On the exact completion of the homotopy category

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Résumé. On montre que la complétion exacte de la catégorie de l'homotopie des espaces topologiques est un prétopos. Pour cela, on détermine une condition nécessaire et suffisante pour que la complétion exacte d'une catégorie à limites finies faibles soit extensive.

Introduction

The exact completion $(\mathrm{Top})_{ex}$ of Top, the category of topological spaces and continuous maps, has been recently studied with encreasing interest. The reason for this lies in the deep connection between $(\mathrm{Top})_{ex}$ and the category of equilogical spaces (cf. [10] and [1]). It has been proved that $(\mathrm{Top})_{ex}$ is a locally cartesian closed pretopos (cf. [2], [4] and [9]).

Since the category HTop of topological spaces and homotopy classes of continuous maps is equivalent to a full subcategory of $(\mathrm{Top})_{ex}$ and, moreover, the full inclusion behaves well with respect to weak limits (cf. proposition 3.3), it is natural to ask if also $(\mathrm{HTop})_{ex}$ is a locally cartesian closed pretopos.

In this note we show that $(\mathrm{HTop})_{ex}$ is a pretopos (i.e. an extensive exact category). This property is quite simple to prove in the case of $(\mathrm{Top})_{ex}$, because Top itself is extensive and has finite limits; on the contrary, the proof is more delicate if one works with HTop, where only weak limits are available. To overcome this difficulty we propose a weakened notion of lextensivity; this notion can be expressed in any weakly left exact category $\mathbb{C}$, and it turns out that it is necessary and sufficient for $\mathbb{C}_{ex}$ to be lextensive (i.e. left exact and extensive).

For the notion of extensivity the reference is [3]. The theory of the exact completion of a weakly left exact category can be found in [5] (see also [7]) or, in a shorter form, in [11]. We work with finite sums; the generalization to small sums is straightforward.

1 Weakly lextensive categories

From [3] we recall that a category $\mathbb{C}$ with sums is extensive if it has pull-backs along injections in a sum and the following condition holds: in the commutative diagram (where the bottom row is a sum)

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & A \arrow[d] & Y' \arrow[l] \arrow[d] \\ X \arrow[r] & X+Y & Y \arrow[l] \end{tikzcd}$$

the top row is a sum if and only if the two squares are pull-backs. In the following we refer to the (if) part as condition **I** and to the (only if) part as condition **II**. Condition **I** is known as the universality of sums.

If $\mathbb{C}$ has pull-backs along injections, the extensivity is equivalent to have disjoint and universal sums, where disjointness of sums is the following condition **III**:

III.a: the injections in a sum $X \to X+Y \leftarrow Y$ are monic;

III.b: if 0 is an initial object, then the following diagram is a pull-back

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X+Y \end{tikzcd}$$

Moreover, if $\mathbb{C}$ is left exact, condition **I** is equivalent to condition **IV** and to condition **V**:

IV: if the first two diagrams are pull-backs, then also the third one is a pull-back

$$\begin{tikzcd} P_X \arrow[r] \arrow[dd] & X \arrow[d] \\ & X+Y \arrow[d] \\ A \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} P_Y \arrow[r] \arrow[dd] & Y \arrow[d] \\ & X+Y \arrow[d] \\ A \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} P_X+P_Y \arrow[r] \arrow[dd] & X+Y \arrow[dd] \\ & \\ A \arrow[r] & B \end{tikzcd}$$

V.a: the canonical morphism $(X\times Y)+(X\times Z) \to X\times(Y+Z)$ is an isomorphism (i.e. $\mathbb{C}$ is distributive);

V.b: if the first two diagrams are equalizers, then also the third one is an equalizer

$$E_X \to X \rightrightarrows Z \qquad E_Y \to Y \rightrightarrows Z \qquad E_X+E_Y \to X+Y \rightrightarrows Z$$

Recall now from [5] that one can not build up all weak finite limits starting from weak pull-backs and weak terminals; on the contrary, it is possible to do it starting from weak finite products and weak equalizers. Keeping this situation in mind, we give the following definition.

Definition 1.1 *Let $\mathbb{C}$ be a weakly left exact category with sums; we say that $\mathbb{C}$ is weakly lextensive if the following conditions hold:*

III: *sums are disjoint;*

V.aw: *for each choice of weak products $X \times Y$ and $X \times Z$, the sum $(X \times Y) + (X \times Z)$, with the obvious morphisms, is a weak product of X and $Y + Z$;*

V.bw: *(with the same notations as* **V.b***) for each choice of weak equalizers E_X and E_Y, the sum $E_X + E_Y$ is a weak equalizer;*

VI: *initials are strict.*

A first example of weakly lextensive category is given by the sum-completion $\mathrm{Fam}(\mathcal{C})$ of a small category $\mathcal{C}$. This can be directly checked or deduced from proposition 2.1, because the exact completion of $\mathrm{Fam}(\mathcal{C})$ is equivalent to the topos of presheaves on $\mathcal{C}$ (cf. [5]).

