COALGEBRAS, BRAIDINGS, AND DISTRIBUTIVE LAWS

To Aurelio Carboni on his 60th birthday

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ABSTRACT. We show, for a monad $\mathbb{T}$, that coalgebra structures on a $\mathbb{T}$-algebra can be described in terms of "braidings", provided that the monad is equipped with an invertible distributive law satisfying the Yang-Baxter equation.

1. Introduction

The aim of this note is to provide an equivalent description of $\mathbb{T}^*$-coalgebra structures on a $\mathbb{T}$-algebra, for $\mathbb{T}$ a monad — equipped with a special kind of distributive law — on a category $\mathbb{C}$, and $\mathbb{T}^*$ the comonad induced by the adjunction

$$\mathbb{C} \underset{R^{\mathbb{T}}}{\overset{L^{\mathbb{T}}}{\rightleftarrows}} \mathrm{Alg}(\mathbb{T}) \qquad L^{\mathbb{T}} \dashv R^{\mathbb{T}}$$

Our interest for such coalgebras is motivated mainly by classical descent theory: let $f\colon R \to S$ be a morphism of commutative unital rings, and consider the induced functor

$$f!\colon R\text{-mod} \to S\text{-mod}$$

defined by $f!(N) = N \otimes_R S$ (where S is seen as an R-module by restriction of scalars). The descent problem for f consists in recognizing when an S-module is of the form $f!(N)$ for some R-module N. A classical theorem [6, 2, 7], which establishes a deep link between descent theory and the theory of (co)monads, asserts that, if f is faithfully flat, then an S-module is of the form $f!(N)$ if and only if it is equipped with a $\mathbb{T}^*$-coalgebra structure, where $\mathbb{T}^*$ is the comonad on S-mod induced by the adjunction

$$R\text{-mod} \underset{f^*}{\overset{f!}{\rightleftarrows}} S\text{-mod} \qquad f! \dashv f^*$$

and f^* is the restriction of scalars functor. (For this reason, a $\mathbb{T}^*$-coalgebra structure on an S-module M is sometimes called a *descent datum* for M.) It is also well-known that, for any morphism $f\colon R \to S$ of commutative unital rings, there are several equivalent ways of describing what a $\mathbb{T}^*$-coalgebra structure for an S-module is, and a natural problem is to lift to a categorical level these other descriptions of $\mathbb{T}^*$-coalgebra structures.

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In a recent paper [11], Menini and Stefan, extending results by Nuss [12] on non-commutative rings, replace the situation

$$R\text{-mod} \underset{f^*}{\overset{f!}{\rightleftarrows}} S\text{-mod} \qquad f! \dashv f^*$$

by

$$\mathbb{C} \underset{R^{\mathbb{T}}}{\overset{L^{\mathbb{T}}}{\rightleftarrows}} \mathrm{Alg}(\mathbb{T}) \qquad L^{\mathbb{T}} \dashv R^{\mathbb{T}}$$

where $\mathbb{T}$ is a monad on an arbitrary category $\mathbb{C}$ (indeed, even in the non-commutative situation, $f^*\colon S\text{-mod} \to R\text{-mod}$ is a monadic functor, so that S-mod is equivalent to the category of algebras for the monad on R-mod induced by the adjunction $f! \dashv f^*$). In this context, they prove that, if the monad $\mathbb{T}$ is equipped with a "compatible flip" $K\colon T^2 \Rightarrow T^2$, then to give a $\mathbb{T}^*$-coalgebra structure on a $\mathbb{T}$-algebra X is equivalent to giving a "symmetry" on X, that is an involution $TX \to TX$ satisfying some suitable conditions.

Unfortunately, the following natural example, which is a direct generalization of the classical case of commutative rings, does not fit into their general context: let $\mathbb{C}$ be a braided monoidal category and let S be a monoid in $\mathbb{C}$, then the braiding $c_{S,S}\colon S \otimes S \to S \otimes S$ induces a natural isomorphism $K\colon T^2 \Rightarrow T^2$ on the monad $T = -\otimes S\colon \mathbb{C} \to \mathbb{C}$, but this natural isomorphism is not a flip unless the braiding is a symmetry and the monoid is commutative. In this note we adapt the notions of "compatible flip" and "symmetry" to encompass the previous example, as well as another example coming from the theory of bialgebras.

In Section 2 we introduce the notion of *BD-law* on a monad $\mathbb{T}$ as a special case of distributive law in the sense of Beck [1]. Using a BD-law K, we can define *K-braidings* on a $\mathbb{T}$-algebra, and we want to show that K-braidings correspond bijectively to $\mathbb{T}^*$-coalgebra structures. There are two different methods: in Section 3 we use K-braidings to define a category $\mathrm{Brd}(\mathbb{T}, K)$ equipped with a forgetful functor $V\colon \mathrm{Brd}(\mathbb{T}, K) \to \mathrm{Alg}(\mathbb{T})$. We show, using the Beck criterion [10], that V is comonadic; and that the corresponding comonad is $\mathbb{T}^*$, so that $\mathrm{Brd}(\mathbb{T}, K)$ is isomorphic to $\mathrm{Coalg}(\mathbb{T}^*)$. In Section 4 we give a different proof based as far as possible on general facts about monads and distributive laws. This second proof is quite long, but it seems to us of some interest, since it shows that the bijection between K-braidings and $\mathbb{T}^*$-coalgebra structures is the natural bijection induced by a pair of adjoint functors. The description of K-braidings we obtain in this way is slightly different, but in fact equivalent, to that used in Section 3.

2. BD-laws on a monad

To begin, we fix notation. A monad $\mathbb{T}$ on a category $\mathbb{C}$ is a triple

$$\mathbb{T} = (T\colon \mathbb{C} \to \mathbb{C}, m\colon T^2 \Rightarrow T, e\colon Id_{\mathbb{C}} \Rightarrow T)$$

consisting of a functor T, and natural transformations m and e making the diagrams

