & &
\end{tikzcd}$$

Snake Lemma 2.3 *Consider the following commutative diagram in $\mathcal{A}$ and assume that f and f_0 are regular epimorphisms*

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
\mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

If α, β and $\mathrm{K}(\alpha)$ are proper, then there exists a morphism $d\colon \mathrm{Ker}(\beta) \to \mathrm{Cok}(\gamma)$ such that the following sequence is exact

$$\mathrm{Ker}(\mathrm{K}(\alpha)) \xrightarrow{\mathrm{K}(k_f)} \mathrm{Ker}(\alpha) \xrightarrow{\mathrm{K}(f)} \mathrm{Ker}(\beta) \xrightarrow{d} \mathrm{Cok}(\mathrm{K}(\alpha)) \xrightarrow{\mathrm{C}(k_{f_0})} \mathrm{Cok}(\alpha) \xrightarrow{\mathrm{C}(f_0)} \mathrm{Cok}(\beta)$$

Observe that if the category $\mathcal{A}$ is abelian, then any morphism in $\mathcal{A}$ is proper. This is the reason why there are no assumptions on α, β and $\mathrm{K}(\alpha)$ for the snake lemma in an abelian category.

On the way to prove the snake lemma, Bourn establishes in [2] the following facts that we need later.

Proposition 2.4 *Consider the following commutative diagram in $\mathcal{A}$ and assume that f is a regular epimorphism*

$$\begin{tikzcd}
& \mathrm{Ker}(\alpha) \arrow[r, "\mathrm{K}(f)"] \arrow[d, "k_\alpha"'] & \mathrm{Ker}(\beta) \arrow[d, "k_\beta"] \\
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
\mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

1. *If* $\mathrm{K}(\alpha)$ *is a regular epimorphism, then* $\mathrm{K}(f)$ *is a regular epimrphism.*
2. *If* $\mathrm{K}(\alpha)$ *and* β *are regular epimorphisms, then* α *is a regular epimrphism.*

Proposition 2.5 *A morphism* $f\colon A \to B$ *in* $\mathcal{A}$ *is a monomorphism if and only if its kernel* $k_f\colon \mathrm{Ker}(f) \to A$ *is the zero morphism.*

3. The snail lemma

In this section, $\mathcal{A}$ is a pointed, regular and protomodular category. Consider a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

and construct the following diagram, where :
- $A_0 \times_{f_0,\beta} B$ is the pullback of f_0 and β,
- $\gamma = \langle \alpha, f \rangle$ is the unique morphism such that $\gamma \cdot \beta' = \alpha$ and $\gamma \cdot f_0' = f$,
- $k_{\beta'} = \langle 0, k_\beta \rangle$ is the unique morphism such that $k_{\beta'} \cdot \beta' = 0$ and $k_{\beta'} \cdot f_0' = k_\beta$, and it is a kernel of β' because β' is a pullback of β,
- $\mathrm{C}(\beta')$ is the unique morphism such that $c_\gamma \cdot \mathrm{C}(\beta') = \beta' \cdot c_\alpha$.

$$\begin{tikzcd}
 & & & \mathrm{Ker}(\beta') = \mathrm{Ker}(\beta) \arrow[d, "k_\beta"] \arrow[dll, "k_{\beta'}"'] \\
A \arrow[rrr, "f"] \arrow[dr, "\gamma"] \arrow[ddd, "\alpha"'] & & & B \arrow[ddd, "\beta"] \\
 & A_0 \times_{f_0,\beta} B \arrow[urr, "f_0'"'] \arrow[dr, "c_\gamma"] \arrow[ddl, "\beta'"] & & \\
 & & \mathrm{Cok}(\gamma) \arrow[ddll, "\mathrm{C}(\beta')"] & \\
A_0 \arrow[rrr, "f_0"'] \arrow[d, "c_\alpha"'] & & & B_0 \\
\mathrm{Cok}(\alpha) & & &
\end{tikzcd}$$

Snail Lemma 3.1 *Consider the following commutative diagram in $\mathcal{A}$*

$$
\begin{tikzcd}
\mathrm{Ker}(\gamma) \arrow[r, "\mathrm{K(id)}"] \arrow[d, "k_\gamma"'] & \mathrm{Ker}(\alpha) \arrow[r, "\mathrm{K}(f)"] \arrow[d, "k_\alpha"'] & \mathrm{Ker}(\beta) \arrow[d, "k_\beta"] \\
A \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A_0 \times_{f_0,\beta} B \arrow[r, "\beta'"'] \arrow[d, "c_\gamma"'] & A_0 \arrow[r, "f_0"'] \arrow[d, "c_\alpha"'] & B_0 \arrow[d, "c_\beta"] \\
\mathrm{Cok}(\gamma) \arrow[r, "\mathrm{C}(\beta')"'] & \mathrm{Cok}(\alpha) \arrow[r, "\mathrm{C}(f_0)"'] & \mathrm{Cok}(\beta)
\end{tikzcd}
$$

