Factorization systems for symmetric cat-groups

S. Kasangian E.M. Vitale

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Abstract. This paper is a first step in the study of symmetric cat-groups as the 2-dimensional analogue of abelian groups. We show that a morphism of symmetric cat-groups can be factorized as an essentially surjective functor followed by a full and faithful one, as well as a full and essentially surjective functor followed by a faithful one. Both these factorizations give rise to a factorization system, in a suitable 2-categorical sense, in the 2-category of symmetric cat-groups. An application to exact sequences is given.

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Introduction

A cat-group is a monoidal groupoid in which each object is invertible, up to isomorphisms, with respect to the tensor product [7, 10, 16, 24]. Cat-groups are a useful tool for ring theory, group cohomology and algebraic topology (for example, small and strict cat-groups correspond to crossed modules) [2, 5, 6, 13, 15, 21, 22, 25]. Symmetric cat-groups, together with symmetric monoidal functors and monoidal natural transformations, constitute a 2-category which can be seen as a 2-dimensional analogue of the category of abelian groups. Several algebraic problems can be considered in this setting, and some of them, especially the study of exact sequences and of extensions, lead in a natural way to the search of convenient classes of surjections and injections between cat-groups. The aim of this note is to discuss two different factorization systems for symmetric cat-groups. The problem of factorizing a (monoidal) functor has been discussed, from different points of view, also in [1, 8, 18, 19, 20].

After recalling some preliminary facts on cat-groups (section 1), in section 2 we show how to factorize a morphism of symmetric cat-groups. The idea is quite simple : in any abelian category, a morphism can be factorized through the kernel of its cokernel or through the cokernel of its kernel, and these two factorizations are essentially the same. But this is a gift of the one-dimensional (abelian) world ! Since we have the appropriate notion of kernel and cokernel for symmetric cat-groups, we can do the same with a morphism between symmetric cat-groups, and we obtain two different factorizations. In section 3 we develop a little bit of the theory of factorization systems in a 2-category and finally, in section 4, we prove that both factorizations constructed in section 2 give rise to factorization systems. In the last section we use our results to discuss the notion of 2-exact sequence of cat-groups. This notion has been introduced in [24] in order to study the Brauer and Picard cat-groups of a commutative ring.

The reason why the two factorizations obtained in section 2 are different is a quite surprising fact, explained in detail in section 5 : in the 2-category of symmetric cat-groups, we can give a suitable definition of mono, epi, kernel and cokernel. As expected in the 2-dimensional analogue of the category of abelian groups, each epi is a cokernel and each mono is a kernel, but in general cokernels fail to be epis and kernels fail to be monos.

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1 Categorical groups

We start recalling some basic facts about cat-groups.

Definition 1 *A (symmetric) cat-group* $\mathbb{G}$ *is a (symmetric) monoidal groupoid* $\mathbb{G} = (\mathbb{G}, \otimes, I, \dots)$ *such that for each object* A *there exists an object* A^* *and an arrow* $\eta_A \colon I \to A \otimes A^*$.

The asymmetry in the previous definition is only apparent, in fact we have the following

Proposition 2 *Let* $\mathbb{G}$ *be a cat-group ; for each object* A *it is possible to find a morphism* $\epsilon_A \colon A^* \otimes A \to I$ *such that*

$$\mathcal{D} = (A^* \dashv A, \eta_A, \epsilon_A)$$

is a duality in $\mathbb{G}$. *The choice, for each* A, *of such a duality induces a monoidal equivalence*

$$(\ \ )^* \colon \mathbb{G}^{op} \to \mathbb{G} \ .$$

One of the main tools of this paper is the cokernel of a morphism of cat-groups. Since we have an explicit description of it only in the case of symmetric cat-groups, from now on we limit our attention to symmetric cat-groups. We denote by SCG the 2-category having symmetric cat-groups as objects, monoidal functors preserving the symmetry as arrows, and monoidal natural transformations as 2-cells. Observe that in this 2-category, 2-cells are invertible. We use the same name, SCG, for the category obtained from the previous 2-category forgetting 2-cells. We write $\mathcal{H}(SCG)$ for the category with the same objects as SCG, but with 2-isomorphism classes of morphisms as arrows. There is an obvious functor

$$\mathcal{H} \colon SCG \to \mathcal{H}(SCG) \ .$$

If $\mathbb{G}$ is a symmetric cat-group, $\pi_0(\mathbb{G})$ is the (possibly large) set of isomorphism classes of objects of $\mathbb{G}$, and $\pi_1(\mathbb{G}) = \mathbb{G}(I, I)$ is the set of endomorphisms of the unit object. Both $\pi_0(\mathbb{G})$ and $\pi_1(\mathbb{G})$ are abelian groups and give rise to two functors

$$\pi_0 \colon SCG \to Ab \ \ , \ \ \pi_1 \colon SCG \to Ab$$

(where Ab is the category of abelian groups) which factor through $\mathcal{H}$.

If G is an abelian group, $\underline{D}(G)$ is the discrete symmetric cat-group with the elements of G as objects, and $G!$ is the symmetric cat-group with a single object and with the elements of G as arrows. In this way we obtain two functors

$$\underline{D} \colon Ab \to SCG \ \ , \ \ ! \colon Ab \to SCG$$

which are full and faithful, as well as their composite with $\mathcal{H}$. Moreover, the following adjunctions hold :

$$! \cdot \mathcal{H} \dashv \pi_1 \ \ , \ \ \pi_0 \dashv \underline{D} \cdot \mathcal{H} \dashv \pi_0 \ .$$

More precisely, for each symmetric cat-group $\mathbb{G}$ we have

- a full and faithful morphism $\pi_1(\mathbb{G})! \to \mathbb{G}$ which is an equivalence iff $\pi_0(\mathbb{G}) = 0$;
- a full and essentially surjective morphism $\mathbb{G} \to \underline{D}(\pi_0(\mathbb{G}))$ which is an equivalence iff $\pi_1(\mathbb{G}) = 0$.

Let $F \colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups. From [24] we recall an explicit description of its kernel and its cokernel.

Kernel: the kernel of F is given by a symmetric cat-group $KerF$, a morphism $e_F \colon KerF \to \mathbb{G}$, and a 2-cell $\lambda_F \colon e_F \cdot F \Rightarrow 0$ (where $0 \colon KerF \to \mathbb{H}$ is the zero-morphism, i.e. the functor which sends each arrow of $KerF$ to the identity of the unit object of $\mathbb{H}$) :

- the objects of $KerF$ are pairs (X, λ_X) where X is an object of $\mathbb{G}$ and $\lambda_X \colon F(X) \to I$ is an arrow of $\mathbb{H}$;
- an arrow $f \colon (X, \lambda_X) \to (Y, \lambda_Y)$ in $KerF$ is an arrow $f \colon X \to Y$ in $\mathbb{G}$ such that

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_X"'] & & F(Y) \arrow[dl, "\lambda_Y"] \\ & I & \end{tikzcd}$$

  commutes;
- the functor e_F forgets the arrow λ_X of (X, λ_X);
- the component at (X, λ_X) of λ_F is given by λ_X.

Cokernel: the cokernel of F is given by a symmetric cat-group $CokerF$, a morphism $P_F \colon \mathbb{H} \to CokerF$, and a 2-cell $\pi_F \colon F \cdot P_F \Rightarrow 0$:

- the objects of $CokerF$ are those of $\mathbb{H}$;

- an arrow $[f,N]\colon\ X \longrightarrow\!\!\!\!\!\!\bullet\ \ Y$ in $CokerF$ is a class of equivalence of pair (f,N) with N an object of $\mathbb{G}$ and $f\colon X \to Y \otimes F(N)$ an arrow of $\mathbb{H}$; two pairs (f,N) and (g,M) are equivalent if there exists an arrow $\alpha\colon N \to M$ in $\mathbb{G}$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \otimes F(N) \arrow[rr, "1\otimes F(\alpha)"'] & & Y \otimes F(M) \end{tikzcd}$$

  commutes;

- the functor P_F sends an arrow $X \to Y$ of $\mathbb{H}$ to $[X \to Y \simeq Y \otimes I \simeq Y \otimes F(I), I]\colon\ X \longrightarrow\!\!\!\!\!\!\bullet\ \ Y$;

- if N is an object of $\mathbb{G}$, the component at N of π_F is $[F(N) \to I \otimes F(N), N]\colon\ F(N) \longrightarrow\!\!\!\!\!\!\bullet\ \ I$.

