A Picard-Brauer exact sequence of categorical groups

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Dedicated to Max Kelly on the occasion of his 70th birthday

Abstract. A categorical group is a monoidal groupoid in which each object has a tensorial inverse. Two main examples are the Picard categorical group of a monoidal category and the Brauer categorical group of a braided monoidal category with stable coequalizers. After discussing the notions of kernel, cokernel and exact sequence for categorical groups, we show that, given a suitable monoidal functor between two symmetric monoidal categories with stable coequalizers, it is possible to build up a five-term Picard-Brauer exact sequence of categorical groups. The usual Units-Picard and Picard-Brauer exact sequences of abelian groups follow from this exact sequence of categorical groups. We also discuss the direct sum decomposition of the Brauer-Long group.

Introduction

It is well-known that, given a homomorphism $f\colon R \to S$ between two unital commutative rings, it is possible to build up an exact sequence

$$\mathrm{Pic}(R) \to \mathrm{Pic}(S) \to F_0 \to \mathrm{Br}(R) \to \mathrm{Br}(S)$$

where $\mathrm{Pic}(R)$ is the Picard group of R (i.e. the group of projective R-modules of constant rank 1) and $\mathrm{Br}(R)$ is the Brauer group of R (i.e. the group of Morita-equivalence classes of Azumaya R-algebras). A wide generalization of this situation occurs in the context of monoidal categories : given a suitable monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ between symmetric monoidal categories with stable coequalizers, it is possible to build up an exact sequence

$$\mathrm{Pic}(\mathbf{C}) \to \mathrm{Pic}(\mathbf{D}) \to F_0 \to \mathrm{Br}(\mathbf{C}) \to \mathrm{Br}(\mathbf{D}).$$

This exact sequence generalizes the previous one, which is recovered taking as $F\colon \mathbf{C} \to \mathbf{D}$ the functor $S \otimes_R -\colon R\text{-mod} \to S\text{-mod}$ between module categories induced by $f\colon R \to S$, as well as several other classical exact sequences built up "à la Brauer" (see [25, 22, 52]) :

- the exact sequence induced by a morphism of ringed spaces (B. Auslander [1]);
- the exact sequence induced by the inclusion of two fibered subcategories of the category of divisorial lattices over a Krull domain (M. Orzech [36] and A. Verschoren [49]) ;
- the exact sequence induced by a change of idempotent kernel functor in R-mod (A. Verschoren [48]) ;
- some exact sequences arising in the context of modules over a separated scheme (A. Verschoren [50]) ;
- the exact sequence connecting the Picard and the Brauer groups of dimodule algebras (F.W. Long [34]) ;
- the exact sequence induced by a morphism of cocommutative finite Hopf algebras (Fernandez Vilaboa et al. [22]).

The present work has two motivations.The first one is to show that the link between "Picard" and "Brauer" is deeper than that expressed by the Picard-Brauer exact sequence. In fact this sequence only explains the relation between the objects of the monoidal categories involved in the construction of the Picard and of the Brauer groups. But a similar relation holds also for the morphisms of these categories. For this reason, invariants taking into account objects and morphisms should be considered.

The second motivation concerns another classical exact sequence built up from a ring homomorphism $f\colon R \to S$, that is

$$\mathrm{U}(R) \to \mathrm{U}(S) \to F_1 \to \mathrm{Pic}(R) \to \mathrm{Pic}(S)$$

where $\mathrm{U}(R)$ is the group of units of R (i.e. the elements which are invertible with respect to the multiplicative structure of R). We will show that the Units-Picard and the Picard-Brauer exact sequences are two traces in the category of abelian groups of a single exact sequence which lives at a higher level.

This higher level is provided by the so-called categorical groups. A categorical group (for short, a cat-group) is a monoidal groupoid in which each object has a tensorial inverse. Two main examples are the Picard cat-group $\mathcal{P}(\mathbf{C})$ of a monoidal category $\mathbf{C}$, which is the subcategory of isomorphisms between invertible objects, and the Brauer cat-group $\mathcal{B}(\mathbf{C})$ of a braided monoidal category $\mathbf{C}$ with stable coequalizers, which is the Picard cat-group of the classyfing category of the bicategory of monoids and bimodules in $\mathbf{C}$.

Given two morphisms (that is two monoidal functors) of cat-groups

$$\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{G} \mathbf{K}$$

such that the composite $F \cdot G$ is naturally isomorphic to the zero-morphism, we can factorize F through the "kernel" of G

$$\begin{tikzcd} \mathbf{G} \arrow[r, "F"] \arrow[dr, "F'"'] & \mathbf{H} \\ & \mathrm{Ker}(G) \arrow[u] \end{tikzcd}$$

and we can define various kinds of exactness looking at the surjectivity of the functor F'. To explain the relation between the Picard and the Brauer cat-groups, an appropriate notion of exactness is what I call 2-exactness: F' must be essentially surjective on objects and full, so that 2-exactness is a kind of surjectivity *on objects and on arrows.* In fact, given a good monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ between symmetric monoidal categories with stable coequalizers, we obtain a sequence of symmetric cat-groups

$$\mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}) \to \overline{\mathcal{F}} \to \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D})$$

which is 2-exact in $\mathcal{P}(\mathbf{D})$, $\overline{\mathcal{F}}$ and $\mathcal{B}(\mathbf{C})$ (proposition 6.1). (The cat-group $\overline{\mathcal{F}}$ in the previous sequence is built up using a quotient ; for this reason we introduce also the "cokernel" of a morphism of symmetric cat-groups.)

Now we can come back to groups. There are two groups canonically associated with a cat-group $\mathbf{G}$: the group $\pi_0(\mathbf{G})$ of the connected components of $\mathbf{G}$ (it is abelian if $\mathbf{G}$ is braided) and the abelian group $\pi_1(\mathbf{G})$ of the endomorphisms of the unit object of $\mathbf{G}$. Both π_0 and π_1 give rise to functors from cat-groups to groups, and applying these functors to the Picard-Brauer exact sequence of cat-groups we obtain, respectively, the usual Picard-Brauer and Units-Picard exact sequences of abelian groups induced by a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ (corollary 6.1).

Categorical groups draw their origins from algebraic geometry and ring theory. In a sense which can be made precise, they are the same that crossed modules. They have been used in ring theory (in particular in connection with ring extensions and Hattory-Villamayor-Zelinsky sequences), in homological algebra (to describe cohomology groups and to classify various kinds of extensions) and in algebraic topology. (For all these aspects, see the subdivision of the items at the end of the bibliography.)

In the first section, we recall the basic definitions : cat-groups and their morphisms, functors π_0 and π_1. In section 2 we discuss the notions of kernel and cokernel for morphisms of cat-groups ; the construction of the kernel is already in [41] and that of the cokernel is to be compared with the similar problem studied in [38]. In section 3 we introduce the notion of 2-exact sequence of cat-groups : it is a quite strong condition justified by the main example of the Picard-Brauer sequence ; some basic facts on 2-exact sequences between (symmetric) cat-groups are established in [29]. In section 4 we describe the Picard cat-group of a monoidal category and the Brauer cat-group of a braided monoidal category with stable coequalizers. We also point out how to recapture the classical Units, Picard and Brauer groups from the above cat-groups using the functors π_0 and π_1. Sections 5 and 6 are devoted to the construction of a five-terms Picard-Brauer 2-exact sequence of symmetric cat-groups. In the last section, we give another example of our approach to Picard and Brauer groups. We discuss, from a categorical point of view, Beattie's decomposition of the Brauer-Long group of a Hopf algebra [2] and Caenepeel's decomposition of the Picard group of a Hopf algebra [11].

Bicategories appear in three different places : in the definition of the Brauer cat-group, in the construction of the cokernel and to build up the cat-group in the middle of the Picard-Brauer sequence. Each time we have in fact a monoidal bicategory, so that its classifying category is a monoidal category ; but, to avoid any tricategorical complexity, I have choosen to consider them as bicategories and to introduce a posteriori the monoidal structure on the classifying category.

I have omitted (almost) all the proofs, which involve general arguments on duality in monoidal categories and, especially in the last two sections, are quite technical.

Finally, I would like to thank J. Bénabou, A. Carboni, G. Janelidze and G.M. Kelly for discussions on these topics and for their encouragement.

1 Categorical groups

In this section we recall the definition of cat-group and some basic properties of cat-groups and their morphisms. First of all, let us fix some notations and conventions :

- in any category, the composite of two arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ is written $f \cdot g$;
- in a monoidal category $\mathbf{C}$, the unit object will be denoted by $I_{\mathbf{C}}$ or simply I, and the tensor product by $\otimes$; if $\mathbf{C}$ is braided, the braiding is denoted by γ ;
- a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ is always such that the transitions $f_I\colon I \to F(I)$ and $f_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B)$ are isomorphisms ; if $\mathbf{C}$ and $\mathbf{D}$ are braided and F preserves the braiding, we say that F is a γ-monoidal functor ;
- natural transformation between monoidal functors always means monoidal natural transformation.