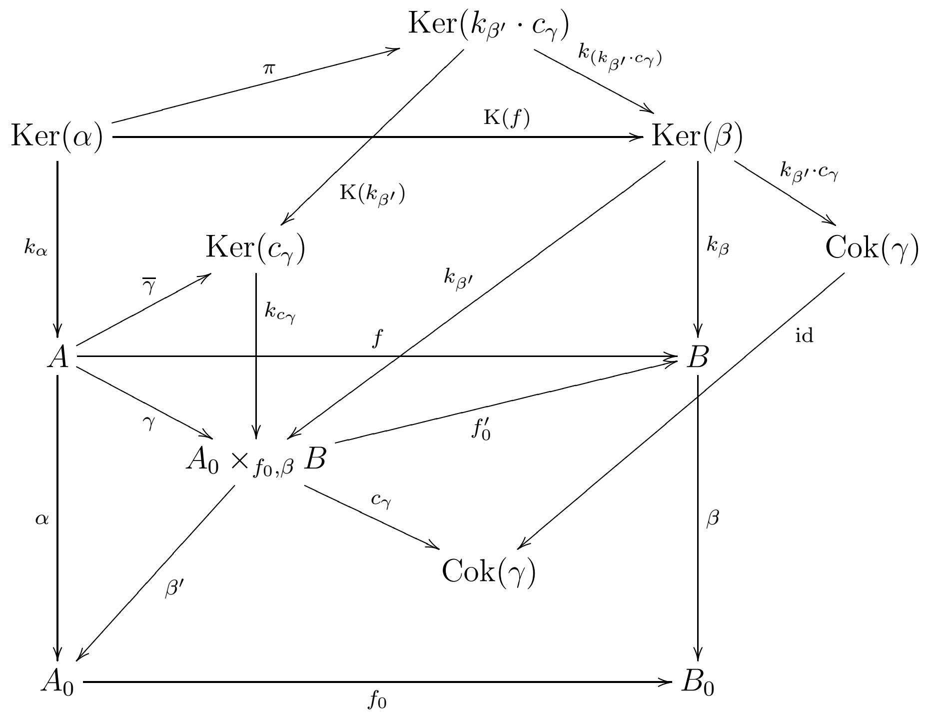
If α, β and γ are proper, then the following sequence is exact

$$\mathrm{Ker}(\gamma) \xrightarrow{\mathrm{K(id)}} \mathrm{Ker}(\alpha) \xrightarrow{\mathrm{K}(f)} \mathrm{Ker}(\beta) \xrightarrow{k_{\beta'} \cdot c_\gamma} \mathrm{Cok}(\gamma) \xrightarrow{\mathrm{C}(\beta')} \mathrm{Cok}(\alpha) \xrightarrow{\mathrm{C}(f_0)} \mathrm{Cok}(\beta)$$

Proof.

• Exactness in $\mathrm{Ker}(\alpha)$: obvious because $\mathrm{K(id)} \colon \mathrm{Ker}(\gamma) \to \mathrm{Ker}(\alpha)$ is a kernel of $\mathrm{K}(f)$ by interchange of limits.

• Exactness in $\mathrm{Ker}(\beta)$: first, observe that $\mathrm{K}(f) \cdot k_{\beta'} = k_\alpha \cdot \gamma$ (compose with the pullback projections) and, therefore, $\mathrm{K}(f) \cdot k_{\beta'} \cdot c_\gamma = 0$. Now, to prove the exactness in $\mathrm{Ker}(\beta)$ we use the following diagram, where π is the unique morphism such that $\pi \cdot k_{(k_{\beta'} \cdot c_\gamma)} = \mathrm{K}(f)$. Observe that $\mathrm{K}(k_{\beta'})$ is a monomorphism because $\mathrm{K}(k_{\beta'}) \cdot k_{c_\gamma} = k_{(k_{\beta'} \cdot c_\gamma)} \cdot k_{\beta'}$ and $k_{(k_{\beta'} \cdot c_\gamma)}$ and $k_{\beta'}$ are monomorphisms.



We have to prove that π is a regular epimorphism. This follows from the fact that $\overline{\gamma}$ is a regular epimorphism (because γ is proper) and the following square is a pullback.

$$\begin{tikzcd}
\mathrm{Ker}(\alpha) \arrow[r, "\pi"] \arrow[d, "k_\alpha"'] & \mathrm{Ker}(k_{\beta'} \cdot c_\gamma) \arrow[d, "\mathrm{K}(k_{\beta'})"] \\
A \arrow[r, "\overline{\gamma}"'] & \mathrm{Ker}(c_\gamma)
\end{tikzcd}$$

To check its commutativity, compose with the monomorphism k_{c_γ}. To check its universality, consider morphisms $x\colon X \to A$ and $y\colon X \to \mathrm{Ker}(k_{\beta'} \cdot c_\gamma)$ such that $x \cdot \overline{\gamma} = y \cdot \mathrm{K}(k_{\beta'})$. We have :

$$x \cdot \alpha = x \cdot \gamma \cdot \beta' = x \cdot \overline{\gamma} \cdot k_{c_\gamma} \cdot \beta' = y \cdot \mathrm{K}(k_{\beta'}) \cdot k_{c_\gamma} \cdot \beta' = y \cdot k_{(k_{\beta'} \cdot c_\gamma)} \cdot k_{\beta'} \cdot \beta' = y \cdot k_{(k_{\beta'} \cdot c_\gamma)} \cdot 0 = 0$$

Therefore, there exists a unique $z\colon X \to \mathrm{Ker}(\alpha)$ such that $z \cdot k_\alpha = x$. The condition $z \cdot \pi = y$ follows from $z \cdot k_\alpha = x$ because $\mathrm{K}(k_{\beta'})$ is a monomorphism.

• Exactness in $\mathrm{Cok}(\gamma)$: first, observe that $k_{\beta'} \cdot c_\gamma \cdot \mathrm{C}(\beta') = k_{\beta'} \cdot \beta' \cdot c_\alpha = 0$. Now, to prove the exactness in $\mathrm{Cok}(\gamma)$ we use the following commutative diagram.

$$\begin{tikzcd}
 & & \mathrm{Ker}(c_\gamma) \arrow[dll, "\mathrm{K}(\beta')"'] \arrow[dd, "k_{c_\gamma}"] & & \mathrm{Ker}(\beta') = \mathrm{Ker}(\beta) \arrow[ddll, "k_{\beta'}"'] \arrow[d, "k_\beta"] \arrow[dr, "\mathrm{K}(c_\gamma)"] & \\
\mathrm{Ker}(c_\alpha) \arrow[dddr, "k_{c_\alpha}"'] & A \arrow[l, "\overline{\alpha}"'] \arrow[ur, "\overline{\gamma}"] \arrow[rrr, "f"] \arrow[dr, "\gamma"] \arrow[ddd, "\alpha"] & & & B \arrow[ddd, "\beta"] & \mathrm{Ker}(\mathrm{C}(\beta')) \arrow[ddll, "k_{\mathrm{C}(\beta')}"] \\
 & & A_0 \times_{f_0,\beta} B \arrow[urr, "f_0'"'] \arrow[dr, "c_\gamma"] \arrow[ddl, "\beta'"] & & & \\
 & & & \mathrm{Cok}(\gamma) \arrow[ddll, "\mathrm{C}(\beta')"] & & \\
 & A_0 \arrow[rrr, "f_0"'] \arrow[d, "c_\alpha"'] & & & B_0 & \\
 & \mathrm{Cok}(\alpha) & & & &
\end{tikzcd}$$