The kernel and the cokernel are special instances of bilimits (see [17]) and are determined, up to monoidal equivalences, by their universal property, which is discussed in detail in [24]. In section 2 we use the explicit description of the kernel and of the cokernel, but in section 4 we essentially use their universal property. For this we recall here the universal property of the kernel (that of the cokernel is dual). For any morphism $G\colon \mathbb{K} \to \mathbb{G}$ and for any 2-cell $\varphi\colon G \cdot F \Rightarrow 0$, there exists a morphism $G'\colon \mathbb{K} \to KerF$ and a 2-cell $\varphi'\colon G' \cdot e_F \Rightarrow G$ such that the following diagram commutes

$$\begin{tikzcd} G' \cdot e_F \cdot F \arrow[r, Rightarrow, "G' \cdot \lambda_F"] \arrow[d, Rightarrow, "\varphi' \cdot F"'] & G' \cdot 0 \arrow[d, Rightarrow] \\ G \cdot F \arrow[r, Rightarrow, "\varphi"'] & 0 \end{tikzcd}$$

If moreover (G'', φ'') satisfies the same condition as (G', φ'), then there exists a unique 2-cell $\psi\colon G'' \Rightarrow G'$ such that

$$\begin{tikzcd} & G & \\ G'' \cdot e_F \arrow[ur, Rightarrow, "\varphi''"] \arrow[rr, Rightarrow, "\psi \cdot e_F"'] & & G' \cdot e_F \arrow[ul, Rightarrow, "\varphi'"'] \end{tikzcd}$$

commutes.

We list here some facts we will use later.

Proposition 3 *Let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups;*

1) $\pi_0(KerF)$ *and* $\pi_1(CokerF)$ *are isomorphic groups;*

2) *the functor* e_F *is faithful; it is full iff* $\pi_1(\mathbb{H})=0$;

3) *the functor* P_F *is essentially surjective; it is full iff* $\pi_0(\mathbb{G})=0$;

4) *the factorization of* $\pi_1(e_F)$ *through the kernel of* $\pi_1(F)$ *is an isomorphism;*

5) *the factorization of* $\pi_0(P_F)$ *through the cokernel of* $\pi_0(F)$ *is an isomorphism;*

6) F *is faithful iff* $\pi_1(F)$ *is injective iff* $\pi_1(KerF)=0$;

7) F *is essentially surjective iff* $\pi_0(F)$ *is surjective iff* $\pi_0(CokerF)=0$;

8) F *is full iff* $\pi_0(F)$ *is injective and* $\pi_1(F)$ *is surjective iff* $\pi_0(KerF)=0$ *iff* $\pi_1(CokerF)=0$;

9) F *is an equivalence iff* $\pi_0(F)$ *and* $\pi_1(F)$ *are isomorphisms iff* $KerF$ *and* $CokerF$ *are equivalent to the cat-group with a single arrow.*

To end this section, let us establish a link with homotopical algebra. The cat-group $KerF$ just described satisfies also the following strict universal property (and a dual one is satisfied by $CokerF$) : given a morphism $G\colon \mathbb{K}\to\mathbb{G}$ and a 2-cell $\varphi\colon G\cdot F\Rightarrow 0$, there exists a unique morphism $G'\colon\mathbb{K}\to KerF$ such that $G'\cdot e_F=G$ and $G'\cdot\lambda_F=\varphi$. In other words, $KerF$ is also a standard homotopy kernel, or homotopy fibre, and $CokerF$ is also a standard homotopy cokernel, or mapping cone (and they are determined up to isomorphisms by these universal properties), while the more general universal properties we are considering can be viewed as providing a "general" homotopy kernel or cokernel, determined up to equivalences, consistently with Mather's original definition for topological spaces [14]. M. Grandis has developed in [9] an axiomatic for homotopical algebra based on homotopy kernels and cokernels. Following his approach, the discrete cat-groups

$$\underline{D}(\pi_1(\mathbb{G}))=Ker(\mathbf{1}\to\mathbb{G})$$

($\mathbf{1}$ is the cat-group with a single arrow) is the object of loops $\Omega(\mathbb{G})$, so that $\pi_1(\mathbb{G})=\pi_0(\Omega(\mathbb{G}))$. Moreover, the cat group

$$\pi_0(\mathbb{G})!=Coker(\mathbb{G}\to\mathbf{1})$$

is the suspension $\Sigma(\mathbb{G})$, so that $\pi_0(\mathbb{G})=\pi_1(\Sigma(\mathbb{G}))$. Finally, the morphisms $\pi_1(\mathbb{G})!\to\mathbb{G}$ and $\mathbb{G}\to\underline{D}(\pi_0(\mathbb{G}))$ are precisely the counit and the unit of the adjunction $\Sigma\dashv\Omega$; this adjunction is a straightforward consequence of the strict universal properties defining these functors.

2 Factorizations of a morphism

In this section we construct two factorizations of a morphism of symmetric cat-groups. Anticipating what will be explained in section 5, we can think of the first factorization as a (epi, regular mono) factorization and of the second one as a (regular epi, mono) factorization.

First factorization: let $F \colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups. Consider its cokernel $P_F \colon \mathbb{H} \to CokerF$ and the kernel of P_F, which we call *image*, $j_F \colon ImF \to \mathbb{H}$. The universal property of ImF gives us a morphism $\hat{F} \colon \mathbb{G} \to ImF$ and a 2-cell $\hat{F} \cdot j_F \Rightarrow F$. Moreover, it is possible to choose (in a unique way) the morphism $\hat{F}$ so that the 2-cell is the identity. In other words, the following is a commutative diagram in SCG

$$\begin{tikzcd} \mathbb{G} \arrow[rr, "F"] \arrow[dr, "\hat{F}"'] & & \mathbb{H} \\ & ImF \arrow[ur, "j_F"'] & \end{tikzcd}$$

Proposition 4 *With the above notations,*

1) the functor j_F is faithful;

2) the functor $\hat{F}$ is full and essentially surjective.

Proof: 1) j_F is a kernel, so it is faithful by point 2 of proposition 3.
2) Explicitely, an object in ImF is a class of triples (X, f, N) with X an object of $\mathbb{H}$, N an object of $\mathbb{G}$ and $f \colon X \to F(N)$; we identify (X, f, N) with (X, f', N') if there exists $\alpha \colon N \to N'$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(N) \arrow[rr, "F(\alpha)"'] & & F(N') \end{tikzcd}$$

commutes. An arrow $\lambda \colon [X, f, N] \longrightarrow\!\!\bullet\!\!\rightarrow [Y, g, M]$ is an arrow $\lambda \colon X \to Y$ such that there exists an arrow $l \colon N \to M$ making commutative the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ F(N) \arrow[r, "F(l)"'] & F(M) \end{tikzcd}$$

The functor $\hat{F} \colon \mathbb{G} \to ImF$ sends $\mu \colon A \to B$ into

$$F(\mu) \colon \; [F(A), 1_{F(A)}, A] \longrightarrow\!\!\bullet\!\!\rightarrow [F(B), 1_{F(B)}, B] \; .$$

- $\hat{F}$ is essentially surjective: let $[X, f, N]$ be an object of ImF, then

$$f\colon\ [X, f, N] \longrightarrow \hat{\mathrm{F}}(N)$$

is an arrow in ImF because

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & F(N) \arrow[d, "1_{F(N)}"] \\ F(N) \arrow[r, "F(1_N)"'] & F(N) \end{tikzcd}$$

commutes.

- $\hat{F}$ is full: let A, B be in $\mathbb{G}$ and consider an arrow $\lambda\colon\ \hat{\mathrm{F}}(A) \longrightarrow \hat{\mathrm{F}}(B)$ in ImF. This means that $\lambda\colon F(A) \to F(B)$ is an arrow in $\mathbb{H}$ and there exists $l\colon A \to B$ in $\mathbb{G}$ such that

$$\begin{tikzcd} F(A) \arrow[r, "\lambda"] \arrow[d, "1_{F(A)}"'] & F(B) \arrow[d, "1_{F(B)}"] \\ F(A) \arrow[r, "F(l)"'] & F(B) \end{tikzcd}$$

commutes. And then $\lambda = \hat{F}(l)$ in ImF.