Recall that a duality $\mathcal{D} = (A^* \dashv A, \eta_A, \epsilon_A)$ in a monoidal category is given by two object A and A^* and two arrows

$$\eta_A\colon I \to A \otimes A^* \quad \epsilon_A\colon A^* \otimes A \to I$$

such that the following compositions are identities :

$$A \longrightarrow I \otimes A \xrightarrow{\eta_A \otimes 1} (A \otimes A^*) \otimes A \longrightarrow A \otimes (A^* \otimes A) \xrightarrow{1 \otimes \epsilon_A} A \otimes I \longrightarrow A$$

$$A^* \longrightarrow A^* \otimes I \xrightarrow{1 \otimes \eta_A} A^* \otimes (A \otimes A^*) \longrightarrow (A^* \otimes A) \otimes A^* \xrightarrow{\epsilon_A \otimes 1} I \otimes A^* \longrightarrow A^*$$

(As a matter of convention, unlabelled arrows in a diagram are arrows built up using the contraints of a monoidal category and the transitions of a monoidal functor, or defined by them in an obvious way.)

Definition 1.1 *a cat-group* $\mathbf{G}$ *is a monoidal category* $\mathbf{G} = (\mathbf{G}, \otimes, I)$ *such that*

- *each arrow is an isomorphism, that is* $\mathbf{G}$ *is a groupoid ;*
- *for each object* A *of* $\mathbf{G}$*, there exists an object* A^* *and a morphism* $\eta_A\colon I \to A \otimes A^*$ *.*

We say that **G** *is a braided (symmetric) cat-group if it is braided (symmetric) as monoidal category.*

If **G** is a cat-group, it is possible, for each object A, to find a morphism $\epsilon_A \colon A^* \otimes A \to I$. In other words, a cat-group is exactly a monoidal groupoid in which each object is invertible, up to isomorphisms, with respect to the tensor product. Moreover, the morphism ϵ_A can be chosen in such a way that $\mathcal{D} = (A^* \dashv A, \eta_A, \epsilon_A)$ is a duality (and then also $\mathcal{D}^{-1} = (A \dashv A^*, \epsilon_A^{-1}, \eta_A^{-1})$ is a duality). The choice, for each A, of such a duality induces an equivalence

$$(\;)^* \colon \mathbf{G}^{\mathrm{op}} \to \mathbf{G} \qquad (f \colon A \to B) \;\;\mapsto\;\; (f^* \colon B^* \to A^*)$$

where f^* is defined by

$$B^* \longrightarrow B^* \otimes I \xrightarrow{1 \otimes \eta_A} B^* \otimes (A \otimes A^*) \xrightarrow{1 \otimes f \otimes 1} B^* \otimes (B \otimes A^*) \longrightarrow (B^* \otimes B) \otimes A^*$$

$$\xrightarrow{\epsilon_B \otimes 1} I \otimes A^* \longrightarrow A^*$$

A morphism of cat-groups $F \colon \mathbf{G} \to \mathbf{H}$ is a monoidal functor ; if **G** and **H** are braided cat-groups and F is a γ-monoidal functor, we say that F is a γ-morphism. Observe that a natural transformation between two morphisms of cat-groups is necessarily a natural isomorphism.

We recall now some simple facts, coming from duality theory in monoidal categories, which will be implicitely used in the rest of the work to make various definitions and constructions well founded : let **G** be a cat-group

- for each pair of objects A, B of **G**, there is a morphism $B^* \otimes A^* \to (A \otimes B)^*$ natural in A and B ;
- for each object A of **G**, there is a morphism $A \to (A^*)^*$ natural in A ;
- if $F \colon \mathbf{G} \to \mathbf{H}$ is a morphism of cat-groups, for each object A of **G** there is a morphism $F(A^*) \to F(A)^*$ natural in A ;
- for each arrow $f \colon A \to B$ of **G**, the following equations hold

$$\eta_A \cdot (f \otimes (f^{-1})^*) = \eta_B \qquad \epsilon_A = ((f^{-1})^* \otimes f) \cdot \epsilon_B$$

Let **G** be a cat-group ; we write $\pi_0(\mathbf{G})$ for the group of isomorphism classes of objects of **G** (that is, the group of connected components of **G**) with the product induced by the tensor product of **G**. In this way we have a functor

$$\pi_0 \colon \text{Cat-groups} \to \text{Groups}$$

which factors through the homotopy category of Cat-groups.

Recall that, if **C** is any monoidal category, the set of endomorphisms $\mathbf{C}(I, I)$ is a commutative monoid. In particular, for a cat-group **G**, the set $\mathbf{G}(I, I)$ is an abelian group (isomorphic to $\mathbf{G}(X,X)$ for any object X of $\mathbf{G}$) which we call $\pi_1(\mathbf{G})$. This easily extends to a functor

$$\pi_1\colon \text{Cat-groups} \to \text{Abelian Groups}$$

which, once again, factors through the homotopy category.

Finally, note that the functor π_0 is the restriction of the classifying functor

$$\mathrm{cl}\colon \text{Bicategories} \to \text{Categories}$$

which assigns to each bicategory $\mathbf{B}$ the category $\mathrm{cl}(\mathbf{B})$ whose objects are those of $\mathbf{B}$ and whose arrows are 2-isomorphism classes of 1-arrows of $\mathbf{B}$ (see [3]).

Several facts concerning morphisms of cat-groups can be checked using π_0 and π_1 ; some of them are listed in the following proposition (recall that a functor $F\colon \mathbf{G} \to \mathbf{H}$ is essentially surjective when for each object Y in $\mathbf{H}$ there exists an object X in $\mathbf{G}$ and an isomorphism $F(X) \to Y$).

Proposition 1.1 *let $F\colon \mathbf{G} \to \mathbf{H}$ be a morphism of cat-groups,*

- *F is essentially surjective iff $\pi_0(F)$ is surjective ;*
- *F is faithful iff $\pi_1(F)$ is injective ;*
- *F is full iff $\pi_0(F)$ is injective and $\pi_1(F)$ is surjective ;*
- *F is an equivalence iff $\pi_0(F)$ and $\pi_1(F)$ are isomorphisms.*

2 Kernel and cokernel

In this section we describe the kernel and the cokernel of a morphism of (symmetric) cat-groups. They are particular instances of bilimits and, as such, they are determined, up to monoidal equivalences, by their universal property (see [40]).

Zero-morphism : let $\mathbf{G}$ and $\mathbf{H}$ be two cat-groups ; the functor $0_{\mathbf{G},\mathbf{H}}\colon \mathbf{G} \to \mathbf{H}$ which sends each arrow in the identity of the unit object of $\mathbf{H}$, is a morphism of cat-groups, called the zero-morphism. Note that for each cat-group $\mathbf{K}$ and for each morphism $F\colon \mathbf{H} \to \mathbf{K}$ or $G\colon \mathbf{K} \to \mathbf{G}$, there are two canonical natural transformations $0_{\mathbf{G},\mathbf{K}} \Rightarrow 0_{\mathbf{G},\mathbf{H}} \cdot F$ and $G \cdot 0_{\mathbf{G},\mathbf{H}} \Rightarrow 0_{\mathbf{K},\mathbf{H}}$. We will write 0 for $0_{\mathbf{G},\mathbf{H}}$.

Kernel : let $F\colon \mathbf{G} \to \mathbf{H}$ be a morphism of cat-groups.The kernel of F is given by a cat-group $\mathrm{Ker}(F)$, a morphism $e_F\colon \mathrm{Ker}(F) \to \mathbf{G}$ and a natural transformation $\epsilon_F\colon e_F \cdot F \Rightarrow 0$

$$\begin{tikzcd} & \mathbf{G} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \epsilon_F"] & \\ \mathrm{Ker}(F) \arrow[ur, "e_F"] \arrow[rr, "0"'] & {} & \mathbf{H} \end{tikzcd}$$

universal in the following sense :
given a cat-group $\mathbf{K}$, a morphism G and a natural transformation $\varphi\colon G\cdot F\Rightarrow 0$

$$\begin{tikzcd} & \mathbf{G} \arrow[dr, "F"] \arrow[d, Rightarrow, "\varphi"'] & \\ \mathbf{K} \arrow[ur, "G"] \arrow[rr, "0"'] & {} & \mathbf{H} \end{tikzcd}$$

there exists a morphism G' and a natural transformation φ'

$$\begin{tikzcd} & \mathrm{Ker}(F) \arrow[dr, "e_F"] \arrow[d, Rightarrow, "\varphi'"'] & \\ \mathbf{K} \arrow[ur, "G'"] \arrow[rr, "G"'] & {} & \mathbf{G} \end{tikzcd}$$

such that

$$\begin{tikzcd} G'\cdot e_F\cdot F \arrow[r, Rightarrow, "G'\cdot\epsilon_F"] \arrow[d, Rightarrow, "\varphi'\cdot F"'] & G'\cdot 0 \arrow[d, Rightarrow] \\ G\cdot F \arrow[r, Rightarrow, "\varphi"'] & 0 \end{tikzcd}$$

commutes. Moreover, if G'' and φ'' satisfy the same condition as G' and φ', then there exists a unique natural transformation $\psi\colon G''\Rightarrow G'$ such that

$$\begin{tikzcd} & G & \\ G''\cdot e_F \arrow[ur, Rightarrow, "\varphi''"] \arrow[rr, Rightarrow, "\psi\cdot e_F"'] & & G'\cdot e_F \arrow[ul, Rightarrow, "\varphi'"'] \end{tikzcd}$$

commutes.