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "eT"] \arrow[dr, Rightarrow, "T"'] & T^2 \arrow[d, Rightarrow, "m"] \arrow[dl, phantom, "(1)", near start] \arrow[dr, phantom, "(2)", near start] & T \arrow[l, Rightarrow, "Te"'] \arrow[dl, Rightarrow, "T"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "Tm"] \arrow[d, Rightarrow, "mT"'] \arrow[dr, phantom, "(3)"] & T^2 \arrow[d, Rightarrow, "m"] \\
T^2 \arrow[r, Rightarrow, "m"'] & T
\end{tikzcd}$$

commute. A $\mathbb{T}$-algebra is a pair $(X, x\colon TX \to X)$ in $\mathbb{C}$ such that the diagrams

$$\begin{tikzcd}
X \arrow[r, "eX"] \arrow[dr, "1"'] & TX \arrow[d, "x"] \arrow[dl, phantom, "(4)", near start] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2X \arrow[r, "Tx"] \arrow[d, "mX"'] \arrow[dr, phantom, "(5)"] & TX \arrow[d, "x"] \\
TX \arrow[r, "x"'] & X
\end{tikzcd}$$

commute. Given two monads $\mathbb{T}$ and $\mathbb{S}$ on the same category $\mathbb{C}$, a distributive law of $\mathbb{T}$ over $\mathbb{S}$ is a natural transformation $K\colon TS \Rightarrow ST$ such that the diagrams

$$\begin{tikzcd}
T \arrow[rr, Rightarrow, "Te"] \arrow[dr, Rightarrow, "eT"'] & \arrow[d, phantom, "(6)"] & TS \arrow[dl, Rightarrow, "K"] \\
& ST &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[rr, Rightarrow, "eS"] \arrow[dr, Rightarrow, "Se"'] & \arrow[d, phantom, "(7)"] & TS \arrow[dl, Rightarrow, "K"] \\
& ST &
\end{tikzcd}$$

$$\begin{tikzcd}
TS^2 \arrow[r, Rightarrow, "KS"] \arrow[d, Rightarrow, "Tm"'] & STS \arrow[r, Rightarrow, "SK"] \arrow[d, phantom, "(8)"] & S^2T \arrow[d, Rightarrow, "mT"] \\
TS \arrow[rr, Rightarrow, "K"'] & & ST
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2S \arrow[r, Rightarrow, "TK"] \arrow[d, Rightarrow, "mS"'] & TST \arrow[r, Rightarrow, "KT"] \arrow[d, phantom, "(9)"] & ST^2 \arrow[d, Rightarrow, "Sm"] \\
TS \arrow[rr, Rightarrow, "K"'] & & ST
\end{tikzcd}$$

commute.

We refer to [1, 3] for more details on monads and distributive laws. When the natural transformation K is an isomorphism, the definition of distributive law can be simplified, as in the following lemma:

Lemma 1 *Consider two monads $\mathbb{T}$ and $\mathbb{S}$ on a category $\mathbb{C}$, and a natural isomorphism $K\colon TS \Rightarrow ST$. Then* (8) *implies* (6) *and* (9) *implies* (7)*. Moreover, K satisfies* (8) *and* (9) *iff K^{-1} does.*

Proof. We prove that (9) implies (7); the proof of the other implication is similar, and the rest of the statement is obvious. The proof is contained in the following diagram, in which unlabelled regions commute by naturality:

$$\begin{tikzcd}
S \arrow[rr, Rightarrow, "eS"] \arrow[dr, Rightarrow, "eS"] \arrow[dd, Rightarrow, "Se"'] & & TS \arrow[rr, Rightarrow, "K"] \arrow[d, phantom, "(2)"] & & ST \\
& TS \arrow[r, Rightarrow, "K"] \arrow[d, Rightarrow, "TSe"'] & ST \arrow[r, Rightarrow, "STe"] & ST^2 \arrow[ur, Rightarrow, "Sm"] & \\
ST \arrow[r, Rightarrow, "eST"'] \arrow[d, Rightarrow, "K^{-1}"'] & TST \arrow[r, Rightarrow, "TK^{-1}"] & T^2S \arrow[r, Rightarrow, "TK"] \arrow[drr, Rightarrow, "mS"] \arrow[d, phantom, "(1)"] & TST \arrow[u, Rightarrow, "KT"] \arrow[r, phantom, "(9)"] & \\
TS \arrow[urr, Rightarrow, "eTS"'] \arrow[rrrr, Rightarrow, "TS"'] & & & & TS. \arrow[uuu, Rightarrow, "K"']
\end{tikzcd}$$

■

Definition 2 Let $\mathbb{T}$ be a monad on a category $\mathbb{C}$. A *BD-law* on $\mathbb{T}$ is a natural transformation $K\colon T^2 \Rightarrow T^2$ such that

(B) K satisfies the Yang-Baxter equation:

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "KT"] \arrow[d, Rightarrow, "TK"'] & T^3 \arrow[r, Rightarrow, "TK"] \arrow[d, phantom, "(10)"] & T^3 \arrow[d, Rightarrow, "KT"] \\
T^3 \arrow[r, Rightarrow, "KT"'] & T^3 \arrow[r, Rightarrow, "TK"'] & T^3
\end{tikzcd}$$

(D) K is a distributive law (that is, it satisfies equations (6–9)).

A *BCD-law* on $\mathbb{T}$ is a BD-law such that

(C) the monad $\mathbb{T}$ is K-commutative:

$$\begin{tikzcd}
T^2 \arrow[rr, Rightarrow, "K"] \arrow[dr, Rightarrow, "m"'] & (11) & T^2 \arrow[dl, Rightarrow, "m"] \\
& T &
\end{tikzcd}$$

A BD-law or BCD-law is said to be *invertible* if the natural transformation K is so.

Remark 3 Once again, as stated in Lemma 1 for the distributivity conditions, a natural isomorphism $K\colon T^2 \Rightarrow T^2$ satisfies conditions (B) or (C) iff K^{-1} does.

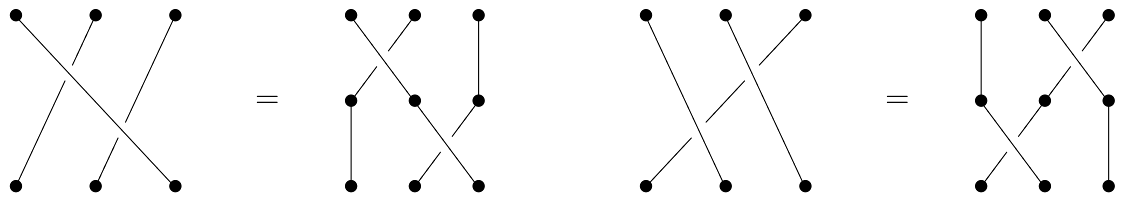
Remark 4 If $\mathbb{C}$ is an arbitrary monoidal category, and $\mathbb{T}$ is a monoid in $\mathbb{C}$, one can define a BD-law or BCD-law on $\mathbb{T}$ as above. A monoid equipped with an invertible BCD-law has been called a *quasi-commutative monoid* by Davydov [5], since the BCD-law provides a kind of "local braiding" with respect to which the monoid is commutative.

Remark 5 If K is involutive — that is, $K^2 = 1$ — each of the conditions (8) and (9) implies the other. This fact, together with Lemma 1, means that compatible flips in the sense of Menini and Stefan [11] are precisely the involutive BCD-laws.