We have to prove that $\mathrm{K}(c_\gamma)$ is a regular epimorphism. For this, observe that $\overline{\gamma} \cdot \mathrm{K}(\beta') = \overline{\alpha}$ (indeed, composing with the monomorphism k_{c_α}, both give α). Since α is proper, $\overline{\alpha}$ is a regular epimorphism, and therefore the equation $\overline{\gamma} \cdot \mathrm{K}(\beta') = \overline{\alpha}$ implies that $\mathrm{K}(\beta')$ is a regular epimorphism. We can conclude that $\mathrm{K}(c_\gamma)$ is a regular epimorphism by applying point 1 of Proposition 2.4 to the following diagram.

$$\begin{tikzcd}
 & \mathrm{Ker}(\beta') \arrow[r, "\mathrm{K}(c_\gamma)"] \arrow[d, "k_{\beta'}"'] & \mathrm{Ker}(\mathrm{C}(\beta')) \arrow[d, "k_{\mathrm{C}(\beta')}"] \\
\mathrm{Ker}(c_\gamma) \arrow[r, "k_{c_\gamma}"] \arrow[d, "\mathrm{K}(\beta')"'] & A_0 \times_{f_0,\beta} B \arrow[r, "c_\gamma"] \arrow[d, "\beta'"] & \mathrm{Cok}(\gamma) \arrow[d, "\mathrm{C}(\beta')"] \\
\mathrm{Ker}(c_\alpha) \arrow[r, "k_{c_\alpha}"'] & A_0 \arrow[r, "c_\alpha"'] & \mathrm{Cok}(\alpha)
\end{tikzcd}$$

• Exactness in $\mathrm{Cok}(\alpha)$: first, observe that $c_\gamma \cdot \mathrm{C}(\beta') \cdot \mathrm{C}(f_0) = 0$ and then, since c_γ is an epimorphism, we have $\mathrm{C}(\beta') \cdot \mathrm{C}(f_0) = 0$. Now, to prove the exactness in $\mathrm{Cok}(\alpha)$ we use the following commutative diagram, where π is the unique morphism such that $\pi \cdot k_{\mathrm{C}(f_0)} = \mathrm{C}(\beta')$.

$$\begin{tikzcd}
A \arrow[rrrr, "f"] \arrow[dr, "\gamma"] \arrow[ddd, "\alpha"'] & & & & B \arrow[dr, "\overline{\beta}"] \arrow[ddd, "\beta"] & \\
 & A_0 \times_{f_0,\beta} B \arrow[urrr, "f'_0"'] \arrow[dr, "c_\gamma"] \arrow[ddl, "\beta'"] & & & & \mathrm{Ker}(c_\beta) \arrow[ddl, "k_{c_\beta}"] \\
 & & \mathrm{Cok}(\gamma) \arrow[dd, "\pi"] \arrow[dddll, "\mathrm{C}(\beta')"'] & & & \\
A_0 \arrow[rrrr, "f_0"'] \arrow[dd, "c_\alpha"'] & & & & B_0 \arrow[dd, "c_\beta"] & \\
 & & \mathrm{Ker}(\mathrm{C}(f_0)) \arrow[dll, "k_{\mathrm{C}(f_0)}"] & & & \\
\mathrm{Cok}(\alpha) \arrow[rrrr, "\mathrm{C}(f_0)"'] & & & & \mathrm{Cok}(\beta) &
\end{tikzcd}$$

We have to prove that π is a regular epimorphism. For this, we split the construction of the pullback $A_0 \times_{f_0,\beta} B$ in two steps :

$$\begin{tikzcd}
A_0 \times_{f_0,\beta} B \arrow[rr, "f'_0"] \arrow[dr, "x"] \arrow[dd, "\beta'"'] & & B \arrow[dr, "\overline{\beta}"] \arrow[dd, "\beta"'] & \\
 & A_0 \times_{f_0,k_{c_\beta}} \mathrm{Ker}(c_\beta) \arrow[rr, "z"] \arrow[dl, "y"] & & \mathrm{Ker}(c_\beta) \arrow[dl, "k_{c_\beta}"] \\
A_0 \arrow[rr, "f_0"'] & & B_0 &
\end{tikzcd}$$

Since β is proper, $\overline{\beta}$ is a regular epimorphism, and then x also is a regular epimorphism because it is a pullback of $\overline{\beta}$. Moreover, since $y \cdot c_\alpha \cdot \mathrm{C}(f_0) = 0$, there exists a unique $t\colon A_0 \times_{f_0,k_{c_\beta}} \mathrm{Ker}(c_\beta) \to \mathrm{Ker}(\mathrm{C}(f_0))$ such that $y \cdot c_\alpha = t \cdot k_{\mathrm{C}(f_0)}$. Now observe that the following square commutes

$$\begin{tikzcd}
A_0 \times_{f_0,\beta} B \arrow[r, "c_\gamma"] \arrow[d, "x"'] & \mathrm{Cok}(\gamma) \arrow[d, "\pi"] \\
A_0 \times_{f_0,k_{c_\beta}} \mathrm{Ker}(c_\beta) \arrow[r, "t"'] & \mathrm{Ker}(\mathrm{C}(f_0))
\end{tikzcd}$$