We list now some properties of weakly left exact and weakly lextensive categories.

Proposition 1.2

1) *let $\mathbb{C}$ be weakly left exact; condition* **V.aw** *can be equivalently restated replacing "for each choice" by "there exists a choice"; the same holds for condition* **V.bw***;*

2) *let $\mathbb{C}$ be weakly left exact; conditions* **V.aw** *and* **V.bw** *imply the weakened version* **IV.w** *of condition* **IV***: (with the same notations as in* **IV***) for each choice of weak pull-backs P_X and P_Y, the sum $P_X + P_Y$ is a weak pull-back;*

3) *let $\mathbb{C}$ be weakly lextensive; then condition* **II** *of extensivity holds;*

4) *let $\mathbb{C}$ be weakly lextensive; if the first two squares are weak pull-backs, then also the third one is a weak pull-back*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd} \qquad \begin{tikzcd} A + A' \arrow[r] \arrow[d] & B + B' \arrow[d] \\ C + C' \arrow[r] & D + D' \end{tikzcd}$$

5) *let $\mathbb{C}$ be weakly left exact with products; if sums are universal in $\mathbb{C}$, then conditions* **V.aw** *and* **V.bw** *hold. In particular, a weakly left exact and extensive category with products is weakly lextensive.*

Proof: 1) and 2) are a routine calculation with weak limits.
3): we have to prove that

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & A+B \arrow[d, "f+g"'] & B \arrow[l, "i_B"'] \arrow[d, "g"] \\ X \arrow[r, "i_X"'] & X+Y & Y \arrow[l, "i_Y"] \end{tikzcd}$$

are pull-backs; since i_A and i_B are monic, it is enough to show that they are weak pull-backs. To find a weak pull-back of i_X and $f+g$ we can use condition **IV.w**; the pull-back of i_X and $i_A \cdot (f+g) = f \cdot i_X$ is A because i_X is a mono; the pull-back of i_X and $i_B \cdot (f+g) = g \cdot i_Y$ factors through the pull-back of i_X and i_Y, and then it is 0 by disjointness of sums and strictness of initials.
4): similar to 3).
5): universality of sums implies the particular case of condition **IV** where the arrow $X+Y \to B$ is the identity of $X+Y$. But, by associativity of weak pull-backs, this particular case implies condition **IV.w**. Moreover, if products exist, one can check that **IV.w** implies **V.aw** and **V.bw**. ■

2 The exact completion

Proposition 2.1 *Let $\mathbb{C}$ be a weakly left exact category with sums and $\Gamma\colon \mathbb{C} \to \mathbb{C}_{ex}$ its exact completion; $\mathbb{C}_{ex}$ is extensive if and only if $\mathbb{C}$ is weakly lextensive.*

Proof: (only if): recall that, up to the full embedding $\Gamma\colon \mathbb{C} \to \mathbb{C}_{ex}$, $\mathbb{C}$ is a projective cover of $\mathbb{C}_{ex}$, that is each object of $\mathbb{C}$ is regular projective in $\mathbb{C}_{ex}$ and each object of $\mathbb{C}_{ex}$ is a regular quotient of an object of $\mathbb{C}$. Then, a weak limit in $\mathbb{C}$ can be recovered performing the corresponding limit in $\mathbb{C}_{ex}$ and then covering it with an object of $\mathbb{C}$. Moreover, the functor Γ preserves all the sums which turn out to exist in $\mathbb{C}$. From this, the first implication easily follows.
(if): $\mathbb{C}_{ex}$ has sums: given two objects in $\mathbb{C}_{ex}$, that is two pseudo equivalence relations in $\mathbb{C}$, $r_0, r_1\colon R \rightrightarrows X$ and $s_0, s_1\colon S \rightrightarrows Y$, its sum in $\mathbb{C}_{ex}$ is $r_0+s_0, r_1+s_1\colon R+S \rightrightarrows X+Y$; the fact that it is a pseudo equivalence relation follows from part 4) of proposition 1.2.
To prove that in $\mathbb{C}_{ex}$ sums are disjoint and universal, we need an explicit description of pull-backs in $\mathbb{C}_{ex}$. The idea is quite simple: write down what a commutative square in $\mathbb{C}_{ex}$ is and then take, at each step, a weak limit in $\mathbb{C}$. Given two arrows in $\mathbb{C}_{ex}$

