A CLASSIFICATION OF GEOMETRIC MORPHISMS AND LOCALIZATIONS FOR PRESHEAF CATEGORIES AND ALGEBRAIC CATEGORIES

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ABSTRACT. We give necessary and sufficient conditions on a functor $k\colon \mathcal{C} \to \mathcal{E}$, where $\mathcal{C}$ is an algebraic theory, in order to the induced functor $\mathcal{E}(k-,-)\colon \mathcal{E} \to \mathrm{Alg}(\mathcal{C})$ being a geometric morphism or a localization. We apply our techniques also to the particular case of module categories and to the case of presheaf categories.

1. Introduction

Let $\mathcal{E}, \mathcal{A}$ be categories with finite limits. A *geometric morphism* from $\mathcal{E}$ to $\mathcal{A}$ is a functor $R\colon \mathcal{E} \to \mathcal{A}$ having a left exact left adjoint $L\colon \mathcal{A} \to \mathcal{E}$. A *localization* of $\mathcal{A}$ is a full and faithful geometric morphism $R\colon \mathcal{E} \to \mathcal{A}$.

Localizations have been intensively studied in the case of $\mathcal{A}$ being a presheaf topos, a module category, or an algebraic category. In all these settings, localizations have been *characterized*: Giraud's theorem establishes that localizations of presheaf toposes are precisely Grothendieck toposes (see Theorem 1 in the Appendix of [7]); Gabriel-Popescu's theorem establishes that localizations of module categories are precisely abelian Grothendieck categories (see [8]); localizations of algebraic categories have been characterized in [9] as those cocomplete exact categories having a regular generator and exact filtered colimits.

The problem of *classifying* geometric morphisms and, in particular, localizations is a slightly different matter. Let us explain it in the context of presheaf categories. Given a functor $k\colon \mathcal{C} \to \mathcal{E}$, where $\mathcal{C}$ is a small category and $\mathcal{E}$ is a cocomplete category, we get an adjunction

$$\mathcal{E} \underset{\mathcal{E}(k-,-)}{\overset{\mathrm{Lan}_{Y_\mathcal{C}}(k)}{\rightleftarrows}} [\mathcal{C}^{op}, \mathrm{Set}] \qquad \mathrm{Lan}_{Y_\mathcal{C}}(k) \dashv \mathcal{E}(k-,-)$$

where $Y_\mathcal{C}\colon \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Set}]$ is the Yoneda embedding and $\mathrm{Lan}_{Y_\mathcal{C}}(k)$ is the left Kan extension of k along $Y_\mathcal{C}$. This produces a functor

$$[\mathcal{C}, \mathcal{E}] \to \mathrm{Adj}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]]$$

where the objects of $\mathrm{Adj}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]]$ are adjoint pairs

$$\mathcal{E} \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{C}^{op}, \mathrm{Set}] \qquad L \dashv R$$

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and the morphisms are the natural transformations between left adjoint functors. Such a functor has a right adjoint

$$\mathrm{Adj}[\mathcal{E},[\mathcal{C}^{op},\mathrm{Set}]] \to [\mathcal{C},\mathcal{E}]$$

which sends an adjoint pair $(L \dashv R)$ to the composite functor $L \cdot Y_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{E}$. Moreover, since the functor $Y_{\mathcal{C}}$ is full, faithful and dense, this is in fact an equivalence

$$\mathrm{Adj}[\mathcal{E},[\mathcal{C}^{op},\mathrm{Set}]] \simeq [\mathcal{C},\mathcal{E}]$$

Therefore, the classification problem for geometric morphisms amounts to restricting the previous equivalence to the full subcategory of $\mathrm{Adj}[\mathcal{E},[\mathcal{C}^{op},\mathrm{Set}]]$ given by geometric morphisms, and it can be stated in the following terms: find necessary and sufficient conditions on a functor $k \colon \mathcal{C} \to \mathcal{E}$ in order to its left Kan extension along Yoneda

$$\mathrm{Lan}_{Y_{\mathcal{C}}}(k) \colon [\mathcal{C}^{op},\mathrm{Set}] \to \mathcal{E}$$

being left exact (and the right adjoint

$$\mathcal{E}(k-,-) \colon \mathcal{E} \to [\mathcal{C}^{op},\mathrm{Set}]$$

being full and faithful, if we wish to classify localizations).

The matter of classifying geometric morphisms into a presheaf topos has been solved in terms of *filtering functors*: the functor $\mathcal{E}(k-,-) \colon \mathcal{E} \to [\mathcal{C}^{op},\mathrm{Set}]$ is a geometric morphism iff $k \colon \mathcal{C} \to \mathcal{E}$ is a filtering functor (see Theorem VII.9.1 in [7], or [3] for a quite different proof). The analogous problem of classifying localizations of the form

$$\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} R\text{-mod}$$

where R is a ring with unit (or, more in general, a small preadditive category) and $\mathcal{E}$ is a Grothendieck category, has been recently solved by Lowen using sheaf theoretical techniques, see [6]. The aim of our paper is to complete the picture. In Section 2 we classify localizations of the form

$$\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} [\mathcal{C}^{op},\mathrm{Set}]$$

We start with presheaf categories because in this case the classification of geometric morphisms in terms of filtering functors is well known, so that it is just a matter of refining the notion of filtering functor to get the classification of localizations. Our proof is quite similar to the one of Lowen for localizations of module categories (see also the comparison lemma in [5]). The case of algebraic categories is more delicate, and it needs some preliminaries on left covering functors. This is the subject of Section 3. Then, in Section 4 we classify geometric morphisms and localizations of the form

$$\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} \mathrm{Alg}(\mathcal{C})$$

where $\mathrm{Alg}(\mathcal{C})$ is the category of algebras for an algebraic theory $\mathcal{C}$. In Section 5 we specialize the result of Section 4 to get a classification of geometric morphisms into a module category.

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2. Localizations of presheaf categories

Throughout the section, let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a cocomplete, exact and extensive category. We refer to Chapter 2 in [1] for the notion of exact category, and to [2, 4] for the notion of extensive category (we underline the need of the infinitary version of extensivity, as in [4]. For a functor $k\colon \mathcal{C} \to \mathcal{E}$, we write

$$k! = \mathrm{Lan}_{Y_{\mathcal{C}}}(k)\colon [\mathcal{C}^{op}, \mathrm{Set}] \to \mathcal{E} \qquad k^* = \mathcal{E}(k-,-)\colon \mathcal{E} \to [\mathcal{C}^{op}, \mathrm{Set}]$$

so that we have $k! \dashv k^*$.

We recall from [7, 3] the definition of filtering functor and the classification of geometric morphisms into a presheaf category.

2.1. Definition.

1. An *epimorphic family* in $\mathcal{E}$ is a collection $\{f_i\colon X_i \to X\}_{i\in I}$ of arrows in $\mathcal{E}$ such that for any pair $u, v\colon X \to Y$, if $u \cdot f_i = v \cdot f_i$ for all $i \in I$, then $u = v$. (Equivalently, such that the induced arrow $\coprod X_i \to X$ is an epimorphism.)