(for this, compose with the monomorphism $k_{\mathrm{C}(f_0)}$). Therefore, since x is a regular epimorphism, to prove that π is a regular epimorphism it remains to show that t is a regular epimorphism. This can be done using the following commutative diagrams

$$\begin{tikzcd}
A_0 \times_{f_0,k_{c_\beta}} \mathrm{Ker}(c_\beta) \arrow[r, "z"] \arrow[d, "y"'] \arrow[dr, phantom, "(1)"] & \mathrm{Ker}(c_\beta) \arrow[r, "0"] \arrow[d, "k_{c_\beta}"'] \arrow[dr, phantom, "(2)"] & 0 \arrow[d, "0"] \\
A_0 \arrow[r, "f_0"'] & B_0 \arrow[r, "c_\beta"'] & \mathrm{Cok}(\beta)
\end{tikzcd}$$

$$\begin{tikzcd}
A_0 \times_{f_0,k_{c_\beta}} \mathrm{Ker}(c_\beta) \arrow[r, "t"] \arrow[d, "y"'] \arrow[dr, phantom, "(3)"] & \mathrm{Ker}(\mathrm{C}(f_0)) \arrow[r, "0"] \arrow[d, "\mathrm{C}(f_0)"'] \arrow[dr, phantom, "(4)"] & 1 \arrow[d, "0"] \\
A_0 \arrow[r, "c_\alpha"'] & \mathrm{Cok}(\alpha) \arrow[r, "\mathrm{C}(f_0)"'] & \mathrm{Cok}(\beta)
\end{tikzcd}$$

Since (1) and (2) are pullbacks, so is (1)+(2), that is, y is a kernel of $f_0 \cdot c_\beta = c_\alpha \cdot \mathrm{C}(f_0)$. Therefore, (3)+(4) is a pullback, but also (4) is a pullback, and then (3) is a pullback. Since c_α is a regular epimorphism, this proves that t is a regular epimorphism. ■

4. Comparing the snake and the snail

As in the previous sections, $\mathcal{A}$ is a pointed, regular and protomodular category. We need a preliminary result.

Lemma 4.1 *Consider the following diagram in* $\mathcal{A}$

$$\begin{tikzcd}
 & \mathrm{Ker}(x) \arrow[d, "k_x"'] & \\
\mathrm{Ker}(y) \arrow[r, "k_y"] & P \arrow[r, "y"] \arrow[d, "x"'] & Y \\
 & X &
\end{tikzcd}$$

If x *and* $k_x \cdot y$ *are regular epimorphisms, then* $k_y \cdot x$ *is a regular epimorphism.*

Proof. Consider the unique factorization $\langle x : y\rangle \colon P \to X \times Y$ of x and y through the product. By applying point 2 of Proposition 2.4 to the diagram

$$\begin{tikzcd} \mathrm{Ker}(x) \arrow[r, "k_x"] \arrow[d, "k_x \cdot y"'] & P \arrow[r, "x"] \arrow[d, "\langle x:y\rangle"'] & X \arrow[d, "\mathrm{id}"] \\ Y \arrow[r, "\langle 0:1\rangle"'] & X \times Y \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

we get that $\langle x : y\rangle$ is a regular epimorphism. Consider now the following commutative diagram

$$\begin{tikzcd} \mathrm{Ker}(y) \arrow[r, "k_y \cdot x"] \arrow[d, "k_y"'] \arrow[dr, phantom, "(1)"] & X \arrow[r, "0"] \arrow[d, "\langle 1:0\rangle"'] \arrow[dr, phantom, "(2)"] & 0 \arrow[d, "0"] \\ P \arrow[r, "\langle x:y\rangle"'] & X \times Y \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Since (2) and (1)+(2) are pullbacks, (1) also is a pullback. Therefore, $k_y \cdot x$ is a regular epimorphism because it is a pullback of $\langle x : y\rangle$. ∎

We will also use the following simple fact, which holds if $\mathcal{A}$ is pointed and has kernels.

Lemma 4.2 *Consider the following commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ \mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

If β is a monomorphism, then the left-hand square is a pullback.

If we compare the snake sequence (Lemma 2.3) and the snail sequence (Lemma 3.1)

$$\mathrm{Ker}(\mathrm{K}(\alpha)) \xrightarrow{\mathrm{K}(k_f)} \mathrm{Ker}(\alpha) \xrightarrow{\mathrm{K}(f)} \mathrm{Ker}(\beta) \xrightarrow{d} \mathrm{Cok}(\mathrm{K}(\alpha)) \xrightarrow{\mathrm{C}(k_{f_0})} \mathrm{Cok}(\alpha) \xrightarrow{\mathrm{C}(f_0)} \mathrm{Cok}(\beta)$$

$$\mathrm{Ker}(\gamma) \xrightarrow{\mathrm{K}(\mathrm{id})} \mathrm{Ker}(\alpha) \xrightarrow{\mathrm{K}(f)} \mathrm{Ker}(\beta) \xrightarrow{k_{\beta'} \cdot c_\gamma} \mathrm{Cok}(\gamma) \xrightarrow{\mathrm{C}(\beta')} \mathrm{Cok}(\alpha) \xrightarrow{\mathrm{C}(f_0)} \mathrm{Cok}(\beta)$$

we see that the only difference is that $\mathrm{Cok}(\mathrm{K}(\alpha))$ is replaced, in the snail sequence, by $\mathrm{Cok}(\gamma)$. Indeed, by exchange of limits, both

$$\mathrm{K}(k_f) \colon \mathrm{Ker}(\mathrm{K}(\alpha)) \to \mathrm{Ker}(\alpha) \quad \text{and} \quad \mathrm{K}(\mathrm{id}) \colon \mathrm{Ker}(\gamma) \to \mathrm{Ker}(\alpha)$$

are kernels of $\mathrm{K}(f) \colon \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$.
In order to show that $\mathrm{Cok}(\mathrm{K}(\alpha))$ and $\mathrm{Cok}(\gamma)$ are isomorphic, let us put together the construction underlying the snake lemma and the construction underlying the snail lemma

in the following diagram, where :
- $k_{f'_0} = \langle k_{f_0}, 0\rangle$ is the unique morphism such that $k_{f'_0} \cdot \beta' = k_{f_0}$ and $k_{f'_0} \cdot f'_0 = 0$, and it is a kernel of f'_0 because f'_0 is a pullback of f_0,
- $\mathrm{C}(k_{f'_0})$ is the unique morphism such that $k_{f'_0} \cdot c_\gamma = c_{\mathrm{K}(\alpha)} \cdot \mathrm{C}(k_{f'_0})$ (such a morphism exists because $k_f \cdot \gamma = \mathrm{K}(\alpha) \cdot k_{f'_0}$, as one easily cheks by composing with the pullback projections).