$\diamondsuit$

Corollary 5 *Let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups; the following conditions are equivalent:*

1) F is a kernel, that is there exists a morphism $G\colon \mathbb{H} \to \mathbb{K}$ such that $F \cdot G$ is isomorphic to the zero-morphism and the induced morphism $\mathbb{G} \to KerG$ is an equivalence;

2) F is the kernel of its cokernel, that is $\hat{F}$ is an equivalence;

3) F is faithful.

Proof: If F is faithful, then $\hat{F}$ is faithful and then it is an equivalence. The implications $2) \Rightarrow 1) \Rightarrow 3)$ are obvious.

$\diamondsuit$

Second factorization: let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups. Consider its kernel $e_F\colon KerF \to \mathbb{G}$ and the cokernel of e_F, which we call *coimage*, $F'\colon \mathbb{G} \to CoimF$. As for the first factorization, the universal property of

$CoimF$ gives us a morphism $i_F\colon CoimF \to \mathbb{H}$ making commutative the following diagram

$$\begin{tikzcd} \mathbb{G} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathbb{H} \\ & CoimF \arrow[ur, "i_F"'] & \end{tikzcd}$$

Proposition 6 *With the above notations,*

1) the functor F' is essentially surjective;

2) the functor i_F is full and faithful.

Proof: 1) F' is a cokernel, so it is essentially surjective by point 3) of proposition 3.
2) Objects of $CoimF$ are those of $\mathbb{G}$. An arrow $[f, N, \lambda_N]\colon X \longrightarrow\!\!\!\!\bullet\;\, Y$ in $CoimF$ is a class of triple (f, N, λ_N) with N in $\mathbb{G}$, $\lambda_N\colon F(N) \to I$ and $f\colon X \to Y \otimes N$; we identify (f, N, λ_N) with (g, M, λ_M) if there exists $\alpha\colon N \to M$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \otimes N \arrow[rr, "1\otimes\alpha"'] & & Y \otimes M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(N) \arrow[rr, "F(\alpha)"] \arrow[dr, "\lambda_N"'] & & F(M) \arrow[dl, "\lambda_M"] \\ & I & \end{tikzcd}$$

commute. The functor i_F sends $[f, N, \lambda_N]\colon X \longrightarrow\!\!\!\!\bullet\;\, Y$ into

$$F(X) \xrightarrow{F(f)} F(Y \otimes N) \simeq F(Y) \otimes F(N) \xrightarrow{1\otimes\lambda_N} F(Y) \otimes I \simeq F(Y).$$

- i_F is faithful : consider two parallel arrows in $CoimF$,

  $$[f, N, \lambda_N], [g, M, \lambda_M]\colon X \longrightarrow\!\!\!\!\bullet\;\, Y$$

  and suppose $i_F[f, N, \lambda_N] = i_F[g, M, \lambda_M]$. Using $f\colon X \to Y \otimes N$ and $g\colon X \to Y \otimes M$, we obtain $\beta = f^{-1} \cdot g\colon Y \otimes N \to Y \otimes M$ and

  $$\alpha\colon N \simeq I \otimes N \xrightarrow{\epsilon_Y^{-1}\otimes 1} Y^* \otimes Y \otimes N \xrightarrow{\beta} Y^* \otimes Y \otimes M \xrightarrow{\epsilon_Y\otimes 1} I \otimes M \simeq M$$

  It remains to check the two conditions on α :
  - the first one, that is the equation $f \cdot (1 \otimes \alpha) = g$, becomes $\beta = 1 \otimes \alpha$, which follows from a diagram chase ;

- as far as the second one is concerned, consider the following diagram

$$
\begin{tikzcd}
 & F(Y \otimes N) \arrow[r] \arrow[dd, "F(1\otimes\alpha)"] & F(Y) \otimes F(N) \arrow[r, "1\otimes\lambda_N"] \arrow[dd, "1\otimes F(\alpha)"] & F(Y) \otimes I \arrow[dd, "1"] \arrow[dr] & \\
F(X) \arrow[ur, "F(f)"] \arrow[dr, "F(g)"'] & & & & F(Y) \\
 & F(Y \otimes M) \arrow[r] & F(Y) \otimes F(M) \arrow[r, "1\otimes\lambda_M"'] & F(Y) \otimes I \arrow[ur] &
\end{tikzcd}
$$

The left-hand triangle commutes because $f \cdot (1 \otimes \alpha) = g$, the first square commutes because F is monoidal, and the hypothesis $i_F[f, N, \lambda_N] = i_F[g, M, \lambda_M]$ precisely means that the outer diagram is commutative. Since all the arrows are isomorphisms, it follows that the second square is commutative. Finally, this commutativity implies the second condition on α, that is $F(\alpha) \cdot \lambda_M = \lambda_N$, because $F(Y)$ is invertible, so that tensoring with $F(Y)$ is an autoequivalence of $\mathbb{H}$.

- i_F is full : recall that we have a natural transformation

$$\pi_{e_F} : e_F \cdot F' \Rightarrow 0$$

and that F' is the identity on objects. Consider two objects X, Y in $CoimF$ and an arrow $h \colon i_F(X) = F(X) \to F(Y) = i_F(Y)$ in $\mathbb{H}$. We can build up an object $(Y^* \otimes X, \lambda)$ of $KerF$ with λ defined by

$$\lambda \colon F(Y^* \otimes X) \simeq F(Y^*) \otimes F(X) \xrightarrow{1\otimes h} F(Y^*) \otimes F(Y) \simeq$$

$$\simeq F(Y^* \otimes Y) \xrightarrow{\quad F(\epsilon_Y) \quad} F(I) \simeq I$$

Using the component at $(Y^* \otimes X, \lambda)$ of the transformation π_{e_F}, we obtain the following arrow in $CoimF$

$$k \colon X \xrightarrow{\ \bullet\ } I \otimes X \xrightarrow{\ \bullet\ } Y \otimes Y^* \otimes X \xrightarrow{1\otimes\pi_{e_F}(Y^*\otimes X, \lambda)} Y \otimes I \xrightarrow{\ \bullet\ } Y.$$

Finally, using the triangular identities of the duality $(Y^* \dashv Y, \eta_Y, \epsilon_Y)$, one checks that $i_F(k) = h$.

$\diamondsuit$

Corollary 7 *Let $F \colon \mathbb{G} \to \mathbb{H}$ be a morphism of symmetric cat-groups; the following conditions are equivalent:*

1) F is a cokernel;

2) F is the cokernel of its kernel;

3) F is essentially surjective.

Proof: If F is essentially surjective, then i_F is an equivalence.

$\diamondsuit$

3 Factorization systems in a 2-category

The notion of factorization system in a category is a well-established one. To give its natural 2-categorical version, and to prove some elementary facts, we closely follow the review of factorization systems given in [3] (see also [12]).

In this section, $\mathcal{C}$ is a 2-category with invertible 2-cells. Given two arrows $f\colon A \to B$ and $g\colon C \to D$, we say that f has the fill-in property with respect to g, in symbols $f \downarrow g$, if for each pair of arrows $u\colon A \to C, v\colon B \to D$ and for each 2-cell $\varphi\colon f \cdot v \Rightarrow u \cdot g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, Rightarrow, "\varphi"'] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

there exists an arrow $w\colon B \to C$ and two 2-cells $\alpha\colon f \cdot w \Rightarrow u$, $\beta\colon w \cdot g \Rightarrow v$ such that the following diagram commutes

$$\begin{tikzcd} f \cdot w \cdot g \arrow[rr, Rightarrow, "f\cdot\beta"] \arrow[dr, Rightarrow, "\alpha\cdot g"'] & & f \cdot v \arrow[dl, Rightarrow, "\varphi"] \\ & u \cdot g & \end{tikzcd}$$