2. A functor $k\colon \mathcal{C} \to \mathcal{E}$ is said to be *filtering* if:

   (F1) the family of arrows $\{kC \to 1 \mid C \in \mathcal{C}\}$, where 1 is a terminal object of $\mathcal{E}$, is epimorphic;

   (F2) for any pair of objects $A, B \in \mathcal{C}$, the family of arrows

   $$\{\langle ku, kv\rangle\colon kC \to kA \times kB \mid \ A \xleftarrow{u} C \xrightarrow{v} B \ \text{ in } \mathcal{C}\}$$

   is epimorphic;

   (F3) for any pair of arrows $u, v\colon A \rightrightarrows B$ in $\mathcal{C}$, the family of arrows

   $$\{w'\colon kC \to E_{u,v} \mid w\colon C \to A \text{ in } \mathcal{C} \text{ such that } u \cdot w = v \cdot w\}$$

   where $e\colon E_{u,v} \to kA$ is an equalizer of (u, v) and $e \cdot w' = kw$, is epimorphic.

2.2. Theorem. *Consider a functor $k\colon \mathcal{C} \to \mathcal{E}$. The following conditions are equivalent:*

1. *$k^*\colon \mathcal{E} \to [\mathcal{C}^{op}, \mathrm{Set}]$ is a geometric morphism;*

2. *$k\colon \mathcal{C} \to \mathcal{E}$ is filtering.*

Let us reformulate the previous result in terms of an equivalence of categories. We denote by $\mathrm{GeoMor}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]]$ the category of geometric morphisms from $\mathcal{E}$ to $[\mathcal{C}^{op}, \mathrm{Set}]$, and geometric transformations between these, that is, natural transformations between the left adjoint functors. We denote by $\mathrm{Filt}[\mathcal{C}, \mathcal{E}]$ the category of filtering functors from $\mathcal{C}$ to $\mathcal{E}$ and natural transformations.

2.3. Corollary. *The equivalence of categories*

$$\mathrm{Adj}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]] \simeq [\mathcal{C}, \mathcal{E}]$$

restricts to an equivalence

$$\mathrm{GeoMor}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]] \simeq \mathrm{Filt}[\mathcal{C}, \mathcal{E}]$$

Our aim is now to refine the previous result in order to achieve a classification of localizations. We need a technical lemma, which adjusts to our context Lemma 3.4 in [6].

2.4. Lemma. *Let $k\colon \mathcal{C} \to \mathcal{E}$ be a functor, and consider an object $C \in \mathcal{C}$, and a subfunctor $r\colon R \to \mathcal{C}(-, C)$. The following conditions are equivalent:*

1. *$k!r$ is an epimorphism;*
2. *The family $kR = \{kd\colon kD \to kC \mid d \in RD\,,\ D \in \mathcal{C}\}$ is epimorphic.*

Proof. Since $r\colon R \to \mathcal{C}(-, C)$ is a subfunctor, for any arrow $d\colon D \to C$ in $R(D)$ the natural transformation $\mathcal{C}(-, d)\colon \mathcal{C}(-, D) \to \mathcal{C}(-, C)$ factors through r. Therefore, the universal property of the coproduct allows us to construct the following commutative diagram

$$\begin{tikzcd}
 & R \arrow[dr, "r"] & \\
\coprod_{d\in R(D), D\in\mathcal{C}} \mathcal{C}(-, D) \arrow[ur, "e"] \arrow[rr, "r'"] & & \mathcal{C}(-, C) \\
 & \mathcal{C}(-, D) \arrow[ul, "\rho_d"] \arrow[ur, "\mathcal{C}(-,d)"'] &
\end{tikzcd}$$

where ρ_d denotes the coproduct injection. Moreover, since the (regular epi-mono) factorization of an arrow in $[\mathcal{C}^{op}, \mathrm{Set}]$ is computed pointwise in Set, the arrow e is a regular epimorphism. When applying to the previous diagram the left adjoint $k!$, we get the following commutative diagram, where $k!\rho_d$ is the coproduct injection and $k!e$ is a regular epimorphism

$$\begin{tikzcd}
 & k!R \arrow[dr, "k!r"] & \\
\coprod_{d\in R(D), D\in\mathcal{C}} kD \arrow[ur, "k!e"] \arrow[rr, "k!r'"] & & kC \\
 & kD \arrow[ul, "k!\rho_d"] \arrow[ur, "kd"'] &
\end{tikzcd}$$

Finally, $k!r$ is an epimorphism iff $k!r'$ is an epimorphism iff the family

$$kR = \{kd\colon kD \to kC \mid d \in RD\,,\ D \in \mathcal{C}\}$$

is epimorphic. ∎

We list here the conditions on $k\colon \mathcal{C} \to \mathcal{E}$ which allow k^* to realize a localization.

2.5. Definition. A functor $k\colon \mathcal{C} \to \mathcal{E}$ is said to be *fully filtering* if

(A) For any object $X \in \mathcal{E}$, the family of arrows $R_X = \{c\colon kC \to X \mid C \in \mathcal{C}\}$ is epimorphic;

(B) For any pair of arrows $kA \xleftarrow{a} X \xrightarrow{b} kB$ in $\mathcal{E}$, the family of arrows

$$R_{a,b} = \{c\colon kC \to X \mid a \cdot c = kf_A\,,\ b \cdot c = kf_B \text{ for some } A \xleftarrow{f_A} C \xrightarrow{f_B} B \text{ in } \mathcal{C}\}$$

is epimorphic;

(F3) As in Definition 2.1.

2.6. Remark. The choice of terminology in the previous definition is justified by the fact that condition (A) implies condition (F1) and condition (B) implies condition (F2), so that any fully filtering functor is indeed filtering.

We are finally able to state and prove the main result of this section.

2.7. Proposition. *Let $k\colon \mathcal{C} \to \mathcal{E}$ be a functor. The following conditions are equivalent:*

1. $k^*\colon \mathcal{E} \to [\mathcal{C}^{op}, \mathrm{Set}]$ *is a localization;*
2. $k\colon \mathcal{C} \to \mathcal{E}$ *is fully filtering.*

Proof. 1 ⇒ 2. Condition (A). Let X be an object in $\mathcal{E}$; if $u, v\colon X \rightrightarrows Y$ are such that $u \cdot c = v \cdot c$ for any $c \in R_X = \{c\colon kC \to X \mid C \in \mathcal{C}\}$, then the natural transformations k^*u and k^*v are equal. Since k^* is faithful, this implies $u = v$, so that R_X is epimorphic. Condition (B). Let $a\colon X \to kA, b\colon X \to kB$ be two arrows in $\mathcal{E}$; we prove that the family

$$R_{a,b} = \{d\colon kD \to X \mid a \cdot d = kf_A\,,\ b \cdot d = kf_B \text{ for some } A \xleftarrow{f_A} D \xrightarrow{f_B} B \text{ in } \mathcal{C}\}$$

is epimorphic. For it, fix an arrow $c\colon kC \to X$ and consider the following diagram

$$\begin{tikzcd}
& P_{a\cdot c} \arrow[rr] \arrow[dl] \arrow[dd, "p"'] & & \mathcal{C}(-,A) \arrow[dd, "\eta_A"] \\
R_{a\cdot c} \arrow[dr, "r"'] & & & \\
& \mathcal{C}(-,C) \arrow[r, "\eta_C"'] & \mathcal{E}(k-,kC) \arrow[r, "\mathcal{E}(k-,a\cdot c)"'] & \mathcal{E}(k-,kA)
\end{tikzcd}$$

where the rectangle is a pullback, the triangle is the (regular epi-mono) factorization, and η is the unit of the adjunction $k! \dashv k^*$. Explicitly,

$$R_{a\cdot c} = \{x \colon C_x \to C \mid a \cdot c \cdot kx = kf_x \text{ for some } f_x \colon C_x \to A\}$$