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[ddd, "\mathrm{K}(\alpha)"'] & A \arrow[rrrr, "f"] \arrow[dr, "\gamma"] \arrow[ddd, "\alpha"'] & & & & B \arrow[ddd, "\beta"] \\
& & A_0 \times_{f_0,\beta} B \arrow[urrr, "f'_0"'] \arrow[dr, "c_\gamma"] \arrow[ddl, "\beta'"] & & & \\
& & & \mathrm{Cok}(\gamma) & & \\
\mathrm{Ker}(f'_0) = \mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] \arrow[uurr, "k_{f'_0}"] \arrow[d, "c_{\mathrm{K}(\alpha)}"'] & A_0 \arrow[rrrr, "f_0"'] & & & & B_0 \\
\mathrm{Cok}(\mathrm{K}(\alpha)) \arrow[uurrr, "\mathrm{C}(k_{f'_0})"'] & & & & &
\end{tikzcd}$$

Proposition 4.3 *(With the previous notation.)*

1. *If* γ *and* $\mathrm{K}(\alpha)$ *are proper, then* $\mathrm{C}(k_{f'_0})$ *is a monomorphism;*

2. *If* f *is a regular epimorphism, then* $\mathrm{C}(k_{f'_0})$ *is a regular epimorphism.*

Proof.
Proof of 1. By Lemma 4.2, the left-hand square in the following commutative diagram is a pullback

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\gamma"] & B \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(f'_0) \arrow[r, "k_{f'_0}"'] & A_0 \times_{f_0,\beta} B \arrow[r, "f'_0"'] & B
\end{tikzcd}$$

Therefore, pulling back $k_{f'_0}$ along the (regular epi, mono)-factorization of γ gives the (regular epi, mono)-factorization of $\mathrm{K}(\alpha)$. Since γ and $\mathrm{K}(\alpha)$ are proper, this means that in the following diagram squares (1) and (2) are pullbacks.

$$\begin{tikzcd}
 & \mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\overline{\mathrm{K}(\alpha)}"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "\overline{\gamma}"] \\
 & \mathrm{Ker}(c_{\mathrm{K}(\alpha)}) \arrow[r, "\mathrm{K}(k_{f_0'})"] \arrow[d, "k_{c_{\mathrm{K}(\alpha)}}"] \arrow[ddl, "0"'] \arrow[dr, phantom, "(2)"] & \mathrm{Ker}(c_\gamma) \arrow[d, "k_{c_\gamma}"] \\
 \arrow[r, phantom, "(3)"] & \mathrm{Ker}(f_0') \arrow[r, "k_{f_0'}"'] \arrow[d, "c_{\mathrm{K}(\alpha)}"'] & A_0 \times_{f_0,\beta} B \arrow[d, "c_\gamma"] \\
\mathrm{Ker}(\mathrm{C}(k_{f_0'})) \arrow[r, "k_{\mathrm{C}(k_{f_0'})}"'] & \mathrm{Cok}(\mathrm{K}(\alpha)) \arrow[r, "\mathrm{C}(k_{f_0'})"'] & \mathrm{Cok}(\gamma)
\end{tikzcd}$$

Now we prove that (3) also is a pullback, so that 0 is a regular epimorphism and then $k_{\mathrm{C}(k_{f_0'})}$ is zero, which implies that $\mathrm{C}(k_{f_0'})$ is a monomorphism (Proposition 2.5). Clearly, (3) commutes. Consider morphisms $x\colon X \to \mathrm{Ker}(\mathrm{C}(k_{f_0'}))$ and $y\colon X \to \mathrm{Ker}(f_0')$ such that $x \cdot k_{\mathrm{C}(k_{f_0'})} = y \cdot c_{\mathrm{K}(\alpha)}$. We have

$$y \cdot k_{f_0'} \cdot c_\gamma = y \cdot c_{\mathrm{K}(\alpha)} \cdot \mathrm{C}(k_{f_0'}) = x \cdot k_{\mathrm{C}(k_{f_0'})} \cdot \mathrm{C}(k_{f_0'}) = x \cdot 0 = 0$$

so that there exists $z\colon X \to \mathrm{Ker}(c_\gamma)$ such that $z \cdot k_{c_\gamma} = y \cdot k_{f_0'}$. Since (2) is a pullback, this implies that there exists $t\colon X \to \mathrm{Ker}(c_{\mathrm{K}(\alpha)})$ such that $t \cdot \mathrm{K}(k_{f_0'}) = z$ and $t \cdot k_{c_{\mathrm{K}(\alpha)}} = y$. Moreover, $t \cdot 0 \cdot k_{\mathrm{C}(k_{f_0'})} = t \cdot k_{c_{\mathrm{K}(\alpha)}} \cdot c_{\mathrm{K}(\alpha)} = y \cdot c_{\mathrm{K}(\alpha)} = x \cdot k_{\mathrm{C}(k_{f_0'})}$, and then $t \cdot 0 = x$ because $k_{\mathrm{C}(k_{f_0'})}$ is a monomorphism.