<u>Existence of kernels</u> : given a morphism $F\colon \mathbf{G}\to\mathbf{H}$ of cat-groups, a kernel of F can be described in the following way :

- an object of $\mathrm{Ker}(F)$ is a pair (X,λ_X) where X is an object of $\mathbf{G}$ and $\lambda_X\colon F(X)\to I$ is an arrow in $\mathbf{H}$;
- an arrow $f\colon (X,\lambda_X)\to(Y,\lambda_Y)$ in $\mathrm{Ker}(F)$ is an arrow $f\colon X\to Y$ in $\mathbf{G}$ such that

  $$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_X"'] & & F(Y) \arrow[dl, "\lambda_Y"] \\ & I & \end{tikzcd}$$

  commutes ;
- identities and composition in $\mathrm{Ker}(F)$ are those of $\mathbf{G}$, so that $\mathrm{Ker}(F)$ is a groupoid ;

- using the transitions $f_I \colon I \to F(I)$ and $f_{X,Y} \colon F(X) \otimes F(Y) \to F(X \otimes Y)$, one defines a monoidal structure on $\mathrm{Ker}(F)$, which in fact is a cat-group.

The faithful functor

$$e_F \colon \mathrm{Ker}(F) \to \mathbf{G} \quad (f \colon (X, \lambda_X) \to (Y, \lambda_Y)) \;\mapsto\; (f \colon X \to Y)$$

is a morphism of cat-groups and the component at (X, λ_X) of the natural transformation ϵ_F is given by λ_X.

Proof of the universality : Define

$$G' \colon \mathbf{K} \to \mathrm{Ker}(F) \quad (f \colon A \to B) \;\mapsto\; (G(f) \colon (G(A), \varphi_A) \to (G(B), \varphi_B))$$

and

$$\varphi' \colon G' \cdot e_F \Rightarrow G \quad \varphi'_A = 1_{G(A)} \colon e_F(G'(A)) = G(A) \to G(A)$$

To have a natural transformation $\psi \colon G'' \Rightarrow G'$ we need, for each object A of $\mathbf{K}$, an arrow

$$\psi_A \colon e_F(G''(A)) \to e_F(G'(A)) = G(A)$$

and we can take $\psi_A = \varphi''_A$; this is possible because the commutativity of

$$\begin{tikzcd}
G'' \cdot e_F \cdot F \arrow[r, Rightarrow, "G'' \cdot \epsilon_F"] \arrow[d, Rightarrow, "\varphi'' \cdot F"'] & G'' \cdot 0 \arrow[d, Rightarrow] \\
G \cdot F \arrow[r, Rightarrow, "\varphi"'] & 0
\end{tikzcd}$$

means exactly that φ''_A is an arrow from $G''(A)$ to $G'(A)$ in $\mathrm{Ker}(F)$. The uniqueness of ψ follows from the faithfulness of e_F.

$\diamondsuit$

Remarks :

1) if $\mathbf{H}$ is braided, $\mathbf{G}$ is braided (symmetric) and F is a γ-morphism, then also $\mathrm{Ker}(F)$ is braided (symmetric) and e_F is a γ-morphism ; if, moreover, $\mathbf{K}$ is braided and G is a γ-morphism, then G' is a γ-morphism.

2) the description of $\mathrm{Ker}(F)$, without its universal property, is already in [41] and, implicitely, in [51].

In proposition 2.1, whose proof uses proposition 1.1 and lemma 2.1, we show in what sense the kernel measures the "injectivity" of a morphism. We write $\mathbf{1}$ for the one-arrow cat-group.

Lemma 2.1 *Consider a morphism of cat-groups together with its kernel*

$$\mathrm{Ker}(F) \xrightarrow{e_F} \mathbf{G} \xrightarrow{F} \mathbf{H}$$

1) the factorization of $\pi_1(e_F)$ through the kernel of $\pi_1(F)$ is an isomorphism;

2) the factorization of $\pi_0(e_F)$ through the kernel of $\pi_0(F)$ is surjective.

(Observe that the factorization of $\pi_0(e_F)$ through the kernel of $\pi_0(F)$ is injective if e_F is full. But e_F is full iff $\pi_1(\mathbf{H}) = 0$.)

Proposition 2.1 *Consider a morphism of cat-groups together with its kernel*

$$\mathrm{Ker}(F) \xrightarrow{e_F} \mathbf{G} \xrightarrow{F} \mathbf{H}$$

1) F is faithful iff $\pi_1(Ker(F)) = 0$;

2) F is full iff $\pi_0(Ker(F)) = 0$;

3) F is full and faithful iff $Ker(F)$ is equivalent to $\mathbf{1}$.

Cokernel : let $F\colon \mathbf{G} \to \mathbf{H}$ be a morphism between cat-groups. The cokernel of F is a cat-group $\mathrm{Coker}(F)$ with a morphism P_F and a natural transformation $\pi_F\colon F \cdot P_F \Rightarrow 0$

$$\begin{tikzcd} & \mathbf{H} \arrow[dr, "P_F"] \arrow[d, phantom, "\Downarrow \pi_F"] & \\ \mathbf{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathrm{Coker}(F) \end{tikzcd}$$

universal in the following sense : given a cat-group $\mathbf{K}$, a morphism G and a natural transformation φ

$$\begin{tikzcd} & \mathbf{H} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow \varphi"] & \\ \mathbf{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbf{K} \end{tikzcd}$$

there exists a morphism G' and a natural transformation φ'

$$\begin{tikzcd} & \mathrm{Coker}(F) \arrow[dr, "G'"] \arrow[d, phantom, "\Downarrow \varphi'"] & \\ \mathbf{H} \arrow[ur, "P_F"] \arrow[rr, "G"'] & {} & \mathbf{K} \end{tikzcd}$$

such that

$$\begin{tikzcd} F \cdot P_F \cdot G' \arrow[r, Rightarrow, "\pi_F \cdot G'"] \arrow[d, Rightarrow, "F \cdot \varphi'"'] & 0 \cdot G' \\ F \cdot G \arrow[r, Rightarrow, "\varphi"'] & 0 \arrow[u, Rightarrow] \end{tikzcd}$$

commutes. Moreover, if G'' and φ'' satisfy the same condition as G' and φ', then there exists a unique natural transformation $\psi\colon G'' \Rightarrow G'$ such that

$$\begin{tikzcd}
 & G & \\
P_F \cdot G'' \arrow[ur, Rightarrow, "\varphi''"] \arrow[rr, Rightarrow, "P_F \cdot \psi"'] & & P_F \cdot G' \arrow[ul, Rightarrow, "\varphi'"']
\end{tikzcd}$$

commutes.

Existence of cokernels : now we describe a cokernel for a γ-morphism $F\colon \mathbf{G} \to \mathbf{H}$ between symmetric cat-groups.

First step. We start the construction of the cokernel introducing the bicategory $\mathrm{Cok}(F)$ as follows :

- the object of $\mathrm{Cok}(F)$ are those of $\mathbf{H}$;
- a 1-arrow $X \longrightarrow\!\!\bullet\!\!\!\to Y$ in $\mathrm{Cok}(F)$ is a pair (f, N) with N an object of $\mathbf{G}$ and $f\colon X \to Y \otimes F(N)$ an arrow in $\mathbf{H}$;
- given two 1-arrows $(f, N)\colon X \longrightarrow\!\!\bullet\!\!\!\to Y$ and $(g, M)\colon Y \longrightarrow\!\!\bullet\!\!\!\to Z$ in $\mathrm{Cok}(F)$, their composition is $(f \bullet g, M \otimes N)\colon X \longrightarrow\!\!\bullet\!\!\!\to Z$, where $f \bullet g\colon X \to Z \otimes F(M \otimes N)$ is given by

$$X \xrightarrow{f} Y \otimes F(N) \xrightarrow{g\otimes 1} (Z \otimes F(M)) \otimes F(N) \longrightarrow Z \otimes F(M \otimes N) \quad ;$$

- the 1-identity on an object X of $\mathrm{Cok}(F)$ is the pair $(X \to X \otimes I \to X \otimes F(I), I)$;
- given two parallel 1-arrows $(f, N), (g, M)\colon X \longrightarrow\!\!\bullet\!\!\!\to Y$ in $\mathrm{Cok}(F)$, a 2-arrow $\alpha\colon (f, N) \Rightarrow (g, M)$ is an arrow $\alpha\colon N \to M$ in $\mathbf{G}$ such that

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y \otimes F(N) \arrow[rr, "1\otimes F(\alpha)"'] & & Y \otimes F(M)
\end{tikzcd}$$

  commutes ;
- 2-identities and vertical 2-composition are identities and composition in $\mathbf{G}$; horizontal 2-composition is the tensor product of $\mathbf{G}$.