$$
\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
X \arrow[r, "f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}
$$

we can choose a weak limit $(P, \underline{f}, \varphi, \underline{g})$ as in the following diagram

$$
\begin{tikzcd}
X \arrow[d, "f"'] & P \arrow[l, "\underline{f}"'] \arrow[r, "\underline{g}"] \arrow[d, "\varphi"'] & Z \arrow[d, "g"] \\
Y & S \arrow[l, "s_0"] \arrow[r, "s_1"'] & Y
\end{tikzcd}
$$

and then a weak limit $(E, \rho, e_0, e_1, \tau)$ as in the following diagram

$$
\begin{tikzcd}
 & & E \arrow[dll, "\rho"'] \arrow[dl, "e_0"] \arrow[dr, "e_1"'] \arrow[drr, "\tau"] & & \\
R \arrow[d, "r_0"'] \arrow[dr, "r_1"] & P \arrow[dl, "\underline{f}"] \arrow[drr, "\underline{g}"] & & P \arrow[dll, "\underline{f}"'] \arrow[dr, "\underline{g}"'] & T \arrow[dl, "t_0"'] \arrow[d, "t_1"] \\
X & X & & Z & Z
\end{tikzcd}
$$

The required pull-back in $\mathbb{C}_{ex}$ is then

$$
\begin{tikzcd}
R \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & E \arrow[l, "\rho"'] \arrow[r, "\tau"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & T \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
X & P \arrow[l, "\underline{f}"] \arrow[r, "\underline{g}"'] & Z
\end{tikzcd}
$$

Observe that a weak limit P can be obtained taking three weak pull-backs. Also the construction of E can be split into two steps: first take a weak limit L over the zig-zag

$$
\begin{tikzcd}
T \arrow[d, "t_0"'] & P \arrow[dl, "\underline{g}"'] \arrow[d, "\underline{f}"'] & R \arrow[dl, "r_0"'] \arrow[d, "r_1"] & P \arrow[dl, "\underline{f}"] \arrow[d, "\underline{g}"] & T \arrow[dl, "t_1"] \\
Z & X & X & Z &
\end{tikzcd}
$$

(which can be computed with iterated weak pull-backs) and then take a weak equalizer $E \to L \rightrightarrows T$.
In $\mathbb{C}_{ex}$ sums are universal: we have to calculate the pull-back in $\mathbb{C}_{ex}$ of

$$\begin{tikzcd}
Q+R \arrow[r, "\overline{f}"] \arrow[d, shift right, "q_0+r_0"'] \arrow[d, shift left, "q_1+r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
V+X \arrow[r, "f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}$$

Following the previous description and using points 2) and 4) of proposition 1.2 for any weak pull-back, and condition **V.bw** for any weak equalizer, we exactly obtain the sum of the pull-backs of

$$\begin{tikzcd}
Q \arrow[r, "i_Q \cdot \overline{f}"] \arrow[d, shift right, "q_0"'] \arrow[d, shift left, "q_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
V \arrow[r, "i_V \cdot f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "i_R \cdot \overline{f}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[l, "\overline{g}"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
X \arrow[r, "i_X \cdot f"'] & Y & Z \arrow[l, "g"]
\end{tikzcd}$$

In $\mathbb{C}_{ex}$ sums are disjoint: we have to calculate the pull-back in $\mathbb{C}_{ex}$ of

$$\begin{tikzcd}
R \arrow[r, "i_R"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & R+S \arrow[d, shift right, "r_0+s_0"'] \arrow[d, shift left, "r_1+s_1"] & S \arrow[l, "i_S"'] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
X \arrow[r, "i_X"'] & X+Y & Y \arrow[l, "i_Y"]
\end{tikzcd}$$

By part 3) of proposition 1.2 and by disjointness of sums in $\mathbb{C}$, the bottom part of the pull-back is

$$\begin{tikzcd}
& & 0 \arrow[dl] \arrow[dr] & & \\
X \arrow[d, "i_X"'] & R \arrow[l, "r_0"'] \arrow[dr, "i_R"] & & S \arrow[r, "s_1"] \arrow[dl, "i_S"'] & Y \arrow[d, "i_Y"] \\
X+Y & & R+S \arrow[ll, "r_0+s_0"] \arrow[rr, "r_1+s_1"'] & & X+Y
\end{tikzcd}$$

and, by strictness of initials, the top part is also 0.
It remains to prove that in $\mathbb{C}_{ex}$ injections are mono. This follows from the fact that they are mono in $\mathbb{C}$, using once again part 3) of proposition 1.2. ■