Proof of 2. Consider the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "\overline{\gamma}"] \arrow[dr, "\gamma"'] \arrow[drr, bend left, "f"] & \mathrm{Ker}(c_\gamma) \arrow[d, "k_{c_\gamma}"] & \\
\mathrm{Ker}(k_{f_0'}) \arrow[rr, "k_{f_0'}"] \arrow[d, "c_{\mathrm{K}(\alpha)}"'] & & A_0 \times_{f_0,\beta} B \arrow[r, "f_0'"'] \arrow[d, "c_\gamma"] & B \\
\mathrm{Cok}(\mathrm{K}(\alpha)) \arrow[rr, "\mathrm{C}(k_{f_0'})"'] & & \mathrm{Cok}(\gamma) &
\end{tikzcd}$$

Since f is a regular epimorphism and $f = \gamma \cdot f_0' = \overline{\gamma} \cdot k_{c_\gamma} \cdot f_0'$, we have that $k_{c_\gamma} \cdot f_0'$ is a regular epimorphism. Therefore, we can apply Lemma 4.1 to

$$\begin{tikzcd} & \mathrm{Ker}(c_\gamma) \arrow[d, "k_{c_\gamma}"] & \\ \mathrm{Ker}(k_{f_0'}) \arrow[r, "k_{f_0'}"] & A_0 \times_{f_0,\beta} B \arrow[r, "f_0'"'] \arrow[d, "c_\gamma"] & B \\ & \mathrm{Cok}(\gamma) & \end{tikzcd}$$

and we have that $k_{f_0'} \cdot c_\gamma$ is a regular epimorphism. Since $k_{f_0'} \cdot c_\gamma = c_{\mathrm{K}(\alpha)} \cdot \mathrm{C}(k_{f_0'})$, we can conclude that $\mathrm{C}(k_{f_0'})$ is a regular epimorphism. ■

5. A 2-categorical explication of the snail lemma

The most economical way to explain the construction involved in the snail lemma is by using strong homotopy kernels. Indeed, to express the universal property of a strong homotopy kernel we only need null-homotopies. In order to formalize null-homotopies and (strong) homotopy kernels, we adopt the following setting, introduced in [7].

Definition 5.1 A *category with null-homotopies* $\underline{\mathcal{B}}$ is given by
- a category $\mathcal{B}$,
- for each morphism $f : A \to B$ in $\mathcal{B}$, a set $\mathcal{N}(f)$ (the set of null-homotopies on f),
- for each triple of composable morphisms $f : A \to B, g : B \to C, h : C \to D$, a map

$$f \circ - \circ h : \mathcal{N}(g) \to \mathcal{N}(f \cdot g \cdot h)\ ,\ \ \mu \mapsto f \circ \mu \circ h$$

(If $f = \mathrm{id}_B$ or $h = \mathrm{id}_C$, we write $\mu \circ h$ or $f \circ \mu$ instead of $f \circ \mu \circ h$.)
These data have to satisfy the following associativity condition : given morphisms

$$A' \xrightarrow{f'} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{h'} D'$$

then for any $\mu \in \mathcal{N}(g)$ one has $(f' \cdot f) \circ \mu \circ (h \cdot h') = f' \circ (f \circ \mu \circ h) \circ h'$.

Definition 5.2 Let $\underline{\mathcal{B}}$ be a category with null-homotopies and let $f : A \to B$ be a morphism in $\mathcal{B}$. A triple

$$\mathrm{Ker}(f),\ \mathrm{K}(f) : \mathrm{Ker}(f) \to A,\ k(f) \in \mathcal{N}(\mathrm{K}(f) \cdot f)$$

1. is a *homotopy kernel* of f if for any triple $D,\ g : D \to A,\ \mu \in \mathcal{N}(g \cdot f)$, there exists a unique morphism $g' : D \to \mathrm{Ker}(f)$ such that $g' \cdot \mathrm{K}(f) = g$ and $g' \circ k(f) = \mu$
2. is a *strong homotopy kernel* of f if it is a homotopy kernel of f and, moreover, for any triple $D,\ h : D \to \mathrm{Ker}(f),\ \mu \in \mathcal{N}(h \cdot \mathrm{K}(f))$ such that $\mu \circ f = h \circ k(f)$, there exists a unique $\lambda \in \mathcal{N}(h)$ such that $\lambda \circ \mathrm{K}(f) = \mu$.

Example 5.3 To help intuition, here is an easy example. The category $\mathcal{B}$ is the category $\mathbf{Grpd}_*$ of groupoids with a distinguished object $*$ and pointed functors (that is, functors preserving the object $*$). For a pointed functor $f\colon A \to B$, the set $\mathcal{N}(f)$ is the set of natural transformations $\lambda\colon 0 \to f$ such that $\lambda_* = \mathrm{id}_*$, where $0\colon A \to B$ is the constant pointed functor. The operation $f \circ \mu \circ g$ is the reduced horizontal composition of natural transformations. The strong homotopy kernel of $f\colon A \to B$ is the usual comma (or slice) groupoid $*/f$ whose objects are pairs $(a_0 \in A, b_1\colon * \to f(a_0) \in B)$.

In the rest of this section, $\mathcal{A}$ is a category with pullbacks.

Notation 5.4 Let us describe the category with null-homotopies $\underline{\mathbf{Arr}}(\mathcal{A})$.
- An object $\mathbb{A}$ is a morphism $\alpha\colon A \to A_0$ in $\mathcal{A}$.
- A morphism $F\colon \mathbb{A} \to \mathbb{B}$ is a pair (f, f_0) of morphisms in $\mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

- Given a morphism $F\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Arr}(\mathcal{A})$, the set of null-homotopies $\mathcal{N}(F)$ is the set of morphisms $\lambda\colon A_0 \to B$ in $\mathcal{A}$ such that $\alpha \cdot \lambda = f$ and $\lambda \cdot \beta = f_0$.
- Given three morphisms $F\colon \mathbb{A} \to \mathbb{B}, G\colon \mathbb{B} \to \mathbb{C}, H\colon \mathbb{C} \to \mathbb{D}$ in $\mathbf{Arr}(\mathcal{A})$, the operation $F \circ - \circ H\colon \mathcal{N}(G) \to \mathcal{N}(F \cdot G \cdot H)$ is given by

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"] \\ B_0 \arrow[r, "g_0"'] \arrow[ur, "\mu"] & C_0 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \arrow[d, "\delta"] \\ A \arrow[r, "f_0"'] \arrow[urrr, "f_0 \cdot \mu \cdot h"] & B_0 \arrow[r, "g_0"'] & C_0 \arrow[r, "h_0"'] & D_0 \end{tikzcd}$$

The first part of the next proposition is obvious (it requires that $\mathcal{A}$ is pointed). It is included in the statement just to stress the relation between snails, snakes, and kernels.