(we say that (α, w, β) is a fill-in for (u, φ, v)); moreover, if (α', w', β') is another fill-in for (u, φ, v), then there exists a unique $\psi\colon w \Rightarrow w'$ such that

$$\begin{tikzcd} f \cdot w \arrow[rr, Rightarrow, "f\cdot\psi"] \arrow[dr, Rightarrow, "\alpha"'] & & f \cdot w' \arrow[dl, Rightarrow, "\alpha'"] \\ & u & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} w \cdot g \arrow[rr, Rightarrow, "\psi\cdot g"] \arrow[dr, Rightarrow, "\beta"'] & & w' \cdot g \arrow[dl, Rightarrow, "\beta'"] \\ & v & \end{tikzcd}$$

commute. (See [4] for an interpretation of this conditions in terms of bilimits.) If $\mathcal{H}$ is a class of arrows of $\mathcal{C}$, we use the following notations :

$$\mathcal{H}^{\downarrow} = \{g \text{ s.t. } h \downarrow g \text{ for each } h \text{ in } \mathcal{H}\}$$

$$\mathcal{H}^{\uparrow} = \{f \text{ s.t. } f \downarrow h \text{ for each } h \text{ in } \mathcal{H}\}$$

Clearly, if $\mathcal{H}_1 \subset \mathcal{H}_2$ then $\mathcal{H}_1^{\downarrow} \supset \mathcal{H}_2^{\downarrow}$ and $\mathcal{H}_1^{\uparrow} \supset \mathcal{H}_2^{\uparrow}$

Definition 8 *A factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$ is given by two classes $\mathcal{E}$ and $\mathcal{M}$ of arrows in $\mathcal{C}$ such that :*

1) $\mathcal{E}$ and $\mathcal{M}$ contain equivalences and are closed under composition with equivalences ;

2) $\mathcal{E}$ and $\mathcal{M}$ are stable under 2-cells (this means that if e is in $\mathcal{E}$ and there is a 2-cell $e' \Rightarrow e$, then also e' is in $\mathcal{E}$, and the same holds for $\mathcal{M}$) ;

3) for each arrow f of $\mathcal{C}$, there exist $e \in \mathcal{E}$, $m \in \mathcal{M}$ and a 2-cell φ

$$\begin{tikzcd} & I \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\varphi"] & \\ A \arrow[ur, "e"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

(we say that (e, φ, m) is a $(\mathcal{E}, \mathcal{M})$-factorization for f) ;

4) for each $e \in \mathcal{E}$ and for each $m \in \mathcal{M}$, one has $e \downarrow m$.

Proposition 9 *Let $(\mathcal{E}, \mathcal{M})$ be a factorization system in $\mathcal{C}$;*

1) if f is in $\mathcal{E}$ and in $\mathcal{M}$, then f is an equivalence ;

2) $\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$;

3) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition ;

4) $\mathcal{E}$ is stable under bi-pushout and $\mathcal{M}$ is stable under bi-pullback ;

5) if $(\mathcal{E}', \mathcal{M}')$ is another factorization system in $\mathcal{C}$, then $\mathcal{E} \subset \mathcal{E}'$ iff $\mathcal{M} \supset \mathcal{M}'$.

Proof: The proof is a quite long but essentially straightforward 2-categorical version of that of the corresponding 1-categorical properties. To give the flavor of the proof, we check that $\mathcal{M}^{\uparrow}$ is closed under composition. Consider the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] & C \arrow[d, "v"] \arrow[dll, Rightarrow, "\varphi"'] \\ D \arrow[rr, "m"'] & & E \end{tikzcd}$$

with $f, g \in \mathcal{M}^{\uparrow}$ and $m \in \mathcal{M}$. Since $f \downarrow m$, there exists a fill-in (α, w, β) for $(u, \varphi, g \cdot v)$. Since $g \downarrow m$, there exists a fill-in (ϵ, z, μ) for (w, β^{-1}, v). In this way we obtain a fill-in

$$(\ f \cdot g \cdot z \xRightarrow{f \cdot \epsilon} f \cdot w \xRightarrow{\alpha} u\ , z, \mu)$$

for (u,φ,v). Now suppose that $(\overline{\eta},\overline{z},\overline{\mu})$ is another fill-in for (u,φ,v). This implies that $(\overline{\eta},g\cdot\overline{z},g\cdot\overline{\mu})$ is a fill-in for $(u,\varphi,g\cdot v)$, so that there exists a unique $\psi\colon w\Rightarrow g\cdot\overline{z}$ such that the following diagrams commute

$$\begin{tikzcd} f\cdot w \arrow[rr, Rightarrow, "f\cdot\psi"] \arrow[dr, Rightarrow, "\alpha"'] & & f\cdot g\cdot\overline{z} \arrow[dl, Rightarrow, "\overline{\eta}"] \\ & u & \end{tikzcd} \qquad \begin{tikzcd} w\cdot m \arrow[rr, Rightarrow, "\psi\cdot m"] \arrow[dr, Rightarrow, "\beta"'] & & g\cdot\overline{z}\cdot m \arrow[dl, Rightarrow, "g\cdot\overline{\mu}"] \\ & g\cdot v & \end{tikzcd}$$

This implies that $(\psi^{-1},\overline{z},\overline{\mu})$ is a fill-in for (w,β^{-1},v), so that there exists a unique $\sigma\colon z\Rightarrow\overline{z}$ such that the following diagrams commute

$$\begin{tikzcd} g\cdot z \arrow[rr, Rightarrow, "g\cdot\sigma"] \arrow[dr, Rightarrow, "\epsilon"'] & & g\cdot\overline{z} \arrow[dl, Rightarrow, "\psi^{-1}"] \\ & w & \end{tikzcd} \qquad \begin{tikzcd} z\cdot m \arrow[rr, Rightarrow, "\sigma\cdot m"] \arrow[dr, Rightarrow, "\mu"'] & & \overline{z}\cdot m \arrow[dl, Rightarrow, "\overline{\mu}"] \\ & v & \end{tikzcd}$$

and then commutes also

$$\begin{tikzcd} f\cdot g\cdot z \arrow[r, Rightarrow, "f\cdot g\cdot\sigma"] \arrow[d, Rightarrow, "f\cdot\epsilon"'] & f\cdot g\cdot\overline{z} \arrow[d, Rightarrow, "\overline{\eta}"] \\ f\cdot w \arrow[r, Rightarrow, "\alpha"'] & u \end{tikzcd}$$

It remains to check that if $\sigma'\colon z\Rightarrow\overline{z}$ is another 2-cell making commutative

$$\begin{tikzcd} f\cdot g\cdot z \arrow[r, Rightarrow, "f\cdot g\cdot\sigma'"] \arrow[d, Rightarrow, "f\cdot\epsilon"'] & f\cdot g\cdot\overline{z} \arrow[d, Rightarrow, "\overline{\eta}"] \\ f\cdot w \arrow[r, Rightarrow, "\alpha"'] & u \end{tikzcd} \qquad \begin{tikzcd} z\cdot m \arrow[rr, Rightarrow, "\sigma'\cdot m"] \arrow[dr, Rightarrow, "\mu"'] & & \overline{z}\cdot m \arrow[dl, Rightarrow, "\overline{\mu}"] \\ & v & \end{tikzcd}$$

then $\sigma=\sigma'$. For this, it suffices to prove the commutativity of

$$\begin{tikzcd} g\cdot z \arrow[rr, Rightarrow, "g\cdot\sigma'"] \arrow[dr, Rightarrow, "\epsilon"'] & & g\cdot\overline{z} \arrow[dl, Rightarrow, "\psi^{-1}"] \\ & w & \end{tikzcd}$$

that is to prove that $w \xRightarrow{\epsilon^{-1}} g\cdot z \xRightarrow{g\cdot\sigma'} g\cdot\overline{z}$ satisfies the conditions which characterize ψ. The first one coincides with the first assumption on σ', the second one follows from the fact that (ϵ,z,μ) is a fill-in for (w,β^{-1},v).

$\diamondsuit$

We leave it as an exercise for the reader to formulate the correct notion of functoriality and uniqueness for the $(\mathcal{E},\mathcal{M})$-factorization. To end this section, observe that in general a factorization system in $\mathcal{C}$ does not induce a factorization system (in the usual sense) neither in the underlying category of $\mathcal{C}$, nor in the homotopy category of $\mathcal{C}$.

4 Factorization systems in SCG

In this section we show that the factorizations of a morphism between symmetric cat-groups constructed in section 2 satisfy the axioms introduced in section 3.

<u>First factorization:</u> we need two simple lemmas.

Lemma 10 *Consider the following morphisms in SCG*

$$\begin{tikzcd} \mathbb{L} \arrow[r, "G"] & \mathbb{K} \arrow[r, shift left, "K"] \arrow[r, shift right, "H"'] & \mathbb{H} \end{tikzcd}$$

and assume that G is full and essentially surjective ; for each 2-cell $\mu\colon G\cdot K \Rightarrow G\cdot H$ there exists a unique 2-cell $\overline{\mu}\colon K \Rightarrow H$ such that $G\cdot\overline{\mu} = \mu$.