Using associativity and unit contraints of $\mathbf{G}$, one makes $\mathrm{Cok}(F)$ a bicategory. Moreover, $\mathrm{Cok}(F)$ is a bigroupoid, that is each 2-arrow is an isomorphism (because it is an isomorphism in $\mathbf{G}$) and each 1-arrow is an equivalence. To see this last fact, consider a 1-arrow $(f,N)\colon\ X \longrightarrow\!\!\!\!\!\bullet\ Y$ in $\mathrm{Cok}(F)$ and fix a duality $(N^* \dashv N, \eta_N, \epsilon_N)$ in $\mathbf{G}$; a quasi-inverse of (f,N) is $(\hat{f}, N^*)$ where $\hat{f}$ is given by

$$Y \longrightarrow Y \otimes I \longrightarrow Y \otimes F(I) \xrightarrow{1\otimes F(\eta_N)} Y \otimes F(N \otimes N^*)$$

$$\longrightarrow (Y \otimes F(N)) \otimes F(N^*) \xrightarrow{f^{-1}\otimes 1} X \otimes F(N^*)$$

Second step. Consider the classifying functor

$$\mathrm{cl}\colon \text{Bicategories} \to \text{Categories}$$

and put $\mathrm{Coker}(F) = \mathrm{cl}(\mathrm{Cok}(F))$. Explicitely, objects of $\mathrm{Coker}(F)$ are those of $\mathbf{H}$ and an arrow $[f,N]\colon\ X \longrightarrow\!\!\!\!\!\bullet\ Y$ in $\mathrm{Coker}(F)$ is a 2-isomorphism class of 1-arrow in $\mathrm{Cok}(F)$, with N an object of $\mathbf{G}$ and $f\colon X \to Y \otimes F(N)$ an arrow in $\mathbf{H}$. Since $\mathrm{Cok}(F)$ is a bigroupoid, $\mathrm{Coker}(F)$ is a groupoid.

We can now define the functor P_F and the natural transformation π_F :

$$P_F\colon \mathbf{H} \to \mathrm{Coker}(F)$$

$$(f\colon X \to Y) \quad \mapsto \quad ([\ X \xrightarrow{f} Y \ \to Y \otimes I \to Y \otimes F(I), I]\colon\ X \longrightarrow\!\!\!\!\!\bullet\ Y) \ ;$$

for each object N of $\mathbf{G}$ we choose

$$\pi_F(N) = [F(N) \to I \otimes F(N), N]\colon P_F(F(N)) = \ F(N) \longrightarrow\!\!\!\!\!\bullet\ I \ .$$

It remains to introduce a monoidal structure on $\mathrm{Coker}(F)$:

- the tensor product of objects and the unit object in $\mathrm{Coker}(F)$ are those of $\mathbf{H}$;
- given two arrows $[f,N]\colon\ X \longrightarrow\!\!\!\!\!\bullet\ Y$ and $[g,M]\colon\ Z \longrightarrow\!\!\!\!\!\bullet\ V$ in $\mathrm{Coker}(F)$, their tensor product is $[f \star g, N \otimes M]\colon X \otimes\ Z \longrightarrow\!\!\!\!\!\bullet\ Y\ \otimes V$, where $f \star g$ is given (up to associativity) by

$$X \otimes\ Z \xrightarrow{f\otimes g} Y\ \otimes F(N) \otimes V \otimes\ F(M) \xrightarrow{1\otimes\gamma_{F(N),V}\otimes 1}$$

$$Y \otimes V \otimes F(N) \otimes F(M) \to Y \otimes V \otimes F(N \otimes M)$$

- the associativity, unit and commutativity contraints in $\mathrm{Coker}(F)$ are the image under P_F of the corresponding contraints in $\mathbf{H}$.

The objects of $\mathrm{Coker}(F)$ are invertible because they are invertible in $\mathbf{H}$. Clearly, P_F is a γ-morphism. The key to prove the universality of $\mathrm{Coker}(F)$ is the following lemma :

Lemma 2.2 *Let* $[f,N]\colon\ X \longrightarrow\!\!\!\!\!\bullet\,\, Y$ *be an arrow in Coker(F) and* $r_Y\colon Y\otimes I\to Y$ *the right-unit contraint in* $\mathbf{H}$ *; the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, "{[f,N]}"] \arrow[d, "P_F(f)"'] & Y \\
Y\otimes F(N) \arrow[r, "1\otimes\pi_F(N)"'] & Y\otimes I \arrow[u, "P_F(r_Y)"']
\end{tikzcd}$$

Proof of the universality : Define

$$G'\colon \mathrm{Coker}(F)\to\mathbf{K} \qquad ([f,N]\colon\ X\longrightarrow\!\!\!\!\!\bullet\,\, Y) \ \mapsto$$

$$(G(X)\xrightarrow{G(f)} G(Y\otimes F(N)) \ \to G(Y)\otimes\ G(F(N))\xrightarrow{1\otimes\varphi_N} G(Y)\ \otimes I\to G(Y))$$

and $\varphi'_X = 1_{G(X)}\colon G'(P_F(X)) = G(X)\to G(X)$. The natural transformation ψ is given by $\psi_X=\varphi''_X\colon G''(X)=G''(P_F(X))\to G(X)=G'(X)$; its naturality follows from the naturality of φ'' using the previous lemma and the commutativity of

$$\begin{tikzcd}
F\cdot P_F\cdot G'' \arrow[r, Rightarrow, "\pi_F\cdot G''"] \arrow[d, Rightarrow, "F\cdot\varphi''"'] & 0\cdot G'' \\
F\cdot G \arrow[r, Rightarrow, "\varphi"'] & 0 \arrow[u, Rightarrow, no head]
\end{tikzcd}$$

The uniqueness of ψ follows from the fact that the functor P_F is the identity on objects.

$\diamondsuit$

Remarks :

1) the only point where I have used a symmetry (and not only a braiding) is to prove that the commutativity contraint of $\mathbf{H}$ is natural also with respect to the arrows of Coker(F) ;

2) a problem, similar to those of the cokernel, is discussed in great detail in Shin's thesis [38], where a γ-monoidal functor $F\colon\mathbf{G}\to\mathbf{H}$ between symmetric monoidal categories is considered. Her construction is more complicated essentially because she does not assume the objects of $\mathbf{G}$ to be invertibles ;

3) in analogy with homotopy pull-backs and push-outs, the kernel and the cokernel just described should be called "standard" kernel and cokernel. This is because they satisfy the following additional universal property "of the first order" : (for the kernel) given G and φ as in the universal property of the kernel, there exists a unique morphism $G'\colon\mathbf{K}\to\mathrm{Ker}(F)$ such that $G'\cdot e_F=G$ and $G'\cdot\epsilon_F=\varphi$. In sections 3 and 6, we occasionally

use this fact, but only to simplify notations. On the contrary, to build up the Picard-Brauer sequence it is essential to use the universal property of the cokernel as a bilimit. In fact the first order property does not help in step 5.3 of section 6.

The next proposition shows in what sense the cokernel measures the "surjectivity" of a morphism ; once again, the proof uses proposition 1.1.

Lemma 2.3 *Consider a γ-morphism of symmetric cat-groups together with its cokernel*

$$\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{P_F} \mathrm{Coker}(F)$$

1) the factorization of $\pi_0(P_F)$ through the cokernel of $\pi_0(F)$ is an isomorphism ;

2) the factorization of $\pi_1(P_F)$ through the cokernel of $\pi_1(F)$ is injective.

(Observe that the factorization of $\pi_1(P_F)$ is surjective if P_F is full. But P_F is full iff $\pi_0(\mathbf{G}) = 0$.)

Proposition 2.2 *Consider a γ-morphism of symmetric cat-groups together with its cokernel*

$$\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{P_F} \mathrm{Coker}(F)$$

1) F is essentially surjective iff $\pi_0(Coker(F)) = 0$;

2) F is full iff $\pi_1(Coker(F)) = 0$;

3) F is full and essentially surjective iff Coker(F) is equivalent to **1**.

Putting together proposition 2.1 and proposition 2.2, we obtain the following proposition.

Proposition 2.3 *Let $F\colon \mathbf{G} \to \mathbf{H}$ be a γ-morphism of symmetric cat-groups ;*

1) F is full and essentially surjective iff $\pi_0(Ker(F)) = 0 = \pi_0(Coker(F))$;

2) F is full and faithful iff $\pi_1(Coker(F)) = 0 = \pi_1(Ker(F))$;

3) F is an equivalence iff Ker(F) and Coker(F) are equivalent to **1**.

Finally observe that, for any γ-morphism F between symmetric cat-groups, $\pi_0(\mathrm{Ker}(F))$ and $\pi_1(\mathrm{Coker}(F))$ are isomorphic groups.