Example 6 Let $\mathbb{C} = (\mathbb{C}, \otimes, I, \ldots)$ be a monoidal category and $S = (S, m_S, e_S)$ a monoid in $\mathbb{C}$. The monoid S induces a monad $\mathbb{T}$ on $\mathbb{C}$ in the following way:

- $T = - \otimes S \colon \mathbb{C} \to \mathbb{C}$
- $mX = 1 \otimes m_S \colon X \otimes S \otimes S \to X \otimes S$
- $eX = 1 \otimes e_S \colon X \simeq X \otimes I \to X \otimes S$

2.1. If $\mathbb{C}$ is braided, with braiding $c = \{c_{X,Y} \colon X \otimes Y \to Y \otimes X\}$, then there is an invertible BD-law K on $\mathbb{T}$ defined by $KX = 1 \otimes c_{S,S} \colon X \otimes S \otimes S \to X \otimes S \otimes S$. In this case, condition (B) is precisely the Yang-Baxter equation; by naturality of the braiding, conditions (8) and (9) reduce to the following equations, which hold in any braided monoidal category:



This BD-law is a BCD-law precisely when the monoid S is commutative; it is involutive if and only if $c_{S,S}$ is so, in particular if the braiding c is a symmetry.

2.2. Let $\mathcal{K}$ be a field and take as $\mathbb{C}$ the category of $\mathcal{K}$-vector spaces. Let H be a cobraided bialgebra with universal form $r \colon H \otimes H \to I$; there is a natural isomorphism of H-comodules $c^r_{V,W} \colon V \otimes W \to W \otimes V$ defined by

$$V \otimes W \xrightarrow{\tau} W \otimes V \xrightarrow{\Delta \otimes \Delta} H \otimes W \otimes H \otimes V \xrightarrow{1 \otimes \tau \otimes 1} H \otimes H \otimes W \otimes V \xrightarrow{r \otimes 1 \otimes 1} W \otimes V$$

where Δ is the coaction and τ is the standard twist: see [9]. If S is any H-comodule algebra (in particular, one can take $S = H$), then we have an invertible BD-law K on $\mathbb{T}$ defined by $KX = 1 \otimes c^r_{S,S} \colon X \otimes S \otimes S \to X \otimes S \otimes S$. Indeed, conditions (8-10) follow from [9, Proposition VIII.5.2], using the fact that the multiplication $m_S \colon S \otimes S \to S$ is a homomorphism of H-comodules.

Example 7 If $f \colon R \to S$ is a morphism of unital rings, with R commutative, we can specialize Example 2.1 by taking $\mathbb{C} = R$-mod, so that K is defined by the standard twist $S \otimes S \to S \otimes S$. This is possible also if R is not commutative, provided its image lies in the centre of S : taking now $\mathbb{C} = R$-R-bimod, the standard twist on S can be defined and gives once again an invertible BD-law on $\mathbb{T}$. If we drop the centrality condition the standard twist can no longer be defined, but one can use the additivity of the category of bimodules to define a different BD-law. In fact, if $\mathbb{C}$ is an additive category and $\mathbb{T}$ is any monad on $\mathbb{C}$, then there is an involutive BCD-law K on $\mathbb{T}$ defined by $K = (eT + Te) \cdot m - T^2$; see [11]. This case generalizes results on non-commutative rings established in [4, 12].

3. Coalgebras and braidings

For the reader's convenience, let us recall how the definition of coalgebra for a comonad specializes when the comonad is of the form $\mathbb{T}^*$.

Definition 8 Let $\mathbb{T} = (T, m, e)$ be a monad on a category $\mathbb{C}$ and consider the comonad $\mathbb{T}^*$ on $\mathrm{Alg}(\mathbb{T})$ induced by the adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "L^{\mathbb{T}}"] & \mathrm{Alg}(\mathbb{T}) \arrow[l, shift left, "R^{\mathbb{T}}"] \end{tikzcd} \qquad L^{\mathbb{T}} \dashv R^{\mathbb{T}}$$

A $\mathbb{T}^*$-*coalgebra structure* on a $\mathbb{T}$-algebra $(X, x\colon TX \to X)$ is a morphism $r\colon X \to TX$ such that

$$\begin{tikzcd} TX \arrow[r, "Tr"] \arrow[d, "x"'] \arrow[dr, phantom, "(12)"] & T^2X \arrow[d, "mX"] \\ X \arrow[r, "r"'] & TX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "1"'] & TX \arrow[d, "x"] \arrow[dl, phantom, "(13)"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] \arrow[dr, phantom, "(14)"] & TX \arrow[d, "Tr"] \\ TX \arrow[r, "TeX"'] & T^2X \end{tikzcd}$$

We denote by $\mathbb{T}^*\text{-coalg}(X, x)$ the set of $\mathbb{T}^*$-coalgebra structures on a $\mathbb{T}$-algebra (X, x).

Remark 9 For all $X \in \mathbb{C}$, the morphism $TeX\colon TX \to T^2X$ is a $\mathbb{T}^*$-coalgebra structure on the free $\mathbb{T}$-algebra $L^{\mathbb{T}}X = (TX, mX)$. This is the (object part of the) canonical comparison functor $\mathbb{C} \to \mathrm{Coalg}(\mathbb{T}^*)$.

Definition 10 Let $\mathbb{T} = (T, m, e)$ be a monad on a category $\mathbb{C}$ and let $K\colon T^2 \Rightarrow T^2$ be a BD-law. A K-*braiding* on a $\mathbb{T}$-algebra $(X, x\colon TX \to X)$ is a morphism $c\colon TX \to TX$ such that

$$\begin{tikzcd} TX \arrow[r, "c"] \arrow[dr, phantom, "(15)"] & TX \arrow[d, "x"] \\ X \arrow[u, "eX"] \arrow[r, "1"'] & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "KX"] \arrow[d, "Tc"'] & T^2X \arrow[r, "Tc"] \arrow[d, phantom, "(16)"] & T^2X \arrow[d, "KX"] \\ T^2X \arrow[r, "KX"'] & T^2X \arrow[r, "Tc"'] & T^2X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "KX"] \arrow[d, "Tx"'] & T^2X \arrow[r, "Tc"] \arrow[d, phantom, "(17)"] & T^2X \arrow[d, "mX"] \\ TX \arrow[rr, "c"'] & & TX \end{tikzcd}$$

We denote by $K\text{-Brd}(X, x)$ the set of K-braidings on a $\mathbb{T}$-algebra (X, x).

Remark 11 If K is invertible, condition (17) means that c is a morphism

$$c\colon \overline{T}(X, x) \to T^*(X, x)$$

in $\mathrm{Alg}(\mathbb{T})$, where $\overline{\mathbb{T}}$ is the lifting of $\mathbb{T}$ on $\mathrm{Alg}(\mathbb{T})$ induced by the distributive law K^{-1}.

Remark 12 We shall see in Proposition 16 that if K is invertible or involutive then the same is true of any K-braiding, whence by Corollary 18 it will follow that if K is an involutive BCD-law, then K-braidings are precisely symmetries in the sense of Menini and Stefan [11].