Since k^* is full and faithful, the counit of the adjunction $k! \dashv k^*$ is an isomorphism. Because of the triangular identities, also $k!\eta_A$ is an isomorphism. Since $k!$ is left exact, this implies that $k!p$ is an isomorphism, and then $k!r$ is an epimorphism. By Lemma 2.4, this means that the family $kR_{a\cdot c} = \{kx \colon kC_x \to kC \mid x \in R_{a\cdot c}\}$ is epimorphic. Fix now a morphism $x \colon C_x \to C$ in $R_{a\cdot c}$; using the previous argument, we obtain a family

$$R_{b\cdot c\cdot kx} = \{y_x \colon C_{y_x} \to C_x \mid b \cdot c \cdot kx \cdot ky_x = kf_{y_x} \text{ for some} f_{y_x} \colon C_{y_x} \to B\}$$

such that $kR_{b\cdot c\cdot kx} = \{ky_x \colon kC_{y_x} \to kC_x \mid y_x \in R_{b\cdot c\cdot kx}\}$ is epimorphic. Pasting together R_X (which is epimorphic by condition (A)), $kR_{a\cdot c}$ and $kR_{b\cdot c\cdot kx}$, we get a new epimorphic family

$$M_{a,b} = \{c \cdot kx \cdot ky_x \mid y_x \in R_{b\cdot c\cdot kx}, x \in R_{a\cdot c}, c \in R_X\}$$

Moreover, by definition of $R_{b\cdot c\cdot kx}$ and $R_{a\cdot c}$, the collection $M_{a,b}$ is contained in $R_{a,b}$, so that also $R_{a,b}$ is epimorphic.

2 $\Rightarrow$ 1. We prove first that if $k \colon \mathcal{C} \to \mathcal{E}$ satisfies conditions (A) and (B), then k^* is full and faithful. Let X, Y be objects in $\mathcal{E}$ and $\alpha \colon k^*X \to k^*Y$ an arrow in $[\mathcal{C}^{op}, \mathrm{Set}]$. By condition (A), the family $R_X = \{h \colon kC \to X \mid C \in \mathcal{C}\}$ is epimorphic. This means that the canonical arrow λ induced by the arrows $h \in R_X$ via the universal property of the coproduct

$$\begin{tikzcd} \coprod_{h\in R_X} kC \arrow[r, "\lambda"] & X \\ kC \arrow[u, "\sigma_h"] \arrow[ur, "h"'] & \end{tikzcd}$$

is an epimorphism (we denote by σ_h the coproduct injection). Since $\mathcal{E}$ is exact and extensive, any epimorphism is a regular epimorphism (Lemma 4.7 in [4]), so that λ is the coequalizer of its kernel pair $\lambda_0, \lambda_1 \colon N(\lambda) \rightrightarrows \coprod kC$. On the other hand, for any $h \in R_X$, we have an arrow $\alpha_C(h) \colon kC \to Y$, and therefore a canonical morphism μ from the coproduct

$$\begin{tikzcd} \coprod_{h\in R_X} kC \arrow[r, "\mu"] & Y \\ kC \arrow[u, "\sigma_h"] \arrow[ur, "\alpha_C(h)"'] & \end{tikzcd}$$

It suffices to prove then that the arrow μ coequalizes λ_0 and λ_1, in order to get a unique arrow $b \colon X \to Y$ such that $b \cdot \lambda = \mu$, that is a unique arrow b such that $k^*b = \alpha$, as desired. For any pair $h, h' \in R_X$, consider the following diagram, where the outer square is a pullback and the dotted arrow is the canonical factorization

$$\begin{tikzcd}
P(h,h') \arrow[rr, "p_h"] \arrow[dd, "p_{h'}"'] \arrow[dr, dotted, "s_{h,h'}"'] & & kC \arrow[d, "\sigma_h"] \\
& N(\lambda) \arrow[r, "\lambda_0"] \arrow[d, "\lambda_1"'] & \coprod kC \arrow[d, "\lambda"] \\
kC' \arrow[r, "\sigma_{h'}"'] & \coprod kC \arrow[r, "\lambda"'] & X
\end{tikzcd}$$

By extensivity of $\mathcal{E}$, the diagram $\langle s_{h,h'} \colon P(h,h') \to N(\lambda) \mid h, h' \in R_X \rangle$ is a coproduct, so that in order to check the equation $\mu \cdot \lambda_0 = \mu \cdot \lambda_1$ we just have to pre-compose with all the $s_{h,h'}$. By condition (B), the family

$$R_{p_h, p_{h'}} = \{d \colon kD \to P(h,h') \mid p_h \cdot d = kf \ , \ p_{h'} \cdot d = kf' \text{ for some } C \xleftarrow{f} D \xrightarrow{f'} C' \text{ in } \mathcal{C}\}$$

is epimorphic, then to verify the equation $\mu \cdot \lambda_0 \cdot s_{h,h'} = \mu \cdot \lambda_1 \cdot s_{h,h'}$ it is enough to pre-compose with all the $d \in R_{p_h, p_{h'}}$. Finally, using the naturality of α, we have
$\mu \cdot \lambda_0 \cdot s_{h,h'} \cdot d = \mu \cdot \sigma_h \cdot p_h \cdot d = \mu \cdot \sigma_h \cdot kf = \alpha_C(h) \cdot kf = \alpha_D(h \cdot kf) = \alpha_D(h \cdot p_h \cdot d) = \alpha_D(h' \cdot p_{h'} \cdot d) = \alpha_D(h' \cdot kf') = \alpha_{C'}(h') \cdot kf' = \mu \cdot \sigma_{h'} \cdot kf' = \mu \cdot \sigma_{h'} \cdot p_{h'} \cdot d = \mu \cdot \lambda_1 \cdot s_{h,h'} \cdot d.$ ■

The previous result can be reformulated in terms of an equivalence of categories. We denote by $\mathrm{Loc}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]]$ the full subcategory of $\mathrm{GeoMor}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]]$ given by the localizations. We denote by $\mathrm{FullyFilt}[\mathcal{C}, \mathcal{E}]$ the full subcategory of $\mathrm{Filt}[\mathcal{C}, \mathcal{E}]$ given by the fully filtering functors.

2.8. Corollary. *The equivalence of categories*

$$\mathrm{Adj}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]] \simeq [\mathcal{C}, \mathcal{E}]$$

restricts to an equivalence

$$\mathrm{Loc}[\mathcal{E}, [\mathcal{C}^{op}, \mathrm{Set}]] \simeq \mathrm{FullyFilt}[\mathcal{C}, \mathcal{E}]$$

3. Left covering functors on categories of free algebras

Throughout the section, let $\mathcal{C} = \langle 0, T, 2T, \ldots, nT, \ldots \rangle$ be an algebraic theory (we refer to Chapter 3 in [1]). We denote by $\mathrm{Alg}(\mathcal{C})$ the category of finite product preserving functors $\mathcal{C}^{op} \to \mathrm{Set}$, and by $\mathcal{F}(\mathcal{C})$ the category of free algebras, which is (equivalent to) a full subcategory of $\mathrm{Alg}(\mathcal{C})$. The full embedding $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{F}(\mathcal{C})$, assigning to any nT in $\mathcal{C}$ the free algebra $\mathcal{C}(-, nT)$, satisfies the following properties.