Proof: Let X be an object of $\mathbb{K}$ and choose an object A_X in $\mathbb{L}$ and a morphism $x\colon X \to G(A_X)$; we can put

$$\overline{\mu}_X\colon K(X) \xrightarrow{K(x)} K(G(A_X)) \xrightarrow{\mu_{A_X}} H(G(A_X)) \xrightarrow{H(x^{-1})} H(X)$$

This definition does not depend on the choice of A_X and x because G is full: if $x'\colon X \to G(A'_X)$ is another choice, there exists a $z\colon A_X \to A'_X$ such that $G(z) = x^{-1}\cdot x'$; now the naturality of μ implies that $K(x)\cdot\mu_{A_X}\cdot H(x^{-1}) = K(x')\cdot\mu_{A'_X}\cdot H(x'^{-1})$. In particular, if $X = G(A)$, we can choose $A_X = A$ and $x = 1_A$, so that $\overline{\mu}_{G(A)} = \mu_A$, that is $G\cdot\overline{\mu} = \mu$. Now let $f\colon X \to Y$ be a morphism in $\mathbb{K}$ and choose $x\colon X \to G(A_X)$, $y\colon Y \to G(A_Y)$. Once again, the fullness of G gives us a morphism $\varphi\colon A_X \to A_Y$ such that $G(\varphi) = x^{-1}\cdot f\cdot y$. The following diagram commutes in each part, so that $\overline{\mu}$ is natural

$$\begin{tikzcd} K(X) \arrow[r, "K(x)"] \arrow[d, "K(f)"'] & K(G(A_X)) \arrow[r, "\mu_{A_X}"] \arrow[d, "K(G(\varphi))"'] & H(G(A_X)) \arrow[r, "K(x^{-1})"] \arrow[d, "H(G(\varphi))"'] & H(X) \arrow[d, "H(f)"] \\ K(Y) \arrow[r, "K(y)"'] & K(G(A_Y)) \arrow[r, "\mu_{A_Y}"'] & H(G(A_Y)) \arrow[r, "H(y^{-1})"'] & H(Y) \end{tikzcd}$$

As far as the uniqueness of $\overline{\mu}$ is concerned, observe that its naturality implies the commutativity of

$$\begin{tikzcd} K(X) \arrow[r, "\overline{\mu}_X"] \arrow[d, "K(x)"'] & H(X) \arrow[d, "H(x)"] \\ K(G(A_X)) \arrow[r, "\overline{\mu}_{G(A_X)}"'] & H(G(A_X)) \end{tikzcd}$$

so that the condition $G\cdot\overline{\mu} = \mu$ forces our definition of $\overline{\mu}$. Finally, it is easy to check that if μ is monoidal, then also $\overline{\mu}$ is monoidal.

$\diamondsuit$

Lemma 11 *Consider three morphisms H, K and F in SCG and the kernel of F, as in the diagram*

$$\begin{tikzcd} & & \mathbb{G} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \lambda_F"] & \\ \mathbb{L} \arrow[r, shift left, "K"] \arrow[r, shift right, "H"'] & KerF \arrow[ur, "e_F"] \arrow[rr, "0"'] & & \mathbb{H} \end{tikzcd}$$

let $\rho \colon K \cdot e_F \Rightarrow H \cdot e_F$ be a 2-cell in SCG such that

$$\begin{tikzcd} K \cdot e_F \cdot F \arrow[r, Rightarrow, "\rho \cdot F"] \arrow[d, Rightarrow, "K \cdot \lambda_F"'] & H \cdot e_F \cdot F \arrow[d, Rightarrow, "H \cdot \lambda_F"] \\ K \cdot 0 \arrow[r, Rightarrow] & H \cdot 0 \end{tikzcd}$$

commutes. There exists a unique 2-cell $\overline{\rho} \colon K \Rightarrow H$ such that $\overline{\rho} \cdot e_F = \rho$.

Proof: It suffices to check the particular case where $H = 0$, and this case follows easily from the universal property of the kernel.

$\diamondsuit$

Proposition 12 *Consider the following classes of morphisms in SCG :*

$\mathcal{E}_1$ = full and essentially surjective morphisms ;

$\mathcal{M}_1$ = faithful morphisms.

$(\mathcal{E}_1, \mathcal{M}_1)$ is a factorization system in the sense of Definition 8.

Proof: Clearly, $\mathcal{E}_1$ and $\mathcal{M}_1$ contain equivalences, are closed under composition with equivalences and are stable under 2-cells. The existence of an $(\mathcal{E}_1, \mathcal{M}_1)$-factorization has been proved in Proposition 4, so that it remains to prove the fill-in condition. For this, consider $G \colon \mathbb{L} \to \mathbb{K}$ in $\mathcal{E}_1$ and let

$$\begin{tikzcd} & \mathbb{G} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \lambda_F"] & \\ KerF \arrow[ur, "e_F"] \arrow[rr, "0"'] & & \mathbb{H} \end{tikzcd}$$

be the kernel of a morphism F. We start showing that $G \downarrow e_F$. Consider two morphisms $U \colon \mathbb{L} \to KerF$, $V \colon \mathbb{K} \to \mathbb{G}$ and a 2-cell $\varphi \colon G \cdot V \Rightarrow U \cdot e_F$. We obtain a new 2-cell

$$\mu \colon G \cdot V \cdot F \xRightarrow{\varphi \cdot F} U \cdot e_F \cdot F \xRightarrow{U \cdot \lambda_F} U \cdot 0 \Longrightarrow G \cdot 0$$

and, by Lemma 10, there exists a unique 2-cell $\overline{\mu}\colon V \cdot F \Rightarrow 0$ such that $G \cdot \overline{\mu} = \mu$. Now the universal property of the kernel gives us a morphism $W\colon \mathbb{K} \to KerF$ and a 2-cell $\beta\colon W \cdot e_F \Rightarrow V$ such that

$$\begin{tikzcd}
W \cdot e_F \cdot F \arrow[r, Rightarrow, "\beta \cdot e_F"] \arrow[d, Rightarrow, "W \cdot \lambda_F"'] & V \cdot F \arrow[d, Rightarrow, "\overline{\mu}"] \\
W \cdot 0 \arrow[r, Rightarrow] & 0
\end{tikzcd}$$

commutes. We obtain a 2-cell

$$\rho\colon G \cdot W \cdot e_F \xRightarrow{G \cdot \beta} G \cdot V \xRightarrow{\varphi} U \cdot e_F$$

making commutative the following diagram

$$\begin{tikzcd}
G \cdot W \cdot e_F \cdot F \arrow[r, Rightarrow, "\rho \cdot F"] \arrow[d, Rightarrow, "G \cdot W \cdot \lambda_F"'] & U \cdot e_F \cdot F \arrow[d, Rightarrow, "U \cdot \lambda_F"] \\
G \cdot W \cdot 0 \arrow[r, Rightarrow] & U \cdot 0
\end{tikzcd}$$

By Lemma 11, there exists a unique 2-cell $\overline{\rho}\colon G \cdot W \Rightarrow U$ such that $\overline{\rho} \cdot e_F = \rho$. This means that $(\overline{\rho}, W, \beta)$ is a fill-in for (U, φ, V). Suppose now that $(\overline{\alpha}, \overline{W}, \overline{\beta})$ is another fill-in for (U, φ, V). The condition to be a fill-in implies the commutativity of

$$\begin{tikzcd}
G \cdot \overline{W} \cdot e_F \cdot F \arrow[r, Rightarrow, "G \cdot \overline{\beta} \cdot F"] \arrow[d, Rightarrow, "G \cdot \overline{W} \cdot \lambda_F"'] & G \cdot V \cdot F \arrow[d, Rightarrow, "G \cdot \overline{\mu}"] \\
G \cdot \overline{W} \cdot 0 \arrow[r, Rightarrow] & G \cdot 0
\end{tikzcd}$$