Remark 13 If K is a BD-law on $\mathbb{T}$, then KX is a K-braiding on $L^{\mathbb{T}}X$, for all $X \in \mathbb{C}$. Indeed, conditions (16) and (17) correspond respectively to conditions (10) and (9), while condition (15) is the pasting of (1) and (6). In the bijection stated in Corollary 15, KX corresponds to the $\mathbb{T}^*$-coalgebra structure TeX of Remark 9.

If $\mathbb{T}$ is a monad on a category $\mathbb{C}$ and $K\colon T^2 \Rightarrow T^2$ is a BD-law, we write $\mathrm{Brd}(\mathbb{T}, K)$ for the category having pairs

$$\langle\, (X,x) \in \mathrm{Alg}(\mathbb{T}), c \in K\text{-}\mathrm{Brd}(X,x) \,\rangle$$

as objects. An arrow $f\colon \langle\, (X,x), c\,\rangle \to \langle\, (X',x'), c'\,\rangle$ in $\mathrm{Brd}(\mathbb{T},K)$ is an arrow between the underlying $\mathbb{T}$-algebras such that the diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "c"'] & TX' \arrow[d, "c'"] \\
TX \arrow[r, "Tf"'] & TX'
\end{tikzcd}$$

commutes.

We are ready to state our main result.

Theorem 14 *Let $\mathbb{T} = (T, m, e)$ be a monad on a category $\mathbb{C}$ and let $K\colon T^2 \Rightarrow T^2$ be a BD-law on $\mathbb{T}$. The forgetful functor*

$$V\colon \mathrm{Brd}(\mathbb{T}, K) \to \mathrm{Alg}(\mathbb{T})$$

is comonadic, and the corresponding comonad on $\mathrm{Alg}(\mathbb{T})$ *is the comonad* $\mathbb{T}^*$ *induced by the adjunction* $L^{\mathbb{T}} \dashv R^{\mathbb{T}}$ *between* $\mathbb{C}$ *and* $\mathrm{Alg}(\mathbb{T})$.

Proof. We show that V has a left adjoint and then apply Beck's theorem. The free T-algebra functor $L^{\mathbb{T}} : \mathbb{C} \to \mathrm{Alg}(\mathbb{T})$ factorizes as $L^{\mathbb{T}} = VJ$, where $J : \mathbb{C} \to \mathrm{Brd}(\mathbb{T}, K)$ is the functor sending an object X of $\mathbb{C}$ to the algebra (TX, mX) equipped with the K-braiding $KX : T^2X \to T^2X$. The counit $\epsilon : L^{\mathbb{T}}R^{\mathbb{T}} \to 1$ may be seen as a natural transformation $VJR^{\mathbb{T}} = L^{\mathbb{T}}R^{\mathbb{T}} \to 1$. For an object (X, x, c) of $\mathrm{Brd}(\mathbb{T}, K)$, write $\beta : X \to TX$ for the composite

$$X \xrightarrow{eX} TX \xrightarrow{c} TX$$

and observe that, by commutativity of

$$\begin{tikzcd}
TX \arrow[r, "eTX"'] \arrow[rr, bend left, "TeX"] \arrow[d, "x"'] & T^2X \arrow[r, "KX"'] \arrow[d, "Tx"'] & T^2X \arrow[r, "Tc"] \arrow[d, phantom, "(17)"] & T^2X \arrow[d, "mX"] \\
X \arrow[r, "eX"'] & TX \arrow[rr, "c"'] & & TX
\end{tikzcd}$$

and

$$\begin{tikzcd}
TX \arrow[r, "eTX"'] \arrow[rr, bend left, "TeX"] \arrow[d, "c"'] & T^2X \arrow[r, "KX"'] \arrow[d, "Tc"'] & T^2X \arrow[r, "Tc"] \arrow[d, phantom, "(16)"] & T^2X \arrow[d, "KX"] \\
TX \arrow[r, "eTX"] \arrow[rr, bend right, "TeX"'] & T^2X \arrow[r, "KX"] & T^2X \arrow[r, "Tc"'] & T^2X
\end{tikzcd}$$

this makes β a map in $\mathrm{Brd}(\mathbb{T}, K)$ from (X, x, c) to (TX, mX, KX). This gives the component at (X, x, c) of a natural transformation $\beta : 1 \to JR^{\mathbb{T}}V$.

The triangle equation $\epsilon V.V\beta = 1$ is precisely equation (15), while the other triangle equation $JR^{\mathbb{T}}\epsilon.\beta JR^{\mathbb{T}} = 1$ follows easily from the definitions of BD-law and of $\mathbb{T}$-algebra. Thus there is an adjunction $V \dashv JR^{\mathbb{T}}$, which clearly induces the same comonad as $L^{\mathbb{T}} \dashv R^{\mathbb{T}}$.

It remains to verify the Beck condition. Let $f, g : (X, x, c) \to (Z, z, c')$ be morphisms in $\mathrm{Brd}(\mathbb{T}, K)$, and let

$$\begin{tikzcd}
(W, w) \arrow[r, "i"] & (X, x) \arrow[l, bend left, "u"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Z, z) \arrow[l, bend left=60, "v"]
\end{tikzcd}$$

be a split equalizer diagram in $\mathrm{Alg}(\mathbb{T})$, with $ui = 1$, $iu = gv$, and $fv = 1$. Since Ti is (split) monic, the only possibility for a K-braiding $c'' : TW \to TW$ on (W, w) compatible with i is given by

$$TW \xrightarrow{Ti} TX \xrightarrow{c} TX \xrightarrow{Tu} TW$$

and we need only check that this does indeed give a braiding; the fact that the resulting diagram is an equalizer in $\mathrm{Brd}(\mathbb{T}, K)$ is then obvious. Thus we must check equations (15,16,17); instead, we allow the reader to contemplate the following diagrams at his or her leisure:

$$\begin{tikzcd}
TW \arrow[r, "Ti"] & TX \arrow[r, "c"] & TX \arrow[r, "Tu"] \arrow[d, "x"] & TW \arrow[d, "w"] \\
W \arrow[u, "eW"] \arrow[r, "i"] \arrow[rrr, bend right, "1"'] & X \arrow[u, "eX"'] \arrow[r, "1"] & X \arrow[r, "u"] & W
\end{tikzcd}$$

$$\begin{tikzcd}
 & T^2X \arrow[rr, "T^2u"] \arrow[dr, "T^2g"] & & T^2W \arrow[r, "KW"] \arrow[dd, "T^2i"] & T^2W \arrow[dd, "T^2i"] \\
T^2X \arrow[ur, "Tc"] \arrow[r, "T^2g"'] & T^2Z \arrow[r, "Tc'"'] & T^2Z \arrow[dr, "T^2v"] & & \\
T^2W \arrow[u, "T^2i"] \arrow[r, "T^2i"] \arrow[d, "KW"'] & T^2X \arrow[u, "T^2f"] \arrow[r, "Tc"'] \arrow[d, "KX"] & T^2X \arrow[u, "T^2f"] \arrow[r, "1"'] & T^2X \arrow[r, "KX"'] & T^2X \arrow[d, "Tc"] \\
T^2W \arrow[r, "T^2i"'] & T^2X \arrow[rr, "Tc"'] & & T^2X \arrow[r, "KX"] \arrow[d, "T^2u"'] & T^2X \arrow[d, "T^2u"] \\
 & & & T^2W \arrow[r, "KW"'] & T^2W
\end{tikzcd}$$