3.1. Lemma.

1. *$\mathcal{F}(\mathcal{C})$ has coproducts and $\iota_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{F}(\mathcal{C})$ preserves finite coproducts.*

2. *If $\mathcal{E}$ has coproducts and the functor $k\colon \mathcal{C} \to \mathcal{E}$ preserves finite coproducts, then there is an essentially unique coproduct-preserving functor $k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ such that $k' \cdot \iota_{\mathcal{C}} \simeq k$.*

3. *The coproduct-preserving extension k' is the left Kan extension of k along $\iota_{\mathcal{C}}$.*

Proof. 1. Obvious.
2. The functor $k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ is defined as follows: for a set X, we define $k'X = \coprod_X kT$. Let X and Y be sets, a morphism $f\colon X \to Y$ in $\mathcal{F}(\mathcal{C})$ is a morphism $f\colon \coprod_X \mathcal{C}(-,T) \to \coprod_Y \mathcal{C}(-,T)$ in $\mathrm{Alg}(\mathcal{C})$. Assume Y is an infinite set. Since Y is the filtered colimit of its finite subsets and $\mathcal{C}(-,T)$ is a finitely presentable object in $\mathrm{Alg}(\mathcal{C})$, for any $x \in X$ there exist a finite subset S of Y and a morphism $f_x\colon \mathcal{C}(-,T) \to \coprod_S \mathcal{C}(-,T)$ making commutative the following diagram

$$\begin{tikzcd}
\coprod_X \mathcal{C}(-,T) \arrow[r, "f"] & \coprod_Y \mathcal{C}(-,T) \\
\mathcal{C}(-,T) \arrow[u, "\rho_x"] \arrow[r, "f_x"'] & \coprod_S \mathcal{C}(-,T) \arrow[u, "j_S"']
\end{tikzcd}$$

where ρ_x is the coproduct injection and j_S is induced by the inclusion $S \subset Y$. By Yoneda lemma, the natural transformation $f_x\colon \mathcal{C}(-,T) \to \coprod_S \mathcal{C}(-,T) = \mathcal{C}(-,sT)$ (where s is the cardinality of S) corresponds to a unique arrow $f_x\colon T \to sT$ in $\mathcal{C}$. Therefore, we can define $k'f$ as the unique arrow such that the following diagram

$$\begin{tikzcd}
\coprod_X kT \arrow[r, "k'f"] & \coprod_Y kT \\
kT \arrow[u, "\sigma_x"] \arrow[r, "kf_x"'] & k(sT) = \coprod_S kT \arrow[u, "j_S"']
\end{tikzcd}$$

commutes, for any $x \in X$, where σ_x in the coproduct injection. We have to show that the definition of k' does not depend on the choice of the factorization $j_S \cdot f_x$ for $f \cdot \rho_x$. Suppose $f'_x \cdot j_{S'}$ is another such a factorization and consider the union $S \cup S'$, then we have

$$j_{S\cup S'} \cdot u_S \cdot f_x = j_S \cdot f_x = f \cdot \rho_x = j_{S'} \cdot f'_x = j_{S\cup S'} \cdot u_{S'} \cdot f'_x$$

where $u_S\colon \coprod_S \mathcal{C}(-,T) \to \coprod_{S\cup S'} \mathcal{C}(-,T)$ is the arrow induced by the inclusion $S \subset S \cup S'$, and analogously for $u_{S'}$. Since $j_{S\cup S'}$ is a monomorphism (it is induced by the inclusion of $S \cup S'$ into Y, which is a split monomorphism), we have $u_S \cdot f_x = u_{S'} \cdot f'_x$, so that

$$j_S \cdot kf_x = j_{S\cup S'} \cdot ku_S \cdot kf_x = j_{S\cup S'} \cdot ku_{S'} \cdot kf'_x = j_{S'} \cdot kf'_x$$

The rest of the proof is straightforward.
3. Consider a functor $G\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ and a natural transformation $\gamma\colon k \Rightarrow G \cdot \iota_{\mathcal{C}}$. We determine a unique natural transformation $\alpha\colon k' \Rightarrow G$ such that $\alpha \cdot \iota_{\mathcal{C}} = \gamma$. Given a set X, we define α_X as the unique arrow such that the following diagram

$$\begin{tikzcd}
\coprod_X kT \arrow[r, "\alpha_X"] & G(\coprod_X \mathcal{C}(-,T)) \\
kT \arrow[u, "\sigma_x"] \arrow[r, "\gamma_T"'] & G(\mathcal{C}(-,T)) \arrow[u, "G\rho_x"']
\end{tikzcd}$$

commutes, for any $x \in X$. The condition $\alpha \cdot \iota_{\mathcal{C}} = \gamma$ is obviously satisfied; the naturality of α and its uniqueness can be checked using the naturality of γ and the factorisation of an arrow f in $\mathcal{F}(\mathcal{C})$ as $f \cdot \rho_x = j_S \cdot f_x$, as in the proof of part 2. ∎

3.2. Remark. Because of its uniqueness, the coproduct-preserving extension $k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ of Lemma 3.1 coincides with the restriction of the Kan extension $k!\colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{E}$ to free algebras.

Recall that a *weak limit* on a diagram is defined as a limit, except that the factorization involved in its universal property is not necessarily unique. We are interested in weak limits because the category $\mathcal{F}(\mathcal{C})$, in general, fails to have limits, but it has weak limits. In fact, since free algebras are regular projective objects in $\mathrm{Alg}(\mathcal{C})$ and any algebra is in a canonical way a regular quotient of a free one, to construct a weak limit in $\mathcal{F}(\mathcal{C})$ one has just to construct the corresponding limit in $\mathrm{Alg}(\mathcal{C})$ and then to cover it with a free algebra. The functors which behave well with respect to weak finite limits are the left covering ones. Let us recall the definition from [3, 4].

3.3. Definition. Consider a category $\mathcal{W}$ with weak finite limits, an exact category $\mathcal{E}$ and a functor $K\colon \mathcal{W} \to \mathcal{E}$. The functor K is said to be *left covering* if, for any finite diagram $\mathcal{D}$ in $\mathcal{W}$ and for any (equivalently, for one) weak limit W on $\mathcal{D}$, the canonical arrow from KW to the limit of $K(\mathcal{D})$ is a regular epimorphism.

3.4. Lemma. *Consider a cocomplete, exact category $\mathcal{E}$ with exact filtered colimits, a finite coproduct-preserving functor $k\colon \mathcal{C} \to \mathcal{E}$, and its coproduct preserving extension $k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ as in Lemma 3.1. If k' is left covering with respect to a weak terminal object, weak binary products of objects coming from $\mathcal{C}$, and weak equalizers of pairs of parallel arrows coming from $\mathcal{C}$, then k' is left covering.*

Proof. Thanks to Proposition 27 in [3], it suffices to show that k' is left covering with respect to weak binary products and weak equalizers.

Concerning weak binary products: consider the following diagram

$$\begin{tikzcd}
k'(F \times \coprod_Y \mathcal{C}(-,T)) \arrow[rr, "\lambda"] & & k'F \times k'(\coprod_Y \mathcal{C}(-,T)) \\
k!(\mathrm{colim}_S(F \times \coprod_S \mathcal{C}(-,T))) \arrow[u, "k!c"] & & k'F \times \mathrm{colim}_S k'(\coprod_S \mathcal{C}(-,T)) \arrow[u, "b"'] \\
\mathrm{colim}_S k'(F \times \coprod_S \mathcal{C}(-,T)) \arrow[u, "c"] \arrow[rr, "\mathrm{colim}_S \lambda_S"'] & & \mathrm{colim}_S(k'F \times k'(\coprod_S \mathcal{C}(-,T))) \arrow[u, "a"']
\end{tikzcd}$$

where
- F is a free algebra, and Y is a set;
- the filtered colimits are taken over the finite subsets S of Y;
- products in $\mathcal{F}(\mathcal{C})$ are weak products;
- c stays for the canonical arrow induced by the universal property of a colimit;
- a is an isomorphism, because in $\mathcal{E}$ filtered colimits commute with finite limits;
- b is an isomorphism, because k' preserves coproducts;
- by induction, for each S the comparison λ_S is a regular epimorphism, so that also the colimit of all arrows λ_S is a regular epimorphism.