3 Exact sequences

In this section we propose a notion of exactness for morphisms of cat-groups. For this, consider two morphisms of cat-groups and a natural transformation

$$\begin{tikzcd} & \mathbf{H} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\varphi"] & \\ \mathbf{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbf{K} \end{tikzcd}$$

From the universal property of the kernel of G, we obtain a morphism F' making commutative the following diagram

$$\begin{tikzcd} & \mathrm{Ker}(G) \arrow[dr, "e_G"] & \\ \mathbf{G} \arrow[ur, "F'"] \arrow[rr, "F"'] & & \mathbf{H} \end{tikzcd}$$

Definition 3.1 *with the previous notations, we say that the triple* (F, φ, G) *is 2-exact if* F' *is full and essentially surjective.*

If $\mathbf{G}$,$\mathbf{H}$ and $\mathbf{K}$ are symmetric and F and G are γ-morphisms, we can use proposition 2.2 and characterize exactness using the homology as follows :

(F, φ, G) is 2-exact iff $\mathrm{Coker}(F')$ is equivalent to $\mathbf{1}$.

Starting from a 2-exact sequence of cat-groups, we obtain two exact sequences of groups. In fact, we have :

Proposition 3.1 *If* (F, φ, G) *is a 2-exact sequence of cat-groups, then*

$$\pi_0(\,\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{G} \mathbf{K}) \text{ and } \pi_1(\,\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{G} \mathbf{K})$$

are exact sequences of groups.

More precisely :
- $\pi_0(\,\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{G} \mathbf{K})$ is exact iff for each object (X, λ_X) of $\mathrm{Ker}(G)$ there is an object Y of $\mathbf{G}$ such that $F(Y)$ and $e_G(X, \lambda_X)$ are isomorphic, and this clearly follows from the essential surjectivity of F' ;
- $\pi_1(\,\mathbf{G} \xrightarrow{F} \mathbf{H} \xrightarrow{G} \mathbf{K})$ is exact iff $\pi_1(F')$ is surjective, and this follows from the fullness of F'.

Examples 3.1 *1)*

$$\begin{tikzcd} & \mathbf{G} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\epsilon_F"] & \\ \mathrm{Ker}(F) \arrow[ur, "e_F"] \arrow[rr, "0"'] & {} & \mathbf{H} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbf{H} \arrow[dr, "P_F"] \arrow[d, phantom, "\Downarrow\pi_F"] & \\ \mathbf{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathrm{Coker}(F) \end{tikzcd}$$

are 2-exact ;

2) $\mathbf{1} \longrightarrow \mathbf{G} \xrightarrow{F} \mathbf{H}$ *is 2-exact iff F is full and faithful ;* $\mathbf{G} \xrightarrow{F} \mathbf{H} \longrightarrow \mathbf{1}$ *is 2-exact iff F is full and essentially surjective ;*

3) proposition 3.1 can not be inverted ; for a counter-example consider $\mathbf{1} \to \mathbf{1} \to \mathbb{Z}_2!$ *where* $\mathbb{Z}_2!$ *is the cat-group with one object and two arrows.*

4 Picard and Brauer cat-groups

The Picard cat-group : let $\mathbf{C}$ be a monoidal category ; the Picard cat-group $\mathcal{P}(\mathbf{C})$ of $\mathbf{C}$ is the subcategory of $\mathbf{C}$ given by invertible objects and isomorphisms between them. Clearly, $\mathcal{P}(\mathbf{C})$ is a cat-group, and it is braided (symmetric) if $\mathbf{C}$ is braided (symmetric). Any monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ restricts to a morphism $\mathcal{P}(F)\colon \mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D})$ (which is a γ-morphism if F is γ-monoidal). In this way we obtain a functor

$$\mathcal{P}\colon \text{Monoidal Categories} \to \text{Cat-groups}$$

which restricts to braided (symmetric) monoidal categories and braided (symmetric) cat-groups.

The Brauer cat-group : let $\mathbf{C}$ be a monoidal category with coequalizers stable under tensor product. The bicategory Bim$\mathbf{C}$ is defined in the following way:
- objects are monoids in $\mathbf{C}$;
- 1-arrows are bimodules ;
- 2-arrows are bimodule homomorphisms ;
- the 1-identity on a monoid A is A itself seen as an A-A-bimodule ;
- the 1-composition of two bimodules $M\colon A \longrightarrow\!\!\!\bullet\; B$, $N\colon B \longrightarrow\!\!\!\bullet\; C$ is the tensor product over B, that is the coequalizer of the arrows induced by B-actions

$$M \otimes B \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_B N$$

- 2-identities and vertical 2-composition are identities and composition in $\mathbf{C}$;
- horizontal 2-composition is induced by the universal property of the coequalizer $M \otimes_B N$.

We can use the classifying functor

$$\text{cl}\colon \text{Bicategories} \to \text{Categories}$$

and consider the category cl(Bim$\mathbf{C}$). If $\mathbf{C}$ is braided, then the monoidal structure on $\mathbf{C}$ induces a monoidal structure on cl(Bim$\mathbf{C}$). Applying the Picard functor $\mathcal{P}$ we define the Brauer cat-group

$$\mathcal{B}(\mathbf{C}) = \mathcal{P}(\text{cl}(\text{Bim}\mathbf{C}))$$

of the braided monoidal category $\mathbf{C}$. Observe that $\mathbf{C}$ braided does not imply cl(Bim$\mathbf{C}$) braided (because if A and B are monoids in $\mathbf{C}$, we can not prove that the braiding $\gamma_{A,B}\colon A \otimes B \to B \otimes A$ is a monoid homomorphism). But if $\mathbf{C}$ is symmetric, then also cl(Bim$\mathbf{C}$) (and then $\mathcal{B}(\mathbf{C})$) is symmetric.

Now consider another monoidal category $\mathbf{D}$ with stable coequalizers ; let $F\colon \mathbf{C} \to \mathbf{D}$ be a monoidal functor and assume that F preserves coequalizers. Then F induces functors $\mathrm{Bim}F\colon \mathrm{Bim}\mathbf{C} \to \mathrm{Bim}\mathbf{D}$ and $\mathrm{cl}(\mathrm{Bim}F)\colon \mathrm{cl}(\mathrm{Bim}\mathbf{C}) \to \mathrm{cl}(\mathrm{Bim}\mathbf{D})$. If moreover $\mathbf{C}$ and $\mathbf{D}$ are braided and F is γ-monoidal, then $\mathrm{cl}(\mathrm{Bim}F)$ is a γ-monoidal functor and then it induces a γ-morphism

$$\mathcal{P}(\mathrm{cl}(\mathrm{Bim}F))\colon \mathcal{P}(\mathrm{cl}(\mathrm{Bim}\mathbf{C})) \to \mathcal{P}(\mathrm{cl}(\mathrm{Bim}\mathbf{D}))$$

that is

$$\mathcal{B}(F)\colon \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D})\ .$$

In this way, we obtain a functor

$$\mathcal{B}\colon \text{Braided Monoidal Categories w.s.c.} \to \text{Cat-groups}$$

which restricts to symmetric monoidal categories and symmetric cat-groups.

Remark : the usual Picard group of a monoidal category results from the composition of the functors $\mathcal{P}$ and π_0

$$\text{Monoidal Categories} \xrightarrow{\mathcal{P}} \text{Cat} - \text{groups} \xrightarrow{\pi_0} \text{Groups}$$

In the same way, the Brauer group of a braided monoidal category with stable coequalizers is given by the composition

$$\text{Braided Monoidal Categories w.s.c.} \xrightarrow{\mathcal{B}} \text{Cat} - \text{groups} \xrightarrow{\pi_0} \text{Groups}$$

which restricts to symmetric monoidal categories and abelian groups. (The Brauer group of a braided monoidal category is studied also in [47]. The equivalence between our definition and that given in [47] is attested by propositions 1.2 and 1.3 in [52].)

Consider now the composition

$$\text{Monoidal Categories} \xrightarrow{\mathcal{P}} \text{Cat} - \text{groups} \xrightarrow{\pi_1} \text{Abelian Groups}$$

we can call the group $\pi_1(\mathcal{P}(\mathbf{C}))$ the group of units of $\mathbf{C}$. In fact, if $\mathbf{C}$ is the category of modules over a commutative unital ring R, then $\pi_1(\mathcal{P}(\mathbf{C}))$ is the group of units of R.

On the other hand, the composition

$$\text{Braided Monoidal Categories w.s.c.} \xrightarrow{\mathcal{B}} \text{Cat} - \text{groups} \xrightarrow{\pi_1} \text{Abelian Groups}$$

gives us once again the Picard group of a braided monoidal category with stable coequalizers.

5 The cat-group associated with a monoidal functor

In this section we build up a symmetric cat-group $\mathcal{F}$ starting from a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$. We assume that $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal categories with stable coequalizers ; we assume also that F is γ-monoidal and preserves coequalizers. We will proceed in two steps. In the first step, where the symmetry of $\mathbf{C}$ and $\mathbf{D}$ is not required, we define a bicategory $\mathbf{F}$. The construction of $\mathbf{F}$ is related to the bicategory of cylinders introduced in [3]. In the second step we provide the classifying category cl$\mathbf{F}$ of a monoidal structure and we define $\mathcal{F}$ to be $\mathcal{P}(\mathrm{cl}\mathbf{F})$.