$$\begin{tikzcd}
 & T^2W \arrow[r, "T^2i"] & T^2X \arrow[dr, "Tc"] & & \\
T^2W \arrow[ur, "KW"] \arrow[r, "T^2i"'] \arrow[d, "Tw"'] & T^2X \arrow[ur, "KX"] \arrow[d, "Tx"'] & & T^2X \arrow[r, "T^2u"] \arrow[d, "mX"] & T^2W \arrow[d, "mW"] \\
TW \arrow[r, "Ti"'] & TX \arrow[rr, "Tc"'] & & TX \arrow[r, "Tu"'] & TW.
\end{tikzcd}$$

■

Corollary 15 *Let $\mathbb{T} = (T, m, e)$ be a monad on a category $\mathbb{C}$ and let $K\colon T^2 \Rightarrow T^2$ be a BD-law on $\mathbb{T}$. For (X, x) a $\mathbb{T}$-algebra, consider the map*

$$\Psi_{(X,x)}\colon K\text{-Brd}(X, x) \to \mathbb{T}^*\text{-coalg}(X, x)$$

$$(c\colon TX \to TX) \mapsto (r\colon\ X \xrightarrow{eX} TX \xrightarrow{c} TX\ )$$

and the functor

$$\Psi\colon \text{Brd}(\mathbb{T}, K) \to \text{Coalg}(\mathbb{T}^*)$$

$$\langle (X, x), c\rangle \xrightarrow{f} \langle (X', x'), c'\rangle \ \mapsto\ (X, x, \Psi(c)) \xrightarrow{f} (X', x', \Psi(c'))$$

1. *The functor Ψ is an isomorphism of categories;*
2. *The map $\Psi_{(X,x)}$ is bijective.*

Proof. Since the forgetful functor $V\colon \text{Brd}(\mathbb{T}, K) \to \text{Alg}(\mathbb{T})$ is comonadic, the induced comparison functor $\Psi\colon \text{Brd}(\mathbb{T}, K) \to \text{Coalg}(\mathbb{T}^*)$ is an isomorphism of categories and so it induces a bijection on objects. ■

Proposition 16 *For a BD-law $K : T^2 \to T^2$ we have the following facts about a K-braiding $c : TX \to TX$ on an algebra (X, x):*

1. *The diagrams*

$$\begin{tikzcd} T^2X \arrow[r, "Tc"] \arrow[d, "mX"'] & T^2X \arrow[r, "KX"] \arrow[d, phantom, "(18)"] & T^2X \arrow[d, "Tx"] \\ TX \arrow[rr, "c"'] & & TX \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "TKX"] \arrow[d, "T^2x"'] & T^3X \arrow[r, "T^2c"] & T^3X \arrow[d, "TmX"] \\ T^2X \arrow[d, "mX"'] & (19) & T^2X \arrow[d, "KX"] \\ TX \arrow[r, "c"'] & TX & T^2X \arrow[l, "Tx"] \end{tikzcd}$$

commute;

2. *If K is invertible then so is c;*

3. *If K is involutive then so is c;*

4. *If K is a BCD-law then the diagram*

$$\begin{tikzcd} TX \arrow[rr, "c"] \arrow[dr, "x"'] & (20) & TX \arrow[dl, "x"] \\ & X & \end{tikzcd}$$

commutes.

Proof. In each case the proof goes as follows. Modify the definition of K-braiding so that the extra condition is assumed part of the structure, then check that the modified category $\mathrm{Brd}(\mathbb{T}, K)'$ is still comonadic via the same comonad. This involves (i) proving that the cofree objects (TX, mX, KX) have the required property, and (ii) proving that in the split equalizer diagram in the proof of Theorem 14, the induced morphism $c'' = Tu.c.Ti : TW \to TW$ satisfies the condition if $c : TX \to TX$ does so. In each case (i) is entirely straightforward: for example, the fact that the cofree objects satisfy (18) is precisely (9). We therefore check only (ii).

1. If c satisfies (18), then so does c'' by commutativity of the diagram

$$\begin{tikzcd} & & T^2X \arrow[r, "T^2u"] \arrow[dr, "KX"] & T^2W \arrow[dr, "KW"] & \\ T^2W \arrow[r, "T^2i"] \arrow[d, "mW"'] & T^2X \arrow[ur, "Tc"] \arrow[d, "mX"'] & & T^2X \arrow[r, "T^2u"'] \arrow[d, "Tx"'] & T^2W \arrow[d, "Tw"] \\ TW \arrow[r, "Ti"'] & TX \arrow[rr, "c"'] & & TX \arrow[r, "Tu"'] & TW \end{tikzcd}$$

while (19) is obtained by pasting together (17) and (18).

2 and 3. If c is invertible, then a straightforward calculation shows that $Tu.c^{-1}.Ti$ is inverse to $Tu.c.Ti$.

4. If $xc = x$ then the diagram

$$\begin{tikzcd}
& TX \arrow[rr, "c"] \arrow[dr, "x"] & & TX \arrow[dl, "x"'] \arrow[dr, "Tu"] & \\
TW \arrow[ur, "Ti"] \arrow[dr, "w"'] & & X \arrow[dr, "u"] & & TW \arrow[dl, "w"] \\
& W \arrow[ur, "i"] \arrow[rr, "1"'] & & W &
\end{tikzcd}$$

commutes and so $wc'' = w$. ∎

Remark 17 If K is a distributive law, condition (18) means that c is a morphism

$$c\colon T^*(X,x) \to \widetilde{T}(X,x)$$

in $\mathrm{Alg}(\mathbb{T})$, where $\widetilde{\mathbb{T}}$ is the lifting of $\mathbb{T}$ to $\mathrm{Alg}(\mathbb{T})$ induced by the distributive law K.

In fact we can use the Proposition to give two alternative formulations of the definition. One of them will be used in the following section, the other to make the connection with the "symmetries" of [11].

Corollary 18 *In the definition of K-braiding, condition (17) can be replaced by (19), while if K is a BCD-law then (15) can be replaced by (20).*

Proof. We have seen that for a K-braiding (19) always holds; conversely (17) follows easily from (19) by composing with $eTX : TX \to T^2X$. Similarly, if K is a BCD-law then (20) holds for any K-braiding, while (15) follows from (20) by composing with $eX : X \to TX$. ∎

If K is an involutive BCD-law on a monad $\mathbb{T}$, then a symmetry on a $\mathbb{T}$-algebra (X, x), was defined in [11] to be an involution $c : TX \to TX$ satisfying (16, 17, 20); combining the proposition and the corollary one now sees as promised that this is precisely a K-braiding.