By Lemma 10, this implies the commutativity of

$$\begin{tikzcd}
\overline{W} \cdot e_F \cdot F \arrow[r, Rightarrow, "\overline{\beta} \cdot F"] \arrow[d, Rightarrow, "\overline{W} \cdot \lambda_F"'] & V \cdot F \arrow[d, Rightarrow, "\overline{\mu}"] \\
\overline{W} \cdot 0 \arrow[r, Rightarrow] & 0
\end{tikzcd}$$

so that, by the universal property of the kernel, there exists a unique $\psi\colon W \Rightarrow \overline{W}$ such that

$$\begin{tikzcd}
W \cdot e_F \arrow[rr, Rightarrow, "\psi \cdot e_F"] \arrow[dr, Rightarrow, "\beta"'] & & \overline{W} \cdot e_F \arrow[dl, Rightarrow, "\overline{\beta}"] \\
& U &
\end{tikzcd}$$

commutes. It remains only to check the commutativity of

$$\begin{tikzcd} G \cdot W \arrow[rr, Rightarrow, "G \cdot \psi"] \arrow[dr, Rightarrow, "\overline{\rho}"'] & & G \cdot \overline{W} \arrow[dl, Rightarrow, "\overline{\alpha}"] \\ & U & \end{tikzcd}$$

and to do this we use once again the universal property of the kernel. Consider the 2-cell

$$U \cdot e_F \cdot F \xRightarrow{U \cdot \lambda_F} U \cdot 0 \Longrightarrow 0 \ \ ;$$

it factors through the kernel of F in two ways : the obvious one, that is $(U, U \cdot e_F)$, but also $(G \cdot W, \overline{\rho} \cdot e_F)$. By the universal property, there exists a unique $\sigma \colon G \cdot W \Rightarrow U$ such that

$$\begin{tikzcd} G \cdot W \cdot e_F \arrow[rr, Rightarrow, "\sigma \cdot e_F"] \arrow[dr, Rightarrow, "\overline{\rho} \cdot e_F"'] & & U \cdot e_F \arrow[dl, Rightarrow, "U \cdot e_F"] \\ & U \cdot e_F & \end{tikzcd}$$

commutes. Clearly, we can take $\overline{\rho}$ as σ, but also

$$G \cdot W \xRightarrow{G \cdot \psi} G \cdot W \xRightarrow{\overline{\alpha}} U$$

can be taken as σ. By uniqueness of σ, we have finished.
Now we turn to the general case. Let G be in $\mathcal{E}_1$ and F in $\mathcal{M}_1$; we have to prove that $G \downarrow F$. For this we consider the factorization of F

$$\begin{tikzcd} \mathbb{G} \arrow[rr, "F"] \arrow[dr, "\hat{F}"'] & & \mathbb{H} \\ & ImF \arrow[ur, "j_F"'] & \end{tikzcd}$$

By Proposition 4, $\hat{F}$ is an equivalence. By a general argument (see [11]) we can choose a quasi-inverse $\overline{F} \colon ImF \to \mathbb{G}$ and two natural transformations $\epsilon \colon \hat{F} \cdot \overline{F} \Rightarrow Id$, $\eta \colon Id \Rightarrow \overline{F} \cdot \hat{F}$ in such a way that $(\overline{F}, \hat{F}, \eta, \epsilon)$ is an adjoint equivalence in SCG. By the first part of the proof, we have $G \downarrow j_F$; it is now straightforward, even if quite long, to prove $G \downarrow F$.

$\diamondsuit$

Second factorization: as for the first factorization, we need two preliminary lemmas.

Lemma 13 *Consider the following morphisms in SCG*

$$\mathbb{H} \overset{K}{\underset{H}{\rightrightarrows}} \mathbb{K} \xrightarrow{G} \mathbb{L}$$

and assume that G is full and faithful ; for each 2-cell $\mu \colon K \cdot G \Rightarrow H \cdot G$ there exists a unique 2-cell $\overline{\mu} \colon K \Rightarrow H$ such that $\overline{\mu} \cdot G = \mu$.

Proof: For each object A in $\mathbb{H}$, there exists a unique $\overline{\mu}_A \colon K(A) \to H(A)$ such that $G(\overline{\mu}_A) = \mu_A$. The naturality of $\overline{\mu}$ as well as its uniqueness follows from the faithfulness of G. It is easy to check that if μ is monoidal, then also $\overline{\mu}$ is monoidal.

$\diamondsuit$

Lemma 14 *Consider three morphisms H, K and F in SCG and the cokernel of F, as in the diagram*

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "P_F"] \arrow[d, phantom, "\Downarrow \pi_F"] & & \\ \mathbb{G} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & CokerF \arrow[r, shift left, "K"] \arrow[r, shift right, "H"'] & \mathbb{L} \end{tikzcd}$$

let $\rho \colon P_F \cdot K \Rightarrow P_F \cdot H$ be a 2-cell in SCG such that

$$\begin{tikzcd} F \cdot P_F \cdot K \arrow[r, Rightarrow, "F \cdot \rho"] \arrow[d, Rightarrow, "\pi_F \cdot K"'] & F \cdot P_F \cdot H \arrow[d, Rightarrow, "\pi_F \cdot H"] \\ 0 \cdot K \arrow[r, Rightarrow] & 0 \cdot H \end{tikzcd}$$

commutes. There exists a unique 2-cell $\overline{\rho} \colon K \Rightarrow H$ such that $P_F \cdot \overline{\rho} = \rho$.

Proposition 15 *Consider the following classes of morphisms in SCG* :

$\mathcal{E}_2 =$ *essentially surjective morphisms ;*

$\mathcal{M}_2 =$ *full and faithful morphisms.*

$(\mathcal{E}_2, \mathcal{M}_2)$ is a factorization system in the sense of Definition 8.

Proof: The proof follows the same lines as the proof of Proposition 12. Once again, it is easier to start showing that, if G is in $\mathcal{M}_2$ and F is any morphism, then $P_F \downarrow G$. The general case follows from this using that, by Proposition 6, the morphism i_F in the second factorization of F

$$\begin{tikzcd} \mathbb{G} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathbb{H} \\ & CoimF \arrow[ur, "i_F"'] & \end{tikzcd}$$

is an equivalence if F is essentially surjective.

$\diamondsuit$

5 More on morphisms in SCG

In this section we want to invert Lemma 10 and Lemma 13.

Lemma 16 *Let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism in SCG; the following conditions are equivalent :*

1) F is faithful ;

2) for each symmetric cat-group $\mathbb{K}$, the functor induced by F

$$- \cdot F\colon SCG(\mathbb{K}, \mathbb{G}) \to SCG(\mathbb{K}, \mathbb{H})$$

is faithful ;

3) given two morphisms $K, H\colon \mathbb{L} \to \mathbb{G}$ and a 2-cell $\rho\colon K \cdot F \Rightarrow H \cdot F$ such that

$$\begin{tikzcd} K \cdot F \cdot P_F \arrow[r, Rightarrow, "\rho \cdot P_F"] \arrow[d, Rightarrow, "K \cdot \pi_F"'] & H \cdot F \cdot P_F \arrow[d, Rightarrow, "H \cdot \pi_F"] \\ K \cdot 0 \arrow[r, Rightarrow] & H \cdot 0 \end{tikzcd}$$

commutes, then there exists a unique 2-cell $\overline{\rho}\colon K \Rightarrow H$ such that $\overline{\rho} \cdot F = \rho$.

Proof: 1) $\Rightarrow$ 3) : This follows from Lemma 11 because, by Corollary 5, F is the kernel of P_F.
3) $\Rightarrow$ 2) : If ρ is of the form $\alpha \cdot F$ for a 2-cell $\alpha\colon H \Rightarrow H$, then the condition on ρ holds and then $\beta \cdot F = \rho = \alpha \cdot F$ implies $\beta = \overline{\rho} = \alpha$.
2) $\Rightarrow$ 1) : Consider the abelian group $\mathbb{Z}$ and take $\mathbb{K} = \mathbb{Z}!$, so that $- \cdot F$ becomes $\pi_1(F)$. Now $- \cdot F$ faithful means $\pi_1(F)$ injective, so that F is faithful by point 6) in Proposition 3.

$\diamondsuit$

Proposition 17 *Let $F\colon \mathbb{G} \to \mathbb{H}$ be a morphism in SCG; the following conditions are equivalent :*

1) F is full and faithful ;

2) for each symmetric cat-group $\mathbb{K}$, the functor induced by F

$$- \cdot F\colon SCG(\mathbb{K}, \mathbb{G}) \to SCG(\mathbb{K}, \mathbb{H})$$

is full and faithful ;

Proof: 1) $\Rightarrow$ 2) : This is exactly Lemma 13.
2) $\Rightarrow$ 1) : By Lemma 16, we already know that F is faithful. Now take $\mathbb{K} = \underline{D}(\mathbb{Z})$, and the fulness of $- \cdot F$ means that F is full.