Since the diagram commutes, the comparison λ is a regular epimorphism.

Concerning weak equalizers: consider two parallel arrows

$$f, g \colon \coprod_X \mathcal{C}(-,T) \rightrightarrows \coprod_Y \mathcal{C}(-,T)$$

in $\mathcal{F}(\mathcal{C})$, and a finite subset S of X. Since $\coprod_S \mathcal{C}(-,T)$ is a finitely presentable object in $\mathrm{Alg}(\mathcal{C})$, there exist a finite subset R of Y and two arrows f_S, g_S making the following diagram

$$\begin{tikzcd}
\mathrm{colim}_S E_S \arrow[r, "i"] & E \arrow[r, "e"] & \coprod_X \mathcal{C}(-,T) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod_Y \mathcal{C}(-,T) \\
& E_S \arrow[ul, "\epsilon_S"] \arrow[u, "i_S"'] \arrow[r, "e_S"'] & \coprod_S \mathcal{C}(-,T) \arrow[u, "j_S"'] \arrow[r, shift left, "f_S"] \arrow[r, shift right, "g_S"'] & \coprod_R \mathcal{C}(-,T) \arrow[u, "j_R"']
\end{tikzcd}$$

commutative, where the rows are weak equalizers, i_S is induced by the universal property of E, ϵ_S is the colimit injection and $i \cdot \epsilon_S = i_S$. Applying the functor $k' \colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$, we get the following diagram

$$\begin{tikzcd}
D \arrow[rr, "d"] && k'(\coprod_X \mathcal{C}(-,T)) \arrow[r, shift left, "k'f"] \arrow[r, shift right, "k'g"'] & k'(\coprod_Y \mathcal{C}(-,T)) \\
& k'E \arrow[ul, "\lambda"'] \arrow[ur, "k'e"'] \\
& k'E_S \arrow[u, "k'i_S"'] \arrow[dl, "\lambda_S"'] \arrow[dr, "k'e_S"] \\
D_S \arrow[uuu, "\theta_S"] \arrow[rr, "d_S"'] && k'(\coprod_S \mathcal{C}(-,T)) \arrow[uuu, "k'j_S"'] \arrow[r, shift left, "k'f_S"] \arrow[r, shift right, "k'g_S"'] & k'(\coprod_R \mathcal{C}(-,T)) \arrow[uuu, "k'j_R"']
\end{tikzcd}$$

where the rows are equalizers and $\theta_S, \lambda_S, \lambda$ are induced by the universal property of the equalizers. Finally, we obtain the following commutative diagram

$$\begin{tikzcd}
k!(\mathrm{colim}_S E_S) \arrow[r, "k!i"] & k'E \arrow[r, "\lambda"] & D \\
\mathrm{colim}_S k'E_S \arrow[u, "c"] \arrow[rr, "\mathrm{colim}_S \lambda_S"'] && \mathrm{colim}_S D_S \arrow[u, "\theta"']
\end{tikzcd}$$

where θ is induced by the maps θ_S, and c is canonical. Since all λ_S are regular epimorphisms, $\mathrm{colim}_S \lambda_S$ also is a regular epimorphism. Moreover, θ is an isomorphism, and then the comparison λ is a regular epimorphism, as desired. This follows from the fact that, $k'j_R$ being a monomorphism, D_S is the equalizer of $(k'j_R \cdot k'f_S, k'j_R \cdot k'g_S)$, and in $\mathcal{E}$ filtered colimits commute with finite limits. (Note that the arrow $k'j_R$ is a monomorphism because j_R is induced by the inclusion of R into Y. If Y is not the empty set, then we can assume R to be non empty, so that the inclusion $R \subset Y$ is a split monomorphism; if Y is the empty set, then j_R is the identity.) ■

3.5. Remark. To end this section, let us describe explicitly the weak limits involved in Lemma 3.4.

1. $\mathcal{C}(-,T)$ is a weak terminal object in $\mathcal{F}(\mathcal{C})$;

2. Let A, B be objects in $\mathcal{C}$; a weak product of $\mathcal{C}(-,A)$ and $\mathcal{C}(-,B)$ in $\mathcal{F}(\mathcal{C})$ is given as follows
$$\mathcal{C}(-,A) \xleftarrow{\pi_A} \coprod \mathcal{C}(-,T) \xrightarrow{\pi_B} \mathcal{C}(-,B)$$
where the coproduct is indexed by the pairs $(u,v) \in \mathcal{C}(T,A) \times \mathcal{C}(T,B)$, and $\pi_A \cdot \rho_{(u,v)} = \mathcal{C}(-,u), \pi_B \cdot \rho_{(u,v)} = \mathcal{C}(-,v)$ for any such a pair, $\rho_{(u,v)}$ being the coproduct injection;

3. Let $u, v\colon A \rightrightarrows B$ be arrows in $\mathcal{C}$; a weak equalizer of $\mathcal{C}(-,u)$ and $\mathcal{C}(-,v)$ in $\mathcal{F}(\mathcal{C})$ is given as follows

$$\coprod \mathcal{C}(-,T) \xrightarrow{\ l\ } \mathcal{C}(-,A) \underset{\mathcal{C}(-,v)}{\overset{\mathcal{C}(-,u)}{\rightrightarrows}} \mathcal{C}(-,B)$$

where the coproduct is indexed by the set of arrows $w\colon T \to A$ such that $u \cdot w = v \cdot v$, and $l \cdot \rho_w = \mathcal{C}(-,w)$.

4. Geometric morphisms and localizations of algebraic categories

Let $\mathcal{C} = \langle 0, T, 2T, \ldots, nT, \ldots \rangle$ still denote an algebraic theory and $\mathcal{E}$ a cocomplete, exact category with exact filtered colimits. Since the codomain restriction $Y_\mathcal{C}\colon \mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ of the Yoneda embedding is a dense functor preserving finite coproducts, pre-composing with $Y_\mathcal{C}\colon \mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ still induces an equivalence of categories

$$\mathrm{Adj}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})] \to\ {}_{\coprod}[\mathcal{C}, \mathcal{E}]$$

where ${}_{\coprod}[\mathcal{C}, \mathcal{E}]$ is the category of finite coproduct-preserving functor from $\mathcal{C}$ to $\mathcal{E}$.

The first step to classify geometric morphisms and localizations of the form

$$\mathcal{E} \underset{R}{\overset{L}{\leftrightarrows}} \mathrm{Alg}(\mathcal{C})$$

is to adjust to the new setting the notion of filtering functor. We have already mentioned the fact that in an exact and extensive category any epimorphism is regular. This is no longer true if we omit the extensivity condition, as in the current section. This is the reason why the families involved in the next definition are regular epimorphic, and not just epimorphic.