First step : consider two monoidal categories $\mathbf{C}$ and $\mathbf{D}$ with stable coequalizers and let $F\colon \mathbf{C} \to \mathbf{D}$ be a monoidal functor which preserves coequalizers. The bicategory $\mathbf{F}$ is defined as follows :

- an object is a triple (A, X, B), where A and B are two monoids in $\mathbf{C}$ and $X\colon\ FA \longrightarrow\!\!\!\!\!\bullet\!\!> FB$ is a bimodule in $\mathbf{D}$;

- a 1-arrow from (A, X, B) to (C, Y, D) is a triple (M, f, N), with $M\colon\ A \longrightarrow\!\!\!\!\!\bullet\!\!> C$ and $N\colon\ B \longrightarrow\!\!\!\!\!\bullet\!\!> D$ two bimodules in $\mathbf{C}$ and $f\colon X \otimes_{FB} FN \to FM \otimes_{FC} Y$ a homomorphism of FA-FD-bimodules ;

- the composition of two 1-arrows

$$(A, X, B) \xrightarrow{(M,f,N)} (A', X', B') \xrightarrow{(P,g,Q)} (A'', X'', B'')$$

is given by

$$(M \otimes_{A'} P, f * g, N \otimes_{B'} Q)\ :\ (A, X, B) \to (A'', X'', B'')$$

where $f * g$ is given, up to associativity, by

$$X \otimes_{FB} F(N \otimes_{B'} Q) \longrightarrow X \otimes_{FB} FN \otimes_{FB'} FQ \xrightarrow{f\otimes 1} FM \otimes_{FA'} X' \otimes_{FB'} FQ$$

$$\xrightarrow{1\otimes g} FM \otimes_{FA'} FP \otimes_{FA''} X'' \longrightarrow F(M \otimes_{A'} P) \otimes_{FA''} X''$$

- 1-identities are $(A, x, B)\ :\ (A, X, B) \to (A, X, B)$ where $x\colon X \otimes_{FB} FB \to FA \otimes_{FA} X$ is the obvious isomorphism ;

- given two parallel 1-arrows $(M, f, N), (M', f', N')\ :\ (A, X, B) \to (C, Y, D)$, a 2-arrow $(\alpha, \beta)\ :\ (M, f, N) \Longrightarrow (M', f', N')$ is a pair (α, β) with $\alpha\colon M \to M'$ a morphism of A-C-bimodules and $\beta\colon N \to N'$ a morphism of B-D-bimodules such that the following diagram commutes

$$\begin{tikzcd}
X \otimes_{FB} FN \arrow[r, "f"] \arrow[d, "1\otimes F\beta"'] & FM \otimes_{FC} Y \arrow[d, "F\alpha\otimes 1"] \\
X \otimes_{FB} FN' \arrow[r, "f'"'] & FM' \otimes_{FC} Y
\end{tikzcd}$$

- vertical 2-composition and 2-identities are component-wise composition and identities in $\mathbf{C}$;

- horizontal 2-composition is component-wise tensor product of bimodule homomorphisms in $\mathbf{C}$;
- the bicategorical structure of $\mathbf{F}$ is completed by the canonical isomorphisms of the monoidal structure of $\mathbf{C}$.

Second step : we can apply the functor

$$\mathrm{cl}\colon \text{Bicategories} \to \text{Categories}$$

to the bicategory $\mathbf{F}$. Now we want to provide the category $\mathrm{cl}\mathbf{F}$ of a symmetric monoidal structure ; for this, we assume that $\mathbf{C}$ and $\mathbf{D}$ are symmetric and that F is γ-monoidal. The tensor product of two arrows of $\mathrm{cl}\mathbf{F}$

$$[M,f,N]\colon (A,X,B) \to (C,Y,D) \text{ and } [M',f',N']\colon (A',X',B') \to (C',Y',D')$$

(square brackets mean 2-isomorphism classes of 1-arrows in $\mathbf{F}$) is, by definition,

$$[M\otimes M', f@f', N\otimes N']\colon (A\otimes A', X\otimes X', B\otimes B') \to (C\otimes C', Y\otimes Y', D\otimes D')$$

where, for example, $F(A\otimes A')$ acts on $X\otimes X'$ in the following way

$$F(A\otimes A')\otimes X\otimes X' \to FA\otimes FA'\otimes X\otimes X' \xrightarrow{1\otimes\gamma\otimes 1} FA\otimes X\otimes FA'\otimes X' \to X\otimes X'$$

and $f@f'$ is given by

$$\begin{tikzcd}
(X\otimes X')\otimes_{F(B\otimes B')} F(N\otimes N') \arrow[r] & (X\otimes_{FB} FN)\otimes(X'\otimes_{FB'} FN') \arrow[d, "f\otimes f'"] \\
F(M\otimes M')\otimes_{F(C\otimes C')} (Y\otimes Y') & (FM\otimes_{FC} Y)\otimes(FM'\otimes_{FC'} Y') \arrow[l]
\end{tikzcd}$$

the unit object of $\mathrm{cl}\mathbf{F}$ is $(I_{\mathbf{C}}, I_{\mathbf{D}}, I_{\mathbf{C}})$. To complete the symmetric monoidal structure of $\mathrm{cl}\mathbf{F}$ (and to check the axioms) one uses the strict and faithful monoidal functor

$$\mathrm{Mon}\mathbf{C} \to \mathrm{cl}(\mathrm{Bim}\mathbf{C})$$

where $\mathrm{Mon}\mathbf{C}$ is the category of monoids and monoid homomorphisms in $\mathbf{C}$.

Now we can consider the functor

$$\mathcal{P}\colon \text{Monoidal Categories} \to \text{Cat-groups}$$

introduced in the previous section. We take as symmetric cat-group associated to the functor $F\colon \mathbf{C} \to \mathbf{D}$ the cat-group

$$\mathcal{F} = \mathcal{P}(\mathrm{cl}\mathbf{F}) \ .$$

The next proposition gives us the component-wise description of $\mathcal{F}$.

Proposition 5.1 *with the previous notations :*

1) a 2-arrow (α, β) in $\mathbf{F}$ is a 2-isomorphism iff α and β are isomorphisms in $\mathbf{C}$; when this is the case, $(\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1})$;

2) a 1-arrow (M, f, N) in $\mathbf{F}$ is an equivalence iff M and N are equivalences in $Bim\mathbf{C}$ and f is an isomorphism in $\mathbf{D}$; when this is the case, $(M, f, N)^{-1} = (M^{-1}, \tilde{f}, N^{-1})$, where $\tilde{f}$ is given, up to associativity, by

$$\begin{tikzcd}
Y \otimes_{FD} FN^{-1} \arrow[r] & FM^{-1} \otimes_{FA} FM \otimes_{FC} Y \otimes_{FD} FN^{-1} \arrow[d, "1 \otimes f^{-1} \otimes 1"] \\
FM^{-1} \otimes_{FA} X & FM^{-1} \otimes_{FA} X \otimes_{FB} FN \otimes_{FD} FN^{-1} \arrow[l]
\end{tikzcd}$$

3) an object (A, X, B) is invertible with respect to the tensor product of cl$\mathbf{F}$ iff A and B are invertible with respect to the tensor product of cl($Bim\mathbf{C}$) and X is an equivalence in $Bim\mathbf{D}$; when this is the case, a dual $(A, X, B)^*$ is given by $(A^*, (X^{-1})^*, B^*)$.

6 The Picard-Brauer exact sequence

Consider two symmetric monoidal categories with stable coequalizers and a γ-monoidal functor

$$F \colon \mathbf{C} \to \mathbf{D}$$

and assume that F preserves coequalizers. In this section we build up a sequence of symmetric cat-groups (steps 1 and 2)

$$\mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}) \to \mathcal{F} \to \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D})$$

This sequence is not 2-exact, in fact applying the functor

$$\pi_0 \colon \text{Symmetric Cat-groups} \to \text{Abelian Groups}$$

we obtain a sequence of abelian groups which is not exact at $\pi_0(\mathcal{F})$ (see section 2 in [52]). Then we replace $\mathcal{F}$ by a suitable quotient $\overline{\mathcal{F}}$ (step 3) and we obtain a new sequence of symmetric cat-groups (steps 4 and 5)

$$\mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}) \to \overline{\mathcal{F}} \to \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D})$$

which is 2-exact at $\mathcal{P}(\mathbf{D}), \overline{\mathcal{F}}$ and $\mathcal{B}(\mathbf{C})$.