$\diamondsuit$

Proposition 18 *Let $G\colon \mathbb{L} \to \mathbb{K}$ be a morphism in SCG; the following conditions are equivalent :*

1) *G is full and essentially surjective ;*
2) *for each symmetric cat-group $\mathbb{H}$, the functor induced by G*

$$G \cdot - \colon SCG(\mathbb{K}, \mathbb{H}) \to SCG(\mathbb{L}, \mathbb{H})$$

is full and faithful ;

Proof: 1) $\Rightarrow$ 2) : This is exactly Lemma 10.
2) $\Rightarrow$ 1) : Consider the class $\mathcal{E}'_1$ of the morphisms which satisfy condition 2). Since $\mathcal{E}_1 \subset \mathcal{E}'_1$ (because 1) $\Rightarrow$ 2)), each morphism in SCG has a $(\mathcal{E}'_1, \mathcal{M}_1)$-factorization. Moreover, clearly $\mathcal{E}'_1$ contains equivalences, it is closed under composition with equivalences and it is stable under 2-cells. To prove that $\mathcal{E}_1 = \mathcal{E}'_1$, it remains to show that, for each G in $\mathcal{E}'_1$ and for each F in $\mathcal{M}_1$, one has $G \downarrow F$. But this is exactly what we have done in the proof of Proposition 12.

$\diamondsuit$

Let us close this section with a point of terminology. In view of Proposition 17, it is reasonable to call "monomorphism" a full and faithful morphism (and, dually, "epimorphism" a full and essentially surjective morphism). This is because in any 2-category $\mathcal{C}$ with invertible 2-cells, the following conditions on an arrow $f\colon A \to B$ are equivalent :

i) for each object C of $\mathcal{C}$, the functor

$$- \cdot f \colon \mathcal{C}(C, A) \to \mathcal{C}(C, B)$$

is full and faithful ;

ii) the diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \arrow[dl, Rightarrow, "f"'] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a bi-pullback ;

iii) there exist two morphisms $f_0, f_1\colon P \to A$ and a 2-cell $\varphi\colon f_0 \Rightarrow f_1$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "f_0"] \arrow[d, "f_1"'] & A \arrow[d, "f"] \arrow[dl, Rightarrow, "\varphi \cdot f"'] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a bi-pullback.

In the introduction, we say that SCG can be seen as the 2-dimensional analogue of the category of abelian groups. The results stated in this paper follow this idea, and in fact cokernels coincide with strong epis (= morphisms orthogonal to monos) and kernels coincide with strong monos (= morphisms orthogonal to epis), each morphism can be factorized as an epi followed by a kernel or as a cokernel followed by a mono, each mono is a kernel and each epi is a cokernel. The surprise is that in this 2-dimensional world a kernel fails to be a mono and a cokernel fails to be an epi (the failure being measured respectively by π_1 and π_0).

6 Exact sequences

In order to study some classical exact sequences of abelian groups associated with a morphism of commutative unital rings, in [24] the notion of 2-exact sequence of symmetric cat-groups has been introduced.

For a sequence of morphisms of abelian groups, the notion of exactness can be stated in several equivalent ways, and this is a relevant fact to make exact sequences easy to use. We want to show here that the same can be done in the framework of symmetric cat-groups, but using in the appropriate way both the factorizations we have.

We need some preliminary constructions. Consider two morphisms in SCG together with their factorizations as in the following diagram

$$\begin{tikzcd}
& KerG \arrow[rr, "0"] \arrow[dr, "e_G"'] & & CoimG \arrow[dr, "G'"] \arrow[d, phantom, "\Downarrow g"] & \\
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "\hat{F}"'] & & \mathbb{B} \arrow[rr, "G"'] \arrow[ur, "P_{e_G}"] \arrow[dr, "P_F"] \arrow[u, phantom, "\Uparrow \pi_{e_G}"] \arrow[d, phantom, "\Downarrow \lambda_{P_F}"] & & \mathbb{C} \\
& ImF \arrow[rr, "0"'] \arrow[ur, "e_{P_F}"'] \arrow[u, phantom, "\Uparrow f"] & & CokerF &
\end{tikzcd}$$

with the following commutative diagrams

$$\begin{tikzcd}
\hat{F} \cdot e_{P_F} \cdot P_F \arrow[r, Rightarrow, "f \cdot P_F"] \arrow[d, Rightarrow, "\hat{F} \cdot \lambda_{P_F}"'] & F \cdot P_F \arrow[d, Rightarrow, "\pi_F"] \\
\hat{F} \cdot 0 \arrow[r, Rightarrow] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
e_G \cdot P_{e_G} \cdot G' \arrow[r, Rightarrow, "e_G \cdot g"] \arrow[d, Rightarrow, "\pi_{e_G} \cdot G'"'] & e_G \cdot G \arrow[d, Rightarrow, "\lambda_G"] \\
0 \cdot G' \arrow[r, Rightarrow] & 0
\end{tikzcd}$$

Now fix a 2-cell $\varphi \colon F \cdot G \Rightarrow 0$ and consider the factorizations given by the universal properties of $KerG$ and $CokerF$

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "0"] \arrow[dr, "F"'] & & CokerF \arrow[dr, "\tilde{G}"] \arrow[d, phantom, "\Downarrow \tilde{\varphi}"] & \\
& \mathbb{B} \arrow[ur, "P_F"] \arrow[rr, "G"'] \arrow[u, phantom, "\Uparrow \pi_F"] & & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "\overline{F}"'] & & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow \lambda_G"] & \\
& KerG \arrow[ur, "e_G"] \arrow[rr, "0"'] \arrow[u, phantom, "\Uparrow \overline{\varphi}"] & & \mathbb{C}
\end{tikzcd}$$

with the following commutative diagram

$$\begin{tikzcd}
\overline{F} \cdot e_G \cdot G \arrow[r, Rightarrow, "\overline{\varphi} \cdot G"] \arrow[d, Rightarrow, "\overline{F} \cdot \lambda_G"'] & F \cdot G \arrow[d, Rightarrow, "\varphi"'] & F \cdot P_F \cdot \tilde{G} \arrow[l, Rightarrow, "F \cdot \tilde{\varphi}"'] \arrow[d, Rightarrow, "\pi_F \cdot \tilde{G}"] \\
\overline{F} \cdot 0 \arrow[r, Rightarrow] & 0 & 0 \cdot \tilde{G} \arrow[l, Rightarrow]
\end{tikzcd}$$

Consider the 2-cell

$$\beta \colon \hat{F} \cdot e_{P_F} \cdot G \xRightarrow{f \cdot G} F \cdot G \xRightarrow{\varphi} 0 \Longrightarrow \hat{F} \cdot 0 \quad ;$$

since $\hat{F}$ is full and essentially surjective, by Lemma 10 there exists a unique $\alpha \colon e_{P_F} \cdot G \Rightarrow 0$ such that $\hat{F} \cdot \alpha = \beta$. The universal property of $KerG$ gives us a factorization

$$\begin{tikzcd}
 & KerG \arrow[dr, "e_G"] \arrow[d, phantom, "\Downarrow\mu"] & \\
ImF \arrow[ur, "P"] \arrow[rr, "e_{P_F}"'] & {} & \mathbb{B}
\end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd}
P \cdot e_G \cdot G \arrow[r, Rightarrow, "\mu \cdot G"] \arrow[d, Rightarrow, "P \cdot \lambda_G"'] & e_{P_F} \cdot G \arrow[d, Rightarrow, "\alpha"] \\
P \cdot 0 \arrow[r, Rightarrow] & 0
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
\hat{F} \cdot P \cdot e_G \cdot G \arrow[r, Rightarrow, "\hat{F} \cdot \mu \cdot G"] \arrow[d, Rightarrow, "\hat{F} \cdot P \cdot \lambda_G"'] & \hat{F} \cdot e_{P_F} \cdot G \arrow[r, Rightarrow, "f \cdot G"] & F \cdot G \arrow[d, Rightarrow, "\varphi"] \\
\hat{F} \cdot P \cdot 0 \arrow[rr, Rightarrow] & & 0
\end{tikzcd}$$

commutes, the universal property of $KerG$ gives a unique 2-cell $\nu \colon \overline{F} \Rightarrow \hat{F} \cdot P$ such that the following diagram commutes

$$\begin{tikzcd}
\overline{F} \cdot e_G \arrow[r, Rightarrow, "\nu \cdot e_G"] \arrow[d, Rightarrow, "\overline{\varphi}"'] & \hat{F} \cdot P \cdot e_G \arrow[d, Rightarrow, "\hat{F} \cdot \mu"] \\
F & \hat{F} \cdot e_{P_F} \arrow[l, Rightarrow, "f"]
\end{tikzcd}$$