4.1. Definition.

1. A *regular epimorphic family* in $\mathcal{E}$ is a collection $\{f_i\colon X_i \to X\}_{i \in I}$ of arrows in $\mathcal{E}$ such that the induced arrow $\coprod X_i \to X$ is a regular epimorphism.
2. A functor $k\colon \mathcal{C} \to \mathcal{E}$ is said to be *regular filtering* if:

(RF1) the family of arrows $\{kC \to 1 \mid C \in \mathcal{C}\}$, where 1 is a terminal object of $\mathcal{E}$, is regular epimorphic;

(RF2) for any pair of objects $A, B \in \mathcal{C}$, the family of arrows

$$\{\langle ku, kv \rangle \colon kC \to kA \times kB \mid\ A \xleftarrow{\ u\ } C \xrightarrow{\ v\ } B \text{ in } \mathcal{C}\}$$

is regular epimorphic;

(RF3) for any pair of arrows $u, v\colon A \rightrightarrows B$ in $\mathcal{C}$, the family of arrows

$$\{w'\colon kC \to E_{u,v} \mid w\colon C \to A \text{ in } \mathcal{C} \text{ such that } u \cdot w = v \cdot w\}$$

where $e\colon E_{u,v} \to kA$ is an equalizer of (ku, kv) and $e \cdot w' = kw$, is regular epimorphic.

4.2. Remark. If the functor $k\colon \mathcal{C} \to \mathcal{E}$ involved in the previous definition preserves finite coproducts, then in conditions (RF1–RF3) we can equivalently replace the variable object $C \in \mathcal{C}$ by the base object T of the algebraic theory $\mathcal{C}$.

Here is the announced classification of geometric morphisms.

4.3. Proposition. *Let $k\colon \mathcal{C} \to \mathcal{E}$ be a finite coproduct-preserving functor. The following conditions are equivalent:*

1. *$k^*\colon \mathcal{E} \to \mathrm{Alg}(\mathcal{C})$ is a geometric morphism;*
2. *$k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ is left covering;*
3. *$k\colon \mathcal{C} \to \mathcal{E}$ is regular filtering.*

Proof. $1 \Leftrightarrow 2$: Following the terminology of [3], the full embedding $\mathcal{F}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ is the exact completion of $\mathcal{F}(\mathcal{C})$. The equivalence between condition 1 and condition 2 is then just a particular case of Theorem 29 in [3].

$2 \Leftrightarrow 3$: Thanks to Remark 3.5 and Remark 4.2, $k\colon \mathcal{C} \to \mathcal{E}$ is regular filtering precisely when $k'\colon \mathcal{F}(\mathcal{C}) \to \mathcal{E}$ is left covering with respect to a weak terminal object, and weak binary products and weak equalizers of objects and arrows coming from $\mathcal{C}$. By Lemma 3.4, the proof is complete. ∎

Hence, we denote by $\mathrm{GeoMor}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})]$ the category of geometric morphisms from $\mathcal{E}$ to $\mathrm{Alg}(\mathcal{C})$, and by $\mathrm{RFilt}_{\coprod}[\mathcal{C}, \mathcal{E}]$ the category of those regular filtering functors from $\mathcal{C}$ to $\mathcal{E}$ which preserve finite coproducts.

4.4. Corollary. *The equivalence of categories*

$$\mathrm{Adj}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})] \simeq \ {}_{\coprod}[\mathcal{C}, \mathcal{E}]$$

restricts to an equivalence

$$\mathrm{GeoMor}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})] \simeq \mathrm{RFilt}_{\coprod}[\mathcal{C}, \mathcal{E}]$$

We can move on now to localizations. The algebraic analogue of Lemma 2.4 is given by the following:

4.5. Lemma. *Let $k\colon \mathcal{C} \to \mathcal{E}$ be a functor, $C \in \mathcal{C}$ an object, and $r\colon R \to \mathcal{C}(-, C)$ a subobject in $\mathrm{Alg}(\mathcal{C})$. The following conditions are equivalent:*

1. *$k!r$ is an epimorphism (respectively, a regular epimorphism);*
2. *The family $kR = \{kd\colon kD \to kC \mid d \in RD\,,\ D \in \mathcal{C}\}$ is epimorphic (respectively, regular epimorphic).*

Proof. The proof runs parallel to the one of Lemma 2.4. The only difference accours while proving that

$$e \colon \coprod_{d \in RD, D \in \mathcal{C}} \mathcal{C}(-, D) \to R$$

is a regular epimorphism. For this, apply the forgetful functor $\mathcal{U} \colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Set}$ defined by evaluation at T. The canonical map

$$\coprod_{d \in RD, D \in \mathcal{C}} \mathcal{C}(T, D) \to RT$$

is surjective (just use the identity on T), so that also $\mathcal{U}e$ is surjective. This implies that e is a regular epimorphism because $\mathcal{U}$ reflects regular epimorphisms. ■

4.6. Definition. A functor $k \colon \mathcal{C} \to \mathcal{E}$ is said to be *fully regular filtering* if

(RA) For any object $X \in \mathcal{E}$, the family of arrows $R_X = \{c \colon kC \to X \mid C \in \mathcal{C}\}$ is regular epimorphic;

(RB) For any pair of arrows $kA \xleftarrow{a} X \xrightarrow{b} kB$ in $\mathcal{E}$, the family of arrows

$$R_{a,b} = \{c \colon kC \to X \mid a \cdot c = kf_A\,,\ b \cdot c = kf_B \text{ for some } A \xleftarrow{f_A} C \xrightarrow{f_B} B \text{ in } \mathcal{C}\}$$

is regular epimorphic;

(RF3) As in Definition 4.1.

4.7. Remark. Once again, condition (RA) implies condition (RF1) and condition (RB) implies condition (RF2). Moreover, if $k \colon \mathcal{C} \to \mathcal{E}$ preserves finite coproducts, we can replace in conditions (RA) and (RB) the variable object $C \in \mathcal{C}$ by the base object T. Condition (RA) amounts then to saying that the object kT is a regular generator for $\mathcal{E}$.

4.8. Proposition. *Let $k \colon \mathcal{C} \to \mathcal{E}$ be a finite coproduct-preserving functor. The following conditions are equivalent:*

1. *$k^* \colon \mathcal{E} \to \mathrm{Alg}(\mathcal{C})$ is a localization;*
2. *$k \colon \mathcal{C} \to \mathcal{E}$ is fully regular filtering.*

Proof. $1 \Rightarrow 2$. Condition (RA): Since $\mathcal{C}(-,T)$ is a regular generator for $\mathrm{Alg}(\mathcal{C})$, $k!\mathcal{C}(-,T) = kT$ is a regular generator for $\mathcal{E}$.
Condition (RB): Since pullbacks in $\mathrm{Alg}(\mathcal{C})$ are computed pointwise in Set, we can repeat the same arguments as in the proof of Proposition 2.7 and, using Lemma 4.5, we get three regular epimorphic families $R_X, kR_{a \cdot c}$, and $kR_{b \cdot c \cdot kx}$ (same notations as in the proof of 2.7). We have to show that the family

$$M_{a,b} = \{c \cdot kx \cdot ky_x \mid y_x \in R_{b \cdot c \cdot kx}, x \in R_{a \cdot c}, c \in R_X\}$$

is still regular epimorphic (this immediately implies that $R_{a,b}$ is regular epimorphic, since $M_{a,b}$ is contained in $R_{a,b}$). For it, we consider the following diagram

$$\begin{tikzcd}
\coprod_{M_{a,b}} kC_{y_x} \arrow[r] \arrow[d, dash, "\simeq"'] & X & \coprod_{R_X} kC \arrow[l, "\lambda"'] \\
\coprod_{R_X} \coprod_{kR_{a\cdot c}} \coprod_{kR_{b\cdot c\cdot kx}} kC_{y_x} \arrow[rr, "\coprod_{R_X} \coprod_{kR_{a\cdot c}} \lambda_x"'] & & \coprod_{R_X} \coprod_{kR_{a\cdot c}} kC_x \arrow[u, "\coprod_{R_X} \lambda_c"']
\end{tikzcd}$$

where the morphisms λ are induced by the universal property of the corresponding coproducts, and the isomorphism is given by the associativity isomorphism of the coproduct. Since $R_X, R_{a\cdot c}$ and $R_{b\cdot c\cdot kx}$ are regular epimorphic families, the arrows λ, λ_c and λ_x are regular epimorphisms, and then so are their coproducts. Since the diagram is commutative, the canonical arrow

$$\coprod_{M_{a,b}} kC_{y_x} \to X$$

is a regular epimorphism, as desired.