4. Another proof

We proved our main theorem in the previous section; here we provide an alternative proof, which may be of interest to some readers. It exhibits the bijection which is the object part of the isomorphism Φ of Corollary 15 as being part of the natural bijection (between hom-sets) of an adjunction.

To do this, we use the definition of K-braiding involving (19) rather than (17); see Corollary 18.

Proposition 19 *Let $\mathbb{T} = (T, m, e)$ be a monad on a category $\mathbb{C}$ and let $K\colon T^2 \Rightarrow T^2$ be an invertible BD-law on $\mathbb{T}$. For (X, x) a $\mathbb{T}$-algebra, we have a bijection*

$$K\text{-Brd}(X, x) \cong \mathbb{T}^*\text{-coalg}(X, x)$$

given by

$$\Psi_{(X,x)}\colon K\text{-Brd}(X, x) \to \mathbb{T}^*\text{-coalg}(X, x)$$

$$(c\colon TX \to TX) \mapsto (r\colon\ X \xrightarrow{eX} TX \xrightarrow{c} TX\ )$$

$$\Psi^{-1}_{(X,x)}\colon \mathbb{T}^*\text{-coalg}(X, x) \to K\text{-Brd}(X, x)$$

$$(r\colon X \to TX) \mapsto (c\colon\ TX \xrightarrow{Tr} T^2X \xrightarrow{KX} T^2X \xrightarrow{Tx} TX\ )$$

As explained above, the proposition can be deduced from Corollary 15; our alternative proof occupies the remainder of the paper.

Let $\mathbb{T}$ and $\mathbb{S}$ be monads on a category $\mathbb{C}$. A morphism of monads is a natural transformation $\varphi\colon \mathbb{S} \Rightarrow \mathbb{T}$ such that

$$\begin{tikzcd} S^2 \arrow[r, Rightarrow, "\varphi^2"] \arrow[d, Rightarrow, "m"'] \arrow[dr, phantom, "(21)"] & T^2 \arrow[d, Rightarrow, "m"] \\ S \arrow[r, Rightarrow, "\varphi"'] & T \end{tikzcd} \qquad \begin{tikzcd} Id_{\mathbb{C}} \arrow[r, Rightarrow, "e"] \arrow[dr, Rightarrow, "e"'] & S \arrow[d, Rightarrow, "\varphi"] \arrow[dl, phantom, "(22)"] \\ & T \end{tikzcd}$$

Following [1], such a morphism φ induces a functor $\varphi^*\colon \text{Alg}(\mathbb{T}) \to \text{Alg}(\mathbb{S})$ defined by $\varphi^*(X, x\colon TX \to X) = (X, \varphi X \cdot x\colon SX \to TX \to X)$. Moreover, φ^* has a left adjoint $\varphi!\colon \text{Alg}(\mathbb{S}) \to \text{Alg}(\mathbb{T})$ sending an $\mathbb{S}$-algebra (Y, y) to the object $\varphi!(Y, y)$ in the coequalizer

$$\begin{tikzcd} TSY \arrow[rr, "Ty"] \arrow[dr, "T\varphi Y"'] & & TY \arrow[r, "q"] & \varphi!(Y, y) \\ & T^2Y \arrow[ur, "mY"'] & & \end{tikzcd}$$

in $\text{Alg}(\mathbb{T})$, provided that such a coequalizer exists. Indeed, if $f\colon (Y, y) \to \varphi^*(X, x)$ is a morphism of $\mathbb{S}$-algebras, then $x.Tf\colon TY \to X$ coequalizes Ty and $mY.T\varphi Y$. Conversely, from $g\colon \varphi!(Y, y) \to (X, x)$, we get $g.q.eY : (Y, y) \to \varphi^*(X, x)$.

If $K\colon TS \Rightarrow ST$ is a distributive law of $\mathbb{T}$ over $\mathbb{S}$, we can consider the composite monad $\mathbb{ST}$ on $\mathbb{C}$. Explicitly,

$$\mathbb{ST} = (ST\colon \mathbb{C} \to \mathbb{C},\ STST \xrightarrow{SKT} S^2T^2 \xrightarrow{mm} ST\ ,\ Id_{\mathbb{C}} \xrightarrow{ee} ST\ )$$

Moreover, there are two morphisms of monads as in

$$\mathbb{S} \xrightarrow{\epsilon_1 = Se} \mathbb{ST} \xleftarrow{\epsilon_2 = eT} \mathbb{T}.$$

Now, for any $\mathbb{T}$-algebra $(X, x\colon TX \to X)$ and $\mathbb{S}$-algebra $(Y, y\colon SY \to Y)$, the adjunction $\epsilon_2! \dashv \epsilon_2^*$ gives a natural bijection

$$\Psi\colon \mathrm{Alg}(\mathbb{ST})[\epsilon_2!(X,x), \epsilon_1!(Y,y)] \xrightarrow{\simeq} \mathrm{Alg}(\mathbb{T})[(X,x), \epsilon_2^*\epsilon_1!(Y,y)]$$

The bijection of Proposition 19 will turn out to be a particular case of this natural bijection. To see this, we give an explicit description of Ψ and of the hom-sets involved.

First of all, let us recall from [1] that the coequalizer defining $\epsilon_2!(X,x)$ always exists. In fact, it is given by the solid part of the following diagram in $\mathrm{Alg}(\mathbb{ST})$

$$\begin{tikzcd}
STTX \arrow[r, shift left, "STx"] \arrow[r, shift right, "SmX"'] & STX \arrow[l, bend right, dotted, "SeTX"'] \arrow[r, "Sx"] & SX \arrow[l, bend right, dotted, "SeX"']
\end{tikzcd}$$

with action on SX given by

$$STSX \xrightarrow{SKX} S^2TX \xrightarrow{S^2x} S^2X \xrightarrow{mX} SX$$

Indeed, one easily verifies that the dotted arrows satisfy the equations for a split coequalizer [2]. Recall also that if f is a morphism of $\mathbb{ST}$-algebras coequalizing STx and SmX, then the induced $\mathbb{ST}$-algebra map out of SX is $f.SeX$.