Step 1 : The functor

$$F_1 \colon \mathbf{D} \to \mathrm{cl}\mathbf{F}$$

is defined as follows : if $f \colon X \to Y$ is in $\mathbf{D}$, then $F_1(f)$ is the 2-isomorphism class of the 1-arrow of $\mathbf{F}$

$$(I, F_1(f), I) \colon (I, X, I) \to (I, Y, I)$$

where X and Y are FI-FI-bimodules in the obvious way and $F_1(f)$ is given by

$$X \otimes_{FI} FI \rightarrow \; X \xrightarrow{f} Y \; \rightarrow FI \otimes_{FI} Y$$

The functor F_1 is γ-monoidal, so that we can apply

$$\mathcal{P} \colon \text{Symmetric Monoidal Categories} \rightarrow \text{Symmetric Cat-groups}$$

and we obtain a γ-morphism

$$\mathcal{P}F_1 \colon \mathcal{P}(\mathbf{D}) \rightarrow \mathcal{P}(\mathrm{cl}\mathbf{F}) = \mathcal{F} \; .$$

To define a γ-morphism

$$F_2 \colon \mathcal{F} \rightarrow \mathcal{B}(\mathbf{C})$$

we fix, for each object A of $\mathcal{B}(\mathbf{C})$, a duality $(A^* \dashv A, \eta_A, \epsilon_A)$. Now F_2 sends an arrow

$$[M, f, N] \colon (A, X, B) \rightarrow (C, Y, D)$$

of $\mathcal{F}$ on $M \otimes (N^{-1})^* \colon \; A \otimes B^* \longrightarrow\!\!\!\!\!\!\bullet\;\; C \otimes D^* \; .$

<u>Step 2 :</u> We dispose now of four γ-morphisms

$$\mathcal{P}(\mathbf{C}) \rightarrow \mathcal{P}(\mathbf{D}) \rightarrow \mathcal{F} \rightarrow \mathcal{B}(\mathbf{C}) \rightarrow \mathcal{B}(\mathbf{D})$$

and we can build up three natural transformations towards the zero-morphism.

<u>2.1 :</u>

$$\begin{tikzcd} & \mathcal{P}(\mathbf{D}) \arrow[dr, "\mathcal{P}F_1"] \arrow[d, phantom, "\Downarrow\Psi"] & \\ \mathcal{P}(\mathbf{C}) \arrow[ur, "\mathcal{P}F"] \arrow[rr, "0"'] & {} & \mathcal{F} \end{tikzcd}$$

for each object X of $\mathcal{P}(\mathbf{C})$, we define

$$\Psi_X = [X, \psi_X, I] \colon (I, FX, I) \rightarrow (I, I, I)$$

where ψ_X is the isomorphism in $\mathbf{D}$

$$FX \otimes_{FI} FI \rightarrow FX \otimes_{FI} I$$

induced by the transition $I \rightarrow FI$. Observe that Ψ is natural with respect to isomorphisms of $\mathbf{C}$, and not with respect to any arrow ; for this reason we can not work directly with

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{F_1} \mathrm{cl}\mathbf{F}$$

<u>2.2 :</u>

$$\begin{tikzcd} & \mathcal{F} \arrow[dr, "F_2"] \arrow[d, phantom, "\Downarrow\Phi"] & \\ \mathcal{P}(\mathbf{D}) \arrow[ur, "\mathcal{P}F_1"] \arrow[rr, "0"'] & {} & \mathcal{B}(\mathbf{C}) \end{tikzcd}$$

given an object X in $\mathcal{P}(\mathbf{D})$, we put $\Phi_X = {\eta_I}^{-1}\colon\ I \otimes I^* \longrightarrow\!\!\!\!\!\bullet\ I$

<u>2.3 :</u>

$$\begin{tikzcd} & \mathcal{B}(\mathbf{C}) \arrow[dr, "\mathcal{B}F"] \arrow[d, phantom, "\Downarrow\Sigma"] & \\ \mathcal{F} \arrow[ur, "F_2"] \arrow[rr, "0"'] & & \mathcal{B}(\mathbf{D}) \end{tikzcd}$$

for each object (A, X, B) in $\mathcal{F}$, we define $\Sigma_{(A,X,B)}$ as follows

$$\begin{tikzcd} F(A \otimes B^*) \arrow[r, "\bullet"] & FA \otimes FB^* \arrow[r, "X\otimes 1"] & FB \otimes FB^* \arrow[d, "\bullet"] \\ I & FI \arrow[l, "\bullet"] & F(B \otimes B^*) \arrow[l, "F{\eta_B}^{-1}"'] \end{tikzcd}$$

Observe that this definition is possible because, by proposition 5.1, the monoid B is an invertible object in cl(Bim$\mathbf{C}$).

<u>Step 3 :</u> To obtain a quotient $\overline{\mathcal{F}}$ of $\mathcal{F}$, consider the morphism of bicategories

$$\mathcal{I}\colon \mathrm{Bim}\mathbf{C} \to \mathbf{F}$$

which sends $M\colon\ A \longrightarrow\!\!\!\!\!\bullet\ B$ into $(M, m, M)\colon (A, FA, A) \to (B, FB, B)$ (where m is the isomorphism of FA-FB-bimodules $FA \otimes_{FA} FM \to FM \to FM \otimes_{FB} FB$) and sends a 2-arrow $\alpha\colon M \to N$ into the pair $(\alpha, \alpha)\colon (M, m, M) \to (N, n, N)$. It induces a γ-monoidal functor

$$\mathrm{cl}\mathcal{I}\colon \mathrm{cl}(\mathrm{Bim}\mathbf{C}) \to \mathrm{cl}\mathbf{F}\ .$$

Applying the functor $\mathcal{P}$, we obtain a γ-morphism

$$\mathcal{P}(\mathrm{cl}\mathcal{I})\colon \mathcal{B}(\mathbf{C}) \to \mathcal{F}$$

and we define $\overline{\mathcal{F}}$ as its cokernel

$$\begin{tikzcd} & \mathcal{F} \arrow[dr, "P_{\mathcal{I}}"] \arrow[d, phantom, "\Downarrow\pi_{\mathcal{I}}"] & \\ \mathcal{B}(\mathbf{C}) \arrow[ur, "\mathcal{P}(\mathrm{cl}\mathcal{I})"] \arrow[rr, "0"'] & & \overline{\mathcal{F}} = \mathrm{Coker}\mathcal{P}(\mathrm{cl}\mathcal{I}) \end{tikzcd}$$

<u>Step 4 :</u> We have a natural transformation

$$\begin{tikzcd} & \mathcal{F} \arrow[dr, "F_2"] \arrow[d, phantom, "\Downarrow\Delta"] & \\ \mathcal{B}(\mathbf{C}) \arrow[ur, "\mathcal{P}(\mathrm{cl}\mathcal{I})"] \arrow[rr, "0"'] & & \mathcal{B}(\mathbf{C}) \end{tikzcd}$$

given, for each object A of $\mathcal{B}(\mathbf{C})$, by $\Delta_A = \eta_A^{-1}\colon A \otimes A^* \longrightarrow I$. Following the universal property of the cokernel, we obtain a γ-morphism F_2' making commutative the following diagram

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "P_{\mathcal{I}}"'] \arrow[dr, "F_2"] & \\ \overline{\mathcal{F}} \arrow[rr, "F_2'"'] & & \mathcal{B}(\mathbf{C}) \end{tikzcd}$$

Step 5 : Once again, we dispose of four γ-morphisms of symmetric cat-groups

$$\mathcal{P}(\mathbf{C}) \xrightarrow{\mathcal{P}F} \mathcal{P}(\mathbf{D}) \xrightarrow{(\mathcal{P}F_1)\cdot P_{\mathcal{I}}} \overline{\mathcal{F}} \xrightarrow{F_2'} \mathcal{B}(\mathbf{C}) \xrightarrow{\mathcal{B}F} \mathcal{B}(\mathbf{D})$$

Using the natural transformations defined in step 2, we can build up three natural transformations towards the zero-morphism.
5.1 : by horizontal composition of natural transformations, we obtain

$$\begin{tikzcd} & \mathcal{P}(\mathbf{D}) \arrow[dr, "\mathcal{P}F_1\cdot P_{\mathcal{I}}"] \arrow[d, phantom, "\Downarrow \Psi\cdot P_{\mathcal{I}}"] & \\ \mathcal{P}(\mathbf{C}) \arrow[ur, "\mathcal{P}F"] \arrow[rr, "0"'] & {} & \overline{\mathcal{F}} \end{tikzcd}$$

5.2 : since $P_{\mathcal{I}} \cdot F_2' = F_2$, we have

$$\begin{tikzcd} & \overline{\mathcal{F}} \arrow[dr, "F_2'"] \arrow[d, phantom, "\Downarrow \Phi"] & \\ \mathcal{P}(\mathbf{D}) \arrow[ur, "(\mathcal{P}F_1)\cdot P_{\mathcal{I}}"] \arrow[rr, "0"'] & {} & \mathcal{B}(\mathbf{C}) \end{tikzcd}$$

5.3 : since $P_{\mathcal{I}} \cdot F_2' = F_2$, we have a natural transformation

$$\begin{tikzcd} & \overline{\mathcal{F}} \arrow[dr, "F_2'\cdot\mathcal{B}F"] \arrow[d, phantom, "\Downarrow \Sigma"] & \\ \mathcal{F} \arrow[ur, "P_{\mathcal{I}}"] \arrow[rr, "0"'] & {} & \mathcal{B}(\mathbf{C}) \end{tikzcd}$$

Moreover, the following diagram commutes

$$\begin{tikzcd} \mathcal{P}(\mathrm{cl}\mathcal{I}) \cdot P_{\mathcal{I}} \cdot F_2' \cdot \mathcal{B}F \arrow[r, Rightarrow, "\mathcal{P}(\mathrm{cl}\mathcal{I})\cdot\Sigma"] \arrow[d, Rightarrow, "\pi_{\mathcal{I}}\cdot F_2'\cdot\mathcal{B}F"'] & \mathcal{P}(\mathrm{cl}\mathcal{I}) \cdot 0 \arrow[d, Rightarrow] \\ 0 \cdot F_2' \cdot \mathcal{B}F & 0 \arrow[l, Rightarrow] \end{tikzcd}$$

Therefore, we can use the universal property of the cokernel $\overline{\mathcal{F}}$ to extend Σ to a natural transformation

$$\begin{tikzcd} & \mathcal{B}(\mathbf{C}) \arrow[dr, "\mathcal{B}F"] \arrow[d, phantom, "\overline{\Sigma}\Downarrow"] & \\ \overline{\mathcal{F}} \arrow[ur, "F_2'"] \arrow[rr, "0"'] & & \mathcal{B}(\mathbf{D}) \end{tikzcd}$$

We are ready to state our main result.