2 $\Rightarrow$ 1. We prove that if $k\colon \mathcal{C} \to \mathcal{E}$ satisfies conditions (RA) and (RB), then k^* is full and faithful. Let X, Y be objects in $\mathcal{E}$ and $\alpha\colon k^*X \to k^*Y$ an arrow in $\mathrm{Alg}(\mathcal{C})$. By condition (RA), the canonical map

$$\lambda\colon \coprod_{h\in R_X} kT \to X$$

induced by the arrows in $R_X = \{h\colon kT \to X\}$, is a regular epimorphism, and so it is the coequalizer of its kernel pair $\lambda_0, \lambda_1\colon N(\lambda) \rightrightarrows \coprod kT$. On the other hand, for any $h \in R_X$ we have an arrow $\alpha_T(h)\colon kT \to Y$, and then a canonical morphism

$$\mu\colon \coprod_{h\in R_X} kT \to Y$$

It suffices to prove that μ coequalizes λ_0 and λ_1. For any finite subset $S \subset R_X$, consider the following diagram

$$\begin{tikzcd}
N(S) \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] \arrow[d, "n_S"'] & k(sT) = \coprod_S kT \arrow[d, "j_S"] \arrow[dr, "\lambda_S"] & \\
N(\lambda) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod_{R_X} kT \arrow[r, "\lambda"'] & X
\end{tikzcd}$$

where j_S is the colimit injection, $\lambda_S = \lambda \cdot j_S$, $N(S)$ is the kernel pair of λ_S, n_S is induced by the universal property of $N(\lambda)$, and s is the cardinality of S. By exactness of filtered colimits in $\mathcal{E}$, the diagram $\langle n_S\colon N(S) \to N(\lambda) \mid S \subset R_X, S \text{ finite}\rangle$ is a colimit. Moreover, by condition (RB), the family of arrows

$$R_{s_0,s_1} = \{c\colon kT \to N(S) \mid s_0 \cdot c = kf_0\,,\ s_1 \cdot c = kf_1 \text{ for some } sT \xleftarrow{f_0} T \xrightarrow{f_1} sT \text{ in } \mathcal{C}\}$$

is (regular) epimorphic. Finally, to verify the equation $\mu \cdot \lambda_0 = \mu \cdot \lambda_1$, it is enough to verify the equation $\mu \cdot \lambda_0 \cdot n_S \cdot c = \mu \cdot \lambda_1 \cdot n_S \cdot c$ for all $c \in R_{s_0,s_1}$ and for any finite subset S of R_X, and this last equation holds by naturality of α. ∎

4.9. Remark. Observe that to prove implication $2 \Rightarrow 1$ we just need condition (B) on k, and not condition (RB). So we can replace in the statement of the previous proposition condition (RB) by conditions (B) and (RF2).

We denote by $\mathrm{Loc}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})]$ the full subcategory of $\mathrm{GeoMor}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})]$ given by the localizations. We denote by $\mathrm{FullyRFilt}_{\coprod}[\mathcal{C}, \mathcal{E}]$ the full subcategory of $\mathrm{RFilt}_{\coprod}[\mathcal{C}, \mathcal{E}]$ given by those fully regular filtering functors which preserve finite coproducts.

4.10. Corollary. *The equivalence of categories*

$$\mathrm{Adj}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})] \simeq \ _{\coprod}[\mathcal{C}, \mathcal{E}]$$

restricts to an equivalence

$$\mathrm{Loc}[\mathcal{E}, \mathrm{Alg}(\mathcal{C})] \simeq \mathrm{FullyRFilt}_{\coprod}[\mathcal{C}, \mathcal{E}]$$

5. Geometric morphisms and localizations of module categories

Throughout the section, R is a ring with unit and R-mod is the category of unitary left modules over R. Let us denote by $\mathcal{R}$ the preadditive category with just one object, say T, and with the elements of R as arrows, the composition being given by the product in R.

Given an additive functor $\kappa\colon \mathcal{R} \to \mathcal{E}$, where $\mathcal{E}$ is a cocomplete abelian category, we denote by

$$\kappa!\colon R\text{-mod} \to \mathcal{E}$$

the left Kan extension of κ along the full embedding $\mathcal{R} \to R$-mod, and by

$$\kappa^*\colon \mathcal{E} \to R\text{-mod}$$

the right adjoint of $\kappa!$. The following condition on the functor $\kappa\colon \mathcal{R} \to \mathcal{E}$ is the abelian version of condition (RF3):

(AF3) For any matrix $M = (a_{ij}) \in R^{n\times m}$, the family of arrows

$$\{w'\colon \kappa T \to \mathrm{Ker}(\kappa M) \mid w = \langle w_j \rangle \in R^m \text{ such that for all } i = 1, \dots, n,\ \sum_j a_{ij} \cdot w_j = 0\}$$

where $e\colon \mathrm{Ker}(\kappa M) \to \kappa T^m$ is a kernel of $\kappa M = (\kappa a_{ij})\colon \kappa T^m \to \kappa T^n$ and $e \cdot w' = \langle \kappa w_j \rangle$, is (regular) epimorphic.

(The word "regular" can be avoided because $\mathcal{E}$ is abelian, so that any epimorphism is regular.) As a special case of Proposition 4.3, we get the following:

5.1. Proposition. *Let* $\kappa\colon \mathcal{R} \to \mathcal{E}$ *be an additive functor into a cocomplete abelian category* $\mathcal{E}$ *with exact filtered colimits. The following conditions are equivalent:*

1. $\kappa^*\colon \mathcal{E} \to R\text{-mod}$ *is a geometric morphism;*

2. $\kappa\colon \mathcal{R} \to \mathcal{E}$ *satisfies condition (AF3).*

Proof. Let $\mathcal{C}_R$ be the theory of unitary R-modules (that is, $\mathcal{C}_R$ is equivalent to the full subcategory of R-mod of finitely generated free objects). The category $\mathcal{R}$ embeds into $\mathcal{C}_R$ as $\mathcal{R} = \mathcal{C}_R(T,T)$, and the categories R-mod and $\mathrm{Alg}(\mathcal{C}_R)$ are equivalent. Moreover, the additive functor $\kappa\colon \mathcal{R} \to \mathcal{E}$ extends in a unique way to an additive functor $k\colon \mathcal{C}_R \to \mathrm{Alg}(\mathcal{C}_R)$, and it is easy to check that, up to the equivalence $R\text{-mod} \simeq \mathrm{Alg}(\mathcal{C}_R)$, the Kan extension $\kappa^*\colon R\text{-mod} \to \mathcal{E}$ coincides with the Kan extension $k^*\colon \mathrm{Alg}(\mathcal{C}_R) \to \mathcal{E}$ of k along the Yoneda embedding. To apply Proposition 4.3, it remains to show that k is regular filtering (Definition 4.1) iff κ verifies condition (AF3). In fact, conditions (RF1) and (RF2) are always verified (respectively, because in $\mathcal{E}$ the terminal object is a zero object, and because finite products are biproducts), and the equivalence between (RF3) and (AF3) is just the standard equivalence between kernels and equalizers in an abelian category. ∎

5.2. Remark. When $\mathcal{E}$ is of the form S-mod, for S a ring with unit, to give a functor $\kappa\colon \mathcal{R} \to \mathcal{E}$ amounts to give an S-R-bimodule M, and the Kan extension $\kappa!$ is the functor

$$M \otimes_R -\colon R\text{-mod} \to S\text{-mod}$$

Condition (AF3) amounts then to the flatness of M.