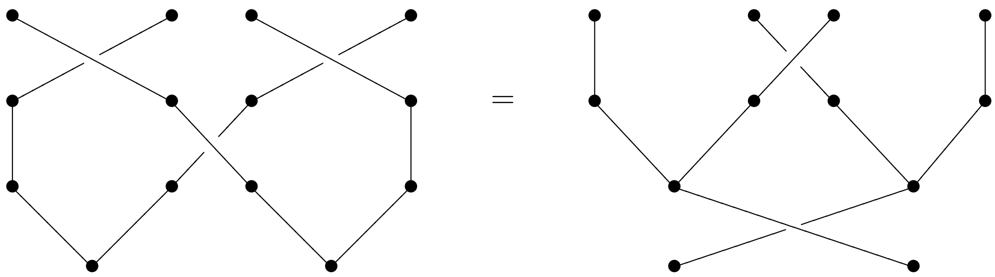
Unfortunately, the existence of the coequalizer defining $\epsilon_1!(Y,y)$ is not automatic. We need the following lemma, which makes sense because of Lemma 1:

Lemma 20 *Consider two monads $\mathbb{S}$ and $\mathbb{T}$ on a category $\mathbb{C}$, and let $K\colon TS \Rightarrow ST$ be an invertible distributive law of $\mathbb{T}$ over $\mathbb{S}$. Consider the composite monad $\mathbb{ST}$ induced by K and the composite monad $\mathbb{TS}$ induced by K^{-1}. Then $K\colon \mathbb{TS} \Rightarrow \mathbb{ST}$ is a morphism of monads, and it satisfies the following equations*

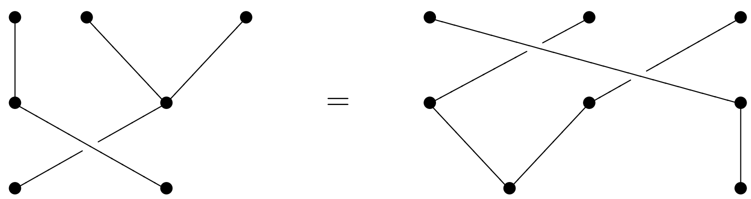
$$\begin{tikzcd}
 & \mathbb{TS} \arrow[dd, Rightarrow, "K"] & \\
\mathbb{S} \arrow[ur, Rightarrow, "\eta_2"] \arrow[dr, Rightarrow, "\epsilon_1"'] & & \mathbb{T} \arrow[ul, Rightarrow, "\eta_1"'] \arrow[dl, Rightarrow, "\epsilon_2"] \\
 & \mathbb{ST} &
\end{tikzcd}$$

where $\eta_1 = Te$ and $\eta_2 = eS$.

Proof. The equations $\epsilon_1 = K\eta_2$ and $\epsilon_2 = K\eta_1$ are conditions (6) and (7), and they imply condition (22). As far as condition (21) is concerned, let us give the idea of the proof, instead of the complete calculation. (This could be formalized using the string calculus of [8].) Think of K as a braiding between $\mathbb{T}$ and $\mathbb{S}$. Then condition (21) amounts to the following equation



which can be proved using the distributivity equations three times. Indeed, the distributivity admits the following graphical representation (this is condition (8))



■

As a consequence of the previous lemma, $\epsilon_1!\colon \mathrm{Alg}(\mathbb{S}) \to \mathrm{Alg}(\mathbb{S}\mathbb{T})$ can be obtained as

$$\mathrm{Alg}(\mathbb{S}) \xrightarrow{\eta_2!} \mathrm{Alg}(\mathbb{T}\mathbb{S}) \xrightarrow{(K^{-1})^*} \mathrm{Alg}(\mathbb{S}\mathbb{T})$$

and then $\epsilon_1!(Y, y)$ can be described by the following coequalizer in $\mathrm{Alg}(\mathbb{S}\mathbb{T})$

$$\begin{tikzcd}
(K^{-1})^*(L^{\mathbb{T}\mathbb{S}}(SY)) \arrow[r, shift left, "TSy"] \arrow[r, shift right, "TmY"'] \arrow[d, "KSY"', "\simeq"] & (K^{-1})^*(L^{\mathbb{T}\mathbb{S}}(Y)) \arrow[r, "Ty"] \arrow[l, dotted, bend right, "TeSY"'] \arrow[d, "\simeq"', "KY"] & TY \arrow[l, dotted, bend right, "TeY"'] \\
L^{\mathbb{S}\mathbb{T}}(SY) \arrow[r, shift left, "STy"] \arrow[r, shift right, "mTY.SKY"'] & L^{\mathbb{S}\mathbb{T}}(Y) &
\end{tikzcd}$$

with action on TY given by

$$ST^2Y \xrightarrow{SmY} STY \xrightarrow{K^{-1}Y} TSY \xrightarrow{Ty} TY.$$

We are now ready to describe the bijection Ψ.

Lemma 21 *Consider two monads $\mathbb{S}$ and $\mathbb{T}$ on a category $\mathbb{C}$, and let $K\colon TS \Rightarrow ST$ be an invertible distributive law of $\mathbb{T}$ over $\mathbb{S}$. For any $\mathbb{T}$-algebra (X, x) and $\mathbb{S}$-algebra (Y, y) :*

(i) The hom-set $\mathrm{Alg}(\mathbb{S})[(Y,y),\epsilon_1^*(\epsilon_2!(X,x))]$ *is the set of morphisms* $r\colon Y\to SX$ *such that*

$$\begin{tikzcd} SY \arrow[r, "Sr"] \arrow[d, "y"'] \arrow[dr, phantom, "(12)"] & S^2X \arrow[d, "mX"] \\ Y \arrow[r, "r"'] & SX \end{tikzcd}$$

commutes.

(ii) The hom-set $\mathrm{Alg}(\mathbb{S}\mathbb{T})[\epsilon_1!(Y,y),\epsilon_2!(X,x)]$ *is the set of morphisms* $c\colon TY\to SX$ *such that*

$$\begin{tikzcd} T^2SY \arrow[r, "TKY"] \arrow[d, "T^2y"'] & TSTY \arrow[r, "TSc"] & TSSX \arrow[d, "TmX"] \\ T^2Y \arrow[d, "mY"'] & (19) & TSX \arrow[d, "KX"] \\ TY \arrow[r, "c"'] & SX & STX \arrow[l, "Sx"] \end{tikzcd}$$

commutes.

(iii) The natural bijection

$$\Psi\colon \mathrm{Alg}(\mathbb{S}\mathbb{T})[\epsilon_1!(Y,y),\epsilon_2!(X,x)] \xrightarrow{\simeq} \mathrm{Alg}(\mathbb{S})[(Y,y),\epsilon_1^*(\epsilon_2!(X,x))]$$

induced by the adjunction $\epsilon_1!\dashv\epsilon_1^*$ *is given by*

$$\Psi(c)\colon\ Y \xrightarrow{eY} TY \xrightarrow{c} SX \qquad \Psi^{-1}(r)\colon\ TY \xrightarrow{Tr} TSX \xrightarrow{KX} STX \xrightarrow{Sx} SX.$$

Proof. Following the description of $\varphi!\dashv\varphi^*$ given at the beginning of this section, we have $\Psi(c)$ and $\Psi^{-1}(r)$ given respectively by

$$Y \xrightarrow{eY} TY \xrightarrow{eTY} STY \xrightarrow{K^{-1}} TSY \xrightarrow{Ty} TY \xrightarrow{c} SX$$

$$TY \xrightarrow{eTY} STY \xrightarrow{STr} STSX \xrightarrow{SKX} SSTX \xrightarrow{nTX} STX \xrightarrow{Sx} SX$$

which are easily seen to reduce to formulas given in (*iii*).