Proposition 6.1 *(with the previous notations) the sequence of symmetric cat-groups and γ-morphisms*

$$\mathcal{P}(\mathbf{C}) \xrightarrow{\mathcal{P}F} \mathcal{P}(\mathbf{D}) \xrightarrow{(\mathcal{P}F_1)\cdot P_{\mathcal{I}}} \overline{\mathcal{F}} \xrightarrow{F_2'} \mathcal{B}(\mathbf{C}) \xrightarrow{\mathcal{B}F} \mathcal{B}(\mathbf{D})$$

together with the natural transformations

$$\Psi \cdot P_{\mathcal{I}} \ , \ \Phi \ \text{ and } \ \overline{\Sigma}$$

is 2-exact in $\mathcal{P}(\mathbf{D})$, $\overline{\mathcal{F}}$ *and* $\mathcal{B}(\mathbf{C})$.

From this proposition and proposition 3.1, we obtain the following corollary.

Corollary 6.1 *the sequences*

$$\pi_0(\mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}) \to \overline{\mathcal{F}} \to \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D}))$$

and

$$\pi_1(\mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}) \to \overline{\mathcal{F}} \to \mathcal{B}(\mathbf{C}) \to \mathcal{B}(\mathbf{D}))$$

are exact sequences of abelian groups.

The first sequence of the previous corollary is the Picard-Brauer exact sequence studied in [25, 52]. As already quoted in the introduction, it contains, as particular cases, several classical exact sequences between groups built up "à la Brauer". The interested reader can refers to the bibliography in [25, 52]. From the remarks in section 4, we have that the second sequence of the previous corollary is the Unit-Picard exact sequence.

7 On the direct sum decomposition of the Brauer-Long group

Fix a commutative unital ring R and a commutative, cocommutative, finitely generated projective Hopf R-algebra H. In [2], M. Beattie established an isomorphism between the Brauer-Long group of H and the direct sum of the Brauer group of R and the Galois group of H

$$\mathrm{BL}(H) \simeq \mathrm{Br}(R) \oplus \mathrm{Gal}(H) \ .$$

In [11], S. Caenepeel established an isomorphism between the Picard group of H and the direct sum of the Picard group of R and the group of group-like elements of the dual Hopf algebra H^*

$$\mathrm{Pic}(H) \simeq \mathrm{Pic}(R) \oplus \mathrm{Gr}(H^*) \ .$$

There is a formal analogy between these two decompositions. We want to make this analogy a precise statement using, once again, categorical groups. We only sketch the argument, details on the algebraic arguments can be found in [2, 11, 16].

We write $\eta\colon R \to H,\ \mu\colon H \otimes_R H \to H$ for the R-algebra structure of H and $\epsilon\colon H \to R,\ \delta\colon H \to H \otimes_R H$ for its R-coalgebra structure. The category H-mod of left H-modules is a symmetric monoidal category, with tensor structure $(\otimes^\delta, R^\epsilon)$ induced by the coalgebra structure of H. There is a monoidal functor

$$i\colon R\text{-mod} \to H\text{-mod} \qquad X \mapsto (X,\ H \otimes_R X \xrightarrow{\epsilon\otimes 1} R \otimes_R X \ \simeq X)$$

which induces a morphism

$$\mathcal{B}(i)\colon \mathcal{B}(R\text{-mod}) \to \mathcal{B}(H\text{-mod}) \ .$$

Moreover, we can look at H as a comonoid in the monoidal category R-alg of R-algebras, and then we can consider the category H-comod-alg of H-comodules in R-alg. The category H-comod-alg is symmetric monoidal : if S and T are R-algebras with coactions $\rho_S\colon S \to S \otimes_R H,\ \rho_T\colon T \to T \otimes_R H$, their tensor product is given by the equalizer of the following pair

$$1\otimes\rho_T\colon S\otimes_R T \to S\otimes_R T\otimes_R H \qquad \rho_S\otimes 1\colon S\otimes_R T \to S\otimes_R H\otimes_R T \simeq S\otimes_R T\otimes_R H \ .$$

We can therefore consider the Picard cat-group $\mathcal{P}(H\text{-comod-alg})$ and its full sub-cat-group $\mathcal{G}(H)$ of Galois H-objects (recall that an object S of H-comod-alg is called a Galois H-object if it is a progenerator in the categry R-mod and if the morphism

$$S \otimes_R S \xrightarrow{1\otimes\rho_S} S \otimes_R S \otimes_R H \xrightarrow{\mu_S\otimes 1} S \otimes_R H$$

is an isomorphism, where μ_S is the multiplication of the R-algebra S). There is a morphism

$$j\colon \mathcal{G}(H) \to \mathcal{B}(H\text{-mod})$$

which sends a Galois H-object S on the smash product $S \bullet H^*$ with the dual Hopf R-algebra H^* (recall that any H-comodule is a H^*-module). Finally, we obtain a morphism

$$F\colon \mathcal{B}(R\text{-mod}) \times \mathcal{G}(H) \to \mathcal{B}(H\text{-mod}) \qquad F(X,S) = \mathcal{B}(i)(X) \otimes^\delta j(S) \ .$$

The next proposition is the categorical formulation of Beattie's and Caenepeel's theorems.

Proposition 7.1 *The morphism $F\colon \mathcal{B}(R\text{-}mod) \times \mathcal{G}(H) \rightarrow \mathcal{B}(H\text{-}mod)$ is an equivalence of symmetric cat-groups.*

Proof: Thanks to proposition 1.1, the situation here is much more easy than in the Pic-Br sequence. We have only to show that $\pi_0(F)$ and $\pi_1(F)$ are isomorphisms of abelian groups. But

$$\pi_0(\mathcal{B}(R\text{-mod}) \times \mathcal{G}(H)) = \pi_0(\mathcal{B}(R\text{-mod})) \times \pi_0(\mathcal{G}(H)) \simeq \mathrm{Br}(R) \times \mathrm{Gal}(H)$$

and $\pi_0(\mathcal{B}(H\text{-mod})) \simeq \mathrm{BL}(H)$, so that $\pi_0(F)$ is an isomorphism by Beattie's theorem. In the same way, $\pi_1(\mathcal{B}(R\text{-mod}) \times \mathcal{G}(H))$ is $\mathrm{Pic}(R) \times \mathrm{Gr}(H^*)$ (see [11] for the isomorphism between $\pi_1(\mathcal{G}(H))$ and $\mathrm{Gr}(H^*)$) and $\pi_1(\mathcal{B}(H\text{-mod}))$ is $\mathrm{Pic}(H)$, so that $\pi_1(F)$ is an isomorphism by Caenepeel's theorem.

$\diamondsuit$

The previous proposition holds if H is a commutative, cocommutative, finitely generated projective Hopf algebra in a symmetric monoidal closed category with equalizers and coequalizers. This is because both Beattie's and Caenepeel's theorems have been generalized in [23] to a Hopf algebra in a closed category.

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Basic facts on bicategories, monoidal categories and duality are in [3, 21, 30, 35, 40].
The origins of cat-groups in algebraic geometry and ring theory come back to [26, 27, 37, 51, 53].
General results about cat-groups can be found in [24, 28, 29, 31, 38, 41].

The coherence problem for cat-groups is studied in [24, 28, 31, 39, 43, 45].
The relation between cat-groups and crossed modules is explained and used in [6, 7, 10, 12, 13, 14, 17, 28, 33, 43].
Cat-groups are of interest in ring theory [12, 15, 19, 20, 41, 42, 43, 44, 46, 51] ;
in group cohomology [5, 9, 12, 13, 14, 15, 17, 28, 32, 33, 38, 46] ;
and in algebraic topology (classification of homotopy types) [8, 10, 12, 13, 14, 33].

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