Corollary 2.2 *Let $\mathbb{C}$ be a weakly left exact category; $\mathbb{C}_{ex}$ is extensive if and only if the Cauchy-completion of $\mathbb{C}$ is weakly lextensive.*

Proof: the Cauchy-completion $\mathrm{cc}(\mathbb{C})$ of $\mathbb{C}$ is equivalent to the full subcategory of regular projectives objects of $\mathbb{C}_{ex}$. Conversely, one has that $(\mathrm{cc}(\mathbb{C}))_{ex}$ is equivalent to $\mathbb{C}_{ex}$. ■

Corollary 2.3 *Let $\mathbb{C}$ be weakly lextensive; then $\Gamma\colon \mathbb{C} \to \mathbb{C}_{ex}$ is the pretopos completion of $\mathbb{C}$.*

Proof: recall that the functor Γ preserves all the sums which turn out to exist in $\mathbb{C}$. Now an exchange argument between sums and coequalizers shows that, for any exact category with sums $\mathbb{A}$, the exact extension $\hat{F}$ of a left covering functor F preserves sums if and only if F preserves sums

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Gamma"] \arrow[dr, "F"'] & & \mathbb{C}_{ex} \arrow[dl, "\hat{F}"] \\ & \mathbb{A} & \end{tikzcd}$$

■

Corollary 2.4 *A left exact category $\mathbb{C}$ is extensive if and only if it is weakly lextensive.*

Proof: One implication is contained in point 5) of proposition 1.2. Conversely, assume that $\mathbb{C}$ is weakly lextensive. By proposition 2.1, it is a full subcategory, closed under sums and finite limits, of the extensive category $\mathbb{C}_{ex}$. ■

3 The exact completion of HTop

Proposition 3.1 *The category HTop is extensive.*

Proof: Since Top is extensive and sums in HTop are topological sums, it is enough to show that topological pull-backs along injections are also pull-backs in HTop. Since injections in Top are fibrations, then topological pull-backs along injections are homotopical pull-backs (cf. [8] and [6]) and then weak pull-backs in HTop. They are "strong" pull-backs in HTop because injections are fibrations and monic in Top, so that they are monic in HTop. ■

Corollary 3.2 *HTop is weakly lextensive, and (HTop)$_{ex}$ is a pretopos.*

Proof: HTop has products, which are topological products, and is extensive. It remains only to prove that it is weakly left exact and, in view of proposition 3.3, we recall here an explicit description of homotopy equalizers (which are weak equalizers in HTop). Consider two continuous maps $f, g \colon X \rightrightarrows Y$ and the evaluations $\mathrm{ev}_0, \mathrm{ev}_1 \colon Y^{[0,1]} \rightrightarrows Y$. An homotopy equalizer $e \colon L \to X$ of f and g is given by the following topological limit

$$\begin{tikzcd}
 & L \arrow[dl, "e"'] \arrow[dr] & \\
X \arrow[d, "f"'] \arrow[drr, "g" near start] & & Y^{[0,1]} \arrow[dll, "\mathrm{ev}_0"' near start] \arrow[d, "\mathrm{ev}_1"] \\
Y & & Y
\end{tikzcd}$$

■

It remains to clear up the relation between HTop and (Top)$_{ex}$. For this, consider the functor $\mathcal{E} \colon \mathrm{Top} \to (\mathrm{Top})_{ex}$ which sends each space X into the pseudo equivalence relation $\mathrm{ev}_0, \mathrm{ev}_1 \colon X^{[0,1]} \rightrightarrows X$.

Proposition 3.3 *The functor $\mathcal{E}$ respects homotopy and its factorization $\mathcal{E}' \colon$ HTop $\to$ (Top)$_{ex}$ is full, faithful and left covering.*

Proof: The left covering character of $\mathcal{E}'$ follows comparing the description of weak equalizers in HTop just given with the description of equalizers in the exact completion. This latter can be found following the same idea used for the description of pull-backs in the proof of proposition 2.1. The rest of the statement is obvious. ■

Remark: it remains for us an open problem to determine if (HTop)$_{ex}$ is cartesian closed. The preliminary work done in [9] for a lextensive category can be generalized to a weakly lextensive one, but we are not able to conclude because of the lack of a good factorization system in HTop.

4 *

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