For $r\colon Y\to SX$, to be a morphism of $\mathbb{S}$-algebras means that

$$\begin{tikzcd} SY \arrow[r, "Sr"] \arrow[d, "y"'] & S^2X \arrow[rr, "SeSX"] & & STSX \arrow[d, "SKX"] \\ Y \arrow[r, "r"'] & SX & S^2X \arrow[l, "mX"] & S^2TX \arrow[l, "S^2x"] \end{tikzcd}$$

commutes, which immediately reduces to (12) by (6) and (4).

For $c\colon TY \to SX$, to be a morphism of $\mathbb{ST}$-algebras means commutativity of

$$\begin{tikzcd}
ST^2Y \arrow[r, "STc"] \arrow[d, "SmY"'] & STSX \arrow[r, "SKX"] & SSTX \arrow[r, "SSx"] & SSX \arrow[d, "mX"] \\
STY \arrow[r, "K^{-1}Y"'] & TSY \arrow[r, "Ty"'] & TY \arrow[r, "c"'] & SX
\end{tikzcd}$$

which, by (8) and (9), reduces to (19). ■

This is the best we can do working at this level of generality. From now on, we assume $\mathbb{S} = \mathbb{T}$ and $(X, x) = (Y, y)$. We now combine the next two lemmas with the previous one to complete the proof of Proposition 19.

Lemma 22 *Let $\mathbb{T}$ be a monad on a category $\mathbb{C}$, and let $K\colon T^2 \Rightarrow T^2$ be an invertible distributive law on $\mathbb{T}$. Fix a $\mathbb{T}$-algebra (X, x). In the bijection of Lemma 21, the map $r\colon X \to TX$ satisfies condition* (13) *iff $c = \Psi^{-1}(r)\colon TX \to TX$ satisfies condition* (15).

Proof. Condition (15) says precisely that $\Psi(c)$ satisfies (13). ■

Lemma 23 *Let $\mathbb{T}$ be a monad on a category $\mathbb{C}$, and let $K\colon T^2 \Rightarrow T^2$ be an invertible BD-law on $\mathbb{T}$. Fix a $\mathbb{T}$-algebra (X, x). In the bijection of Lemma 21, the map $r\colon X \to TX$ satisfies condition* (14) *iff $c = \Psi^{-1}(r)\colon TX \to TX$ satisfies condition* (16).

Proof. (14) $\Rightarrow$ (16) :

$$\begin{tikzcd}
T^2X \arrow[r, "KX"] \arrow[dr, "T^2r"] \arrow[dd, "T^2r"'] & T^2X \arrow[rr, "T^2r"] & & T^3X \arrow[dd, "TKX"] \arrow[ddddl, "TKX"'] \\
& T^3X \arrow[urr, "KTX"] \arrow[d, "T^3eX"'] \arrow[dr, "TKX"] & & \\
T^3X \arrow[u, phantom, "(14)"] \arrow[r, "T^3r"] \arrow[d, "TKX"'] & T^4X \arrow[d, "TKTX"'] & T^3X \arrow[dl, "T^3eX"'] \arrow[ddl, "1"] & T^3X \arrow[d, "T^2x"] \\
T^3X \arrow[r, "T^3r"] \arrow[d, "T^2x"'] \arrow[dr, phantom, "(12)"] & T^4X \arrow[d, "T^2mX"'] \arrow[r, phantom, "(1)"] & & T^2X \arrow[d, "KX"] \\
T^2X \arrow[r, "T^2r"] \arrow[d, "KX"'] & T^3X \arrow[d, "KTX"'] \arrow[r, phantom, "(10)"] & T^3X \arrow[dr, "KTX"'] & T^2X \\
T^2X \arrow[r, "T^2r"'] & T^3X \arrow[rr, "T(KX)"'] & & T^3X \arrow[u, "T^2x"']
\end{tikzcd}$$

(16) $\Rightarrow$ (14) :

$$\begin{tikzcd}
 & X \arrow[dl, "eX"'] \arrow[drr, "eX"] & & \\
TX \arrow[r, "TeX"] \arrow[dr, "eTX"'] \arrow[dd, "c"'] \arrow[dr, phantom, "(7)", shift left=3] & T^2X \arrow[d, "KX"] \arrow[dr, "Tc"] & & TX \arrow[ll, "eTX"'] \arrow[d, "c"] \\
 & T^2X \arrow[d, "Tc"] & T^2X \arrow[ddr, "KX"'] \arrow[ddr, phantom, "(6)", shift left=4] & TX \arrow[l, "eTX"'] \arrow[dd, "TeX"] \\
TX \arrow[r, "eTX"] \arrow[dr, "TeX"'] \arrow[dr, phantom, "(6)", shift left=3] & T^2X \arrow[d, "KX"] & (16) & \\
 & T^2X & & T^2X \arrow[ll, "Tc"]
\end{tikzcd}$$

■

The proof of Proposition 19 is now complete.

References

[1] J. Beck, Distributive laws, in *Seminar on Triples and Categorical Homology Theory* (Lecture Notes in Mathematics 80), pp. 119–140, Springer, Berlin, 1969.

[2] F. Borceux, *Handbook of categorical algebra I*, Cambridge University Press, 1994.

[3] F. Borceux, *Handbook of categorical algebra II*, Cambridge University Press, 1994.

[4] M. Cipolla, Discesa fedelmente piatta dei moduli, *Rend. Circ. Mat. Palermo* 25:43–46, 1976.

[5] A. Davydov, *Quasicommutative monoids*, Lecture at Australian Category Seminar, 21 January 2004.

[6] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique, I : Généralités, descente par morphismes fidèlements plats*, Séminaire Bourbaki 190 (1959-1960).

[7] G. Janelidze and W. Tholen, Facets of descent III: monadic descent for rings and algebras, preprint, 2004.

[8] A. Joyal and R. Street, Geometry of tensor calculus I, *Adv. Math.* 88:55–112, 1991.

[9] C. Kassel, *Quantum groups* (Graduate Texts in Mathematics 155) Springer-Verlag, New York, 1995.

[10] S. Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics 5), Springer, New York-Berlin, 1971.

[11] C. Menini and D. Stefan, Descent theory and Amitsur cohomology of triples, *J. Algebra* 266:261–304, 2003.

[12] P. Nuss, Noncommutative descent and non-abelian cohomology, *K-Theory* 12:23–74, 1997.

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