Proposition 5.5 *Let $F = (f, f_0)\colon \mathbb{A} \to \mathbb{B}$ be a morphism in* $\mathbf{Arr}(\mathcal{A})$.

1. *The construction involved in the snake sequence*

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ \mathrm{Ker}(f_0) \arrow[r, "k_{f_0}"'] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

is the kernel of F in the category $\mathbf{Arr}(\mathcal{A})$.

2. The construction involved in the snail sequence

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[d, "\gamma"'] & & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
A_0 \times_{f_0,\beta} B \arrow[rr, "\beta'"'] \arrow[urrr, "f_0'"] & & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

is the strong homotopy kernel of F in the category with null-homotopies $\underline{\mathbf{Arr}}(\mathcal{A})$.

Proof. Proof of 2. (Homotopy kernel) Consider a triple $\mathbb{D}, G\colon \mathbb{D} \to \mathbb{A}, \mu \in \mathcal{N}(G \cdot F)$ in $\underline{\mathbf{Arr}}(\mathcal{A})$

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "\delta"'] & A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\beta"] \\
D_0 \arrow[r, "g_0"'] \arrow[urr, "\mu"] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

Since $\mu \cdot \beta = g_0 \cdot f_0$, there exists a unique morphism $\overline{\mu}\colon D_0 \to A_0 \times_{f_0,\beta} B$ such that $\overline{\mu} \cdot f_0' = \mu$ and $\overline{\mu} \cdot \beta' = g_0$. We get in this way the following morphism $G'\colon \mathbb{D} \to \mathbb{K}(F)$

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "\delta"'] & A \arrow[d, "\gamma"] \\
D_0 \arrow[r, "\overline{\mu}"'] & A_0 \times_{f_0,\beta} B
\end{tikzcd}$$

Conditions $G' \cdot \mathrm{K}(F) = G$ and $G' \circ k(F) = \mu$ are satisfied : the first one amounts to $g \cdot \mathrm{id} = g$ and $\overline{\mu} \cdot \beta' = g_0$, and the second one amounts to $\overline{\mu} \cdot f_0' = \mu$. As far as the uniqueness of the factorization G' is concerned, given a morphism $H = (h, h_0)\colon \mathbb{D} \to \mathrm{Ker}(F)$, condition $H \cdot \mathrm{K}(F) = G$ means that $h = g$ and $h_0 \cdot \beta' = g_0$, and condition $H \circ k(F) = \mu$ means that $h_0 \cdot f_0' = \mu$. Therefore, $h_0 = \overline{\mu}$ and then $H = G'$.
(Strong homotopy kernel) Consider a triple $\mathbb{D}, H\colon \mathbb{D} \to \mathrm{Ker}(F), \mu \in \mathcal{N}(H \cdot \mathrm{K}(F))$ in $\underline{\mathbf{Arr}}(\mathcal{A})$

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "\delta"'] & A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[d, "\alpha"] \\
D_0 \arrow[r, "h_0"'] \arrow[urr, "\mu"] & A_0 \times_{f_0,\beta} B \arrow[r, "\beta'"'] & A_0
\end{tikzcd}$$

satisfying the condition $\mu \circ F = H \circ k(F)$, which amounts to $\mu \cdot f = h_0 \cdot f_0'$. For a null-homotopy $\lambda \in \mathcal{N}(H)$, the condition $\lambda \circ \mathrm{K}(F) = \mu$ means $\lambda \cdot \mathrm{id} = \mu$. This gives the uniqueness of the null-homotopy λ, and it remains just to check that

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "\delta"'] & A \arrow[d, "\gamma"] \\
D_0 \arrow[r, "h_0"'] \arrow[ur, "\mu"] & A_0 \times_{f_0,\beta} B
\end{tikzcd}$$

is a null-homotopy. In particular, to check that $\mu \cdot \gamma = h_0$, compose with the pullback projections (and use $\mu \cdot f = h_0 \cdot f'_0$ when composing with f'_0). ■

Remark 5.6 We can reconsider Proposition 4.3 in the light of Proposition 5.5. For a given morphism $F = (f, f_0)\colon \mathbb{A} \to \mathbb{B}$ in $\mathbf{Arr}(\mathcal{A})$, the universal property of the strong homotopy kernel gives a canonical comparison from the kernel of F in $\mathbf{Arr}(\mathcal{A})$ to the strong homotopy kernel of F in $\underline{\mathbf{Arr}}(\mathcal{A})$. Explicitly, the canonical comparison is given by

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, "\mathrm{K}(\alpha)"'] & A \arrow[d, "\gamma"] \\ \mathrm{Ker}(f_0) \arrow[r, "k_{f'_0}"'] & A_0 \times_{f_0,\beta} B \end{tikzcd}$$

Now, Proposition 4.3 says that the canonical comparison is a "weak equivalence". More precisely, the canonical comparison induces two morphisms

$$\mathrm{K}(k_f)\colon \mathrm{Ker}(\mathrm{K}(\alpha)) \to \mathrm{Ker}(\gamma) \;\text{ and }\; C(k_{f'_0})\colon \mathrm{Cok}(\mathrm{K}(\alpha)) \to \mathrm{Cok}(\gamma)$$

and we have that
- $\mathrm{K}(k_f)$ always is an isomorphism for a general argument of interchange of limits,
- if γ and $\mathrm{K}(\alpha)$ are proper and if f is a regular epimorphism, then $C(k_{f'_0})$ also is an isomorphism.