Consider again an additive functor $\kappa\colon \mathcal{R} \to \mathcal{E}$. We state now the abelian version of conditions (RA) and (RB) of Definition 4.6, as well as a simplified version of condition (AF3):

(AA) The object κT is a generator for $\mathcal{E}$;

(AB) For any arrow $a\colon \kappa T \to \kappa T$ in $\mathcal{E}$, the family of arrows

$$R_a = \{\kappa r\colon \kappa T \to \kappa T \mid r \in R \text{ and } a \cdot \kappa r = \kappa s \text{ for some } s \in R\}$$

is epimorphic;

(AF3') For any $y \in R$ such that $\kappa y = 0\colon \kappa T \to \kappa T$, the family of arrows

$$E_y = \{\kappa s\colon \kappa T \to \kappa T \mid s \in R \text{ such that } y \cdot s = 0\}$$

is epimorphic.

5.3. Proposition. *Let* $\kappa\colon \mathcal{R} \to \mathcal{E}$ *be an additive functor into a cocomplete abelian category* $\mathcal{E}$ *with exact filtered colimits. The following conditions are equivalent:*

1. $\kappa^*\colon \mathcal{E} \to R\text{-mod}$ *is a localization;*

2. $\kappa\colon \mathcal{R} \to \mathcal{E}$ *satisfies conditions (AA), (AB) and (AF3').*

Proof. With the same notations of the proof of Proposition 5.1, we have to prove that $\kappa\colon \mathcal{R} \to \mathcal{E}$ satisfies conditions (AA), (AB) and (AF3') iff its extension $k\colon \mathcal{C}_R \to \mathcal{E}$ satisfies conditions (RA), (RB) and (AF3). Clearly, (RA) and (AA) are equivalent (see Remark 4.7). Moreover, (RB) implies (AB). To see this, just take $A = B = T, X = \kappa T$ and b the identity on κT in (RB).

We show now that (AA) and (AB) imply (RB). For it, let us start showing that (AA) and (AB) implies the following condition:

($\star$) For any arrow $b\colon X \to \kappa T$ in $\mathcal{E}$, the family of arrows

$$\{c\colon \kappa T \to X \mid b \cdot c = \kappa t \ \text{ for some } \ t \in R\}$$

is epimorphic.

Indeed, for any arrow $x\colon \kappa T \to X$, we get an epimorphic family by applying (AB) to the composite $b \cdot x\colon \kappa T \to \kappa T$. By (AA), we can past together all these families (for x varying in $\mathcal{E}$) and we get a new epimorphic family which is contained in the family under consideration in condition ($\star$). Finally, since the objects A and B in (RB) are finite copowers of T, one can show that (RB) follows from (AB) and ($\star$) by induction.

To apply Proposition 4.8, it remains to compare conditions (AF3) and (AF3'). Clearly, (AF3) implies (AF3'): since $\kappa y = 0$, its kernel is κT. Now condition (AF3) with $m = n = 1$ is precisely (AF3'). Conversely, one can prove, working by induction on n, that (AF3') implies the following condition:

($\star\star$) For any $\langle y_i \rangle \in R^n$ such that $\langle \kappa y_i \rangle = 0\colon \kappa T \to \kappa T^n$, the family of arrows

$$E_{\langle y_i \rangle} = \{\kappa s\colon \kappa T \to \kappa T \mid s \in R \ \text{ such that } \ y_i \cdot s = 0 \ \text{ for all } i\}$$

is epimorphic.

Finally, we prove that (AA), (AB) and ($\star\star$) imply (AF3). Let $M \in R^{n\times m}$ be a matrix as in (AF3) and consider the family

$$\{c\colon \kappa T \to \mathrm{Ker}(\kappa M) \mid e \cdot c = \kappa d_c \ \text{ for some } \ d_c \in R^m\}$$

For any such c, fix an arrow $d_c \in R^m$ such that $e \cdot c = \kappa d_c$, and consider the family

$$\{\kappa b_c\colon \kappa T \to \kappa T \mid M \cdot d_c \cdot b_c = 0, \ b_c \in R\}$$

Now, if we put $w = d_c \cdot b_c$, we get $w'\colon \kappa T \to \mathrm{Ker}(\kappa M)$ as in condition (AF3). This induces an arrow

$$\lambda\colon \coprod_c (\coprod_{b_c} \kappa T) \to \coprod_w \kappa T$$

Consider the diagram

$$\begin{tikzcd}
\coprod_c(\coprod_{b_c} \kappa T) \arrow[r, "\coprod_c \varphi_c"] \arrow[d, "\lambda"'] & \coprod_c \kappa T \arrow[d, "\varphi"] \\
\coprod_w \kappa T \arrow[r, "\psi"'] & \mathrm{Ker}(\kappa M)
\end{tikzcd}$$

where ψ, φ and φ_c are induced by the corresponding families of arrows in $\mathcal{E}$. By condition (RB) (with $a = e$ and b the unique arrow $\mathrm{Ker}(\kappa M) \to \kappa T^0$), φ is an epimorphism. By condition $(\star\star)$ applied to $M \cdot d_c$, each φ_c is an epimorphism. Finally, a diagram chase shows that the previous diagram commutes, so that ψ is an epimorphism. ∎

In [6], the localizations of the form $\kappa^* \colon \mathcal{E} \to \mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}]$, where $\mathcal{A}$ is a small preadditive category and $\mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}]$ is the category of contravariant additive functors from $\mathcal{A}$ to the category of abelian groups, are classified. Proposition 5.3 is precisely the main result of [6] in the particular case of $\mathcal{A}$ being a one-object preadditive category, that is a ring with unit.

5.4. Remark. To finish, we point out that localizations of the form $\mathrm{Loc}[\mathcal{E}, \mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}]]$ can be always reconduced to localizations of the form $\mathrm{Loc}[\mathcal{E}, R\text{-mod}]$. Indeed, by Gabriel-Popescu's theorem, the category $\mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}]$ itself is a localization of R-mod, say

$$\mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}] \underset{r}{\overset{l}{\leftrightarrows}} R\text{-mod}$$

(take as generator G the coproduct of all representable presheaves, and as ring R the ring of endomorphisms of G). Therefore, we can define a functor

$$\mathrm{Adj}[\mathcal{E}, \mathrm{Add}[\mathcal{A}^{op}, \mathrm{Ab}]] \to \mathrm{Adj}[\mathcal{E}, R\text{-mod}] \quad (L \dashv R) \mapsto (L \cdot l \dashv r \cdot R)$$

and the adjunction $(L \dashv R)$ is a geometric morphism (respectively, a localization) if and only if the adjunction $(L \cdot l \dashv r \cdot R)$ is a geometric morphism (respectively, a localization).

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