Analogously, start with the 2-cell

$$\delta \colon F \cdot P_{e_G} \cdot G' \xRightarrow{F \cdot g} F \cdot G \xRightarrow{\varphi} 0 \Longrightarrow 0 \cdot G' \quad ;$$

since G' is full and faithful, by Lemma 13 there exists a unique $\gamma\colon F\cdot P_{e_G}\Rightarrow 0$ such that $\gamma\cdot G'=\delta$. The universal property of $CokerF$ gives a factorization

$$\begin{tikzcd} & CokerF \arrow[dr, "Q"] \arrow[d, phantom, "\Downarrow\theta"] & \\ \mathbb{B} \arrow[ur, "P_F"] \arrow[rr, "P_{e_G}"'] & & CoimG \end{tikzcd}$$

with the commutative diagram

$$\begin{tikzcd} F\cdot P_F\cdot Q \arrow[r, Rightarrow, "F\cdot\theta"] \arrow[d, Rightarrow, "\pi_F\cdot Q"'] & F\cdot P_{e_G} \arrow[d, Rightarrow, "\gamma"] \\ 0\cdot Q \arrow[r, Rightarrow] & 0 \end{tikzcd}$$

Furthermore, the universal property of $CokerF$ gives a unique $\eta\colon \tilde{G}\Rightarrow Q\cdot G'$ with the commutative diagram

$$\begin{tikzcd} P_F\cdot\tilde{G} \arrow[r, Rightarrow, "P_F\cdot\eta"] \arrow[d, Rightarrow, "\tilde{\varphi}"'] & P_F\cdot Q\cdot G' \arrow[d, Rightarrow, "\theta\cdot G'"] \\ G & P_{e_G}\cdot G' \arrow[l, Rightarrow, "g"] \end{tikzcd}$$

Finally, consider the 2-cell

$$x\colon \hat{F}\cdot e_{P_F}\cdot P_{e_G} \xRightarrow{f\cdot P_{e_G}} F\cdot P_{e_G} \xRightarrow{\gamma} 0 \Longrightarrow \hat{F}\cdot 0 \quad ;$$

since $\hat{F}$ is full and essentially surjective, there exists a unique $u\colon e_{P_F}\cdot P_{e_G}\Rightarrow 0$ such that $\hat{F}\cdot u=x$. In the same way, consider

$$y\colon e_{P_F}\cdot P_{e_G}\cdot G' \xRightarrow{e_{P_F}\cdot g} e_{P_F}\cdot G \xRightarrow{\alpha} 0 \Longrightarrow 0\cdot G' \quad ;$$

since G' is full and faithful, there exists a unique $v\colon e_{P_F}\cdot P_{e_G}\Rightarrow 0$ such that $v\cdot G'=y$.

Lemma 19 *With the previous notations, $u=v$.*

Proof: Since $\hat{F}$ is full and essentially surjective and G' is full and faithful, by Lemmas 10 and 13 it is enough to check that $\hat{F}\cdot u\cdot G'=\hat{F}\cdot v\cdot G'$. This easily follows from the previous commutative diagrams.

$\diamondsuit$

Proposition 20 *With the previous notations, the following conditions are equivalent :*

1) $\overline{F}\colon \mathbb{A} \to KerG$ *is full and essentially surjective ;*

2) $P\colon ImF \to KerG$ *is an equivalence ;*

3) $Q\colon CokerF \to CoimG$ *is an equivalence ;*

4) $\tilde{G}\colon CokerF \to \mathbb{C}$ *is full and faithful.*

Proof: 1) $\Rightarrow$ 2) : Using $\mu\colon P \cdot e_G \Rightarrow e_{P_F}$, we can deduce the faithfulness of P from that of e_{P_F}. Using $\nu\colon \overline{F} \Rightarrow \hat{F} \cdot P$, we can deduce the essential surjectivity of P from that of $\overline{F}$, and the fulness of P from the fact that $\overline{F}$ is full and $\hat{F}$ is essentially surjective.
2) $\Rightarrow$ 1) : Since $\hat{F}$ is full and essentially surjective, if P is an equivalence, then the composite $\hat{F} \cdot P$ is full and essentially surjective. But then also $\overline{F}$ is full and essentially surjective because of $\nu\colon \overline{F} \Rightarrow \hat{F} \cdot P$.
In a similar way, one can prove the equivalence of conditions 3) and 4).
3) $\Rightarrow$ 2) : Consider the following diagram in SCG

$$\begin{tikzcd}
KerP_F \arrow[r, "e_{P_F}"] \arrow[d, "P"'] & \mathbb{B} \arrow[r, "P_F"] \arrow[d, "\mathrm{Id}_{\mathbb{B}}"'] & CokerF \arrow[d, "Q"] \\
KerG \arrow[r, "e_G"] \arrow[d] & \mathbb{B} \arrow[r, "P_{e_G}"] \arrow[dr, "G"] & Coker(e_G) \\
KerP_{e_G} \arrow[ur, "e_{P_{e_G}}"'] & & \mathbb{C}
\end{tikzcd}$$

since e_G is faithful, by Corollary 5 we know that the comparison $KerG \to KerP_{e_G}$ is an equivalence. Now if Q is an equivalence, also P is an equivalence.
2) $\Rightarrow$ 3) : Similar to 3) $\Rightarrow$ 2), but using Corollary 7.

$\diamondsuit$

Definition 21 *A sequence in SCG*

$$\begin{tikzcd}
& \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow \varphi"] & \\
\mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C}
\end{tikzcd}$$

is 2-exact if it satisfies one of the equivalent conditions of the previous proposition.

Observe that the first condition in Proposition 20 is the definition of 2-exact sequence used in [24].

Proposition 22 *(with the previous notations) Consider the following sequences in SCG*

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\varphi"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{B} \arrow[dr, "P_{e_G}"] \arrow[d, phantom, "\Downarrow u=v"] & \\ ImF \arrow[ur, "e_{P_F}"] \arrow[rr, "0"'] & {} & CoimG \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ ImF \arrow[ur, "e_{P_F}"] \arrow[rr, "0"'] & {} & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{B} \arrow[dr, "P_{e_G}"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ \mathbb{A} \arrow[ur, "F"] \arrow[rr, "0"'] & {} & CoimG \end{tikzcd}$$

The following conditions are equivalent :

1) the sequence (F, φ, G) *is 2-exact ;*

2) the sequence (e_{P_F}, α, G) *is 2-exact ;*

3) the sequence (F, γ, P_{e_G}) *is 2-exact ;*

4) the sequence $(e_{P_F}, u = v, P_{e_G})$ *is 2-exact.*

Proof: 1) $\iff$ 2) : Same argument as in 1) $\iff$ 2) of Proposition 20, because P is always faithful.
1) $\iff$ 3) : Same argument as in 3) $\iff$ 4) of Proposition 20.
1) $\iff$ 4) : Come back to the diagram in the proof of Proposition 20 and consider the factorization $H\colon KerP_F \to KerP_{e_G}$ of e_{P_F} through $e_{P_{e_G}}$. We have: (F, φ, G) is 2-exact iff P is full and essentially surjective iff H is full and essentially surjective (because $KerG \to KerP_{e_G}$ is an equivalence) iff (e_{P_F}, u, P_{e_G}) is 2-exact.

$\diamondsuit$

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Stefano Kasangian
Dipartimento di Matematica, Università di Milano,
Via Cesare Saldini 50, 20133 Milano, Italy
Stefano.Kasangian@mat.unimi.it

Enrico M. Vitale
Département de Mathématique, Université catholique de Louvain,
Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
vitale@agel.ucl.ac.be