6. The complete snail lemma

From [3], Lemma 4.5.1, recall the following fact (which defines the homology of a complex).

Proposition 6.1 *Let $\mathcal{A}$ be a pointed regular protomodular category. Consider the following commutative diagram in $\mathcal{A}$, with $a \cdot b = 0$ and $a\colon X \to Y$ proper.*

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "a'"'] & & Y \arrow[rr, "b"] \arrow[dr, "c_a"'] & & Z \\ & \mathrm{Ker}(b) \arrow[ur, "k_b"'] \arrow[d, "c_{a'}"'] & & \mathrm{Cok}(a) \arrow[ur, "b'"'] & \\ & \mathrm{Cok}(a') & & \mathrm{Ker}(b') \arrow[u, "k_{b'}"'] & \end{tikzcd}$$

There exists a unique morphism $i\colon \mathrm{Cok}(a') \to \mathrm{Ker}(b')$ such that $c_{a'} \cdot i \cdot k_{b'} = k_b \cdot c_a$. Moreover, the morphism i is a isomorphism.

In the rest of this section $\mathcal{A}$ is abelian. Consider a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

together with its factorization $\langle\alpha, f\rangle$ through the pullback $A_0 \times_{f_0,\beta} B$ and its factorization $[f_0, \beta]$ through the pushout $A_0 +_{\alpha,f} B$ as in the following diagram

$$\begin{tikzcd} A \arrow[rrrr, "f"] \arrow[dr, "\langle\alpha,f\rangle"] \arrow[ddd, "\alpha"'] & & & & B \arrow[ddd, "\beta"] \arrow[ddl, "\alpha'"'] \\ & A_0 \times_{f_0,\beta} B \arrow[urrr, "f_0'"] \arrow[d, "c_{\langle\alpha,f\rangle}"] \arrow[ddl, "\beta'"'] & & \mathrm{Ker}[f_0,\beta] \arrow[d, "k_{[f_0,\beta]}"'] & \\ & \mathrm{Cok}\langle\alpha, f\rangle & & A_0 +_{\alpha,f} B \arrow[dr, "{[f_0,\beta]}"'] & \\ A_0 \arrow[urrr, "f'"'] \arrow[rrrr, "f_0"'] & & & & B_0 \end{tikzcd}$$

Lemma 6.2 *(With the previous notation.) There exists a unique morphism*

$$i\colon \mathrm{Cok}\langle\alpha, f\rangle \to \mathrm{Ker}[f_0, \beta]$$

such that $c_{\langle\alpha,f\rangle} \cdot i \cdot k_{[f_0,\beta]} = \beta' \cdot f' - f_0' \cdot \alpha'$. *Moreover, the morphism i is an isomorphism.*

Proof. Apply Proposition 6.1 to the complex

$$A \xrightarrow{\langle\alpha:f\rangle} A_0 \times B \simeq A_0 + B \xrightarrow{[f_0:\beta]} B$$

where $\langle\alpha : f\rangle$ and $[f_0 : \beta]$ are the factorizations of the pairs (α, f) and (f_0, β) through the product $A_0 \times B$ and the coproduct $A_0 + B$, and the unlabelled isomorphism is given by

$$\begin{pmatrix} \mathrm{id}_{A_0} & 0 \\ 0 & -\mathrm{id}_B \end{pmatrix} : A_0 + B \to A_0 \times B$$

■

Corollary 6.3 *Consider a morphism* $F = (f, f_0)\colon \mathbb{A} \to \mathbb{B}$ *in* $\mathbf{Arr}(\mathcal{A})$ *and construct the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\langle\alpha,f\rangle"'] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "\alpha'"] \arrow[d, "\beta"] & A_0 +_{\alpha,f} B \arrow[d, "{[f_0,\beta]}"] \\ A_0 \times_{f_0,\beta} B \arrow[r, "\beta'"'] & A_0 \arrow[r, "f_0"'] & B_0 \arrow[r, "\mathrm{id}"'] & B_0 \end{tikzcd}$$

1. *The left-hand part is the strong homotopy kernel of F in* $\underline{\mathbf{Arr}}(\mathcal{A})$.
2. *The right-hand part is the strong homotopy cokernel of F in* $\underline{\mathbf{Arr}}(\mathcal{A})$.
3. *There is an exact sequence*

$$\mathrm{Ker}\langle\alpha, f\rangle \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to H(F) \to \mathrm{Cok}(\alpha) \to \mathrm{Cok}(\beta) \to \mathrm{Cok}[f_0, \beta]$$

where $H(F)$, the homology of F, stands for the object $\mathrm{Ker}[f_0, \beta] \simeq \mathrm{Cok}\langle\alpha, f\rangle$.

Proof. Point 1 is just Proposition 5.5, and point 2 follows from point 1 by duality. As far as point 3 is concerned, from the snail lemma and its dual we get two exact sequences that we can past together thanks to Lemma 6.2. ∎

Remark 6.4

1. Since the category $\mathcal{A}$ is abelian, $\underline{\mathbf{Arr}}(\mathcal{A})$ is a 2-category, and it is easy to adapt the proof of Proposition 5.5 to show that the strong homotopy kernel of F is also a bikernel of F in the sense of [1]. Dually, the strong homotopy cokernel of F is also a bicokernel.
2. Moreover, $\underline{\mathbf{Arr}}(\mathcal{A})$ is biequivalent to the 2-category $\underline{\mathbf{Grpd}}(\mathcal{A})$ of internal groupoids, internal functors and internal natural transformations in $\mathcal{A}$. From this point of view, the exact sequence of Corollary 6.3 is a kind of π_1-π_0-sequence associated to F, seen as an internal functor. This is the approach adopted in [10], where non protomodular versions of the snail and of the snake lemma are discussed, in order to get an internalization of the classical exact sequence associated to a fibration of groupoids due to R. Brown (see [5], or [6] for the non fibrational version).

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