Essential localizations and infinitary exact completion

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Abstract. We prove the universal property of the infinitary exact completion of a category with weak small limits. As an application, we slightly weaken the conditions characterizing essential localizations of varieties (in particular, of module categories) and of presheaf categories.

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Introduction

An essential localization is a reflective subcategory such that the reflector has a left adjoint. In [12], Roos gave an abstract characterization of essential localizations of module categories, proving that they are those complete and cocomplete abelian categories with a regular generator, satisfying the following conditions (we write the conditions in a non-abelian style, more convenient for the general framework of this work)

(AB4*) Regular epimorphisms are product-stable ;

(AB5) Filtered colimits are exact, i.e. commute with finite limits ;

(AB6) Given a small family of functors $(H_i \colon \mathcal{A}_i \to \mathbb{A})_I$ defined on small filtered categories, the canonical comparison τ is an isomorphism

$$\tau \colon colim(\prod_I \mathcal{A}_i \to \prod_I \mathbb{A} \to \mathbb{A}) \longrightarrow \prod_I (colim H_i) \ .$$

In a subsequent paper [13], Roos introduced a weaker form of (AB6) :

(WAB6) With the same notations as in (AB6), the comparison morphism τ is a regular epimorphism.

It is then natural to look for a representation of the abelian categories satisfying the same list of conditions as in Roos's theorem, but replacing (AB6) with (WAB6).

The first aim of this paper is to prove that this set of conditions, which seems weaker, still characterizes essential localizations of module categories. In fact we prove a more general result : Roos's theorem has recently been generalized by Adámek, Rosický and the author to a non-additive context [1]. They have characterized essential localizations of (multi-sorted, finitary) varieties as those complete and cocomplete Barr-exact categories with a small regular generator, satisfying (AB4*), (AB5) and (AB6). We show that in this characterization (AB6) can be replaced by (WAB6). Since abelian varieties are exactly module categories, as a particular case we obtain the new formulation of Roos's theorem.

A similar analysis can be done for Roos's characterization of essential Grothendieck topoi (i.e. essential localizations of presheaf categories) [10, 11] (see also Section 4 in [1] for a different approach). In fact we prove that the complete distributivity involved in Roos's theorem can be weakened : instead of an isomorphism, the comparison morphism can be assumed to be a regular epimorphism.

To obtain our results, we use in a systematic way the exact completion of a category with weak limits [5, 6]. This uniform approach stresses one more time the striking analogy between the varietal context and the presheaf context: not only the results are similar, but also the proof technique is essentially the same.

In dealing with essential localizations, we need the infinitary extension of the exact completion. This has been studied in [6]. Unfortunately, the proof of the main theorem in [6] is not complete. The claim that a left exact functor which is left covering with respect to small products is continuous, used to end the proof of Theorem 3.4, is not true, as we show with a counterexample. The second aim of this paper is then to give a complete proof of the universal property of the infinitary exact completion.

I would like to thank P.T. Johnstone for his help concerning Example 8.

1 The infinitary exact completion

For basic definitions and results on regular and exact categories, the reader can see [2, 5]. We assume familiarity with the exact completion as described in [5] and with the theory of left covering functors developed therein. Left covering functors are called flat functors in [6]. We write

$$\Gamma\colon \mathbb{C} \to \mathbb{C}_{ex}$$

for the exact completion of a weakly left exact category $\mathbb{C}$. As far as the infinitary completion is concerned, we will deal only with arbitrary small limits ; in fact the ranked case runs exactly as the general case.

A main difference between limits and weak limits is that in the weak case the usual reduction to small products and equalizers of *pairs* of parallel arrows does not work. One has to use small products and equalizers of small families of parallel arrows (we call them *small equalizers*). The next three lemmas are the infinitary version of Proposition 1, Lemma 28 and Proposition 27 in [5].

Lemma 1 *If a category $\mathbb{C}$ has weak small products and weak small equalizers, then it has weak small limits.*

Proof The proof is similar to that of the finitary case. We give some details in view of Lemma 3.
I) Consider a small diagram of arrows with the same codomain $(f_i \colon X_i \to X)_I$, a weak product $(\pi_i \colon \prod_I X_i \to X_i)_I$ and a weak small equalizer

$$E \xrightarrow{e} \textstyle\prod_I X_i \underset{\vdots}{\xrightarrow{\pi_i \cdot f_i}} X \quad ;$$

then the cone $(e \cdot \pi_i \colon E \to X_i)_I$ is a weak limit over $(f_i \colon X_i \to X)_I$ (a weak *small pullback*).
II) Let $\mathcal{L} \colon \mathcal{D} \to \mathbb{C}$ be a functor with $\mathcal{D}$ a small category ; consider a weak product $(\pi_D \colon \prod_{\mathcal{D}} \mathcal{L}(D) \to \mathcal{L}(D))_{\mathcal{D}}$ and, for each arrow $d \colon D \to D'$ in $\mathcal{D}$, a weak equalizer

$$\begin{tikzcd} E_d \arrow[r, "e_d"] & \prod_{\mathcal{D}} \mathcal{L}(D) \arrow[r, shift left, "\pi_{D'}"] \arrow[r, shift right, "\pi_D \cdot \mathcal{L}(d)"'] & \mathcal{L}(D') \end{tikzcd} \quad ;$$

consider also a weak small pullback $(\epsilon_d \colon E \to E_d)_d$ on the diagram $(e_d \colon E_d \to \prod_{\mathcal{D}} \mathcal{L}(D))_d$. Then $(\epsilon_d \cdot e_d \cdot \pi_D \colon E \to \mathcal{L}(D))_{\mathcal{D}}$ is a weak limit of $\mathcal{L} \colon \mathcal{D} \to \mathbb{C}$. $\square$

Recall that a category $\mathbb{B}$ is completely regular if it is regular, complete and regular epimorphisms are stable under small products (a condition which is redundant in the finitary case).

Lemma 2 *Let $\mathbb{B}$ be a completely regular category.*

1) *If in the following commutative diagram, where the horizontal arrows represent small equalizers, f_1 is a regular epimorphism and f_2 is a mono, then the unique factorization f is a regular epimorphism*

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "f"'] & A_1 \arrow[r, "a_i" above, "\vdots" below] \arrow[d, "f_1"] & A_2 \arrow[d, "f_2"] \\ L \arrow[r, "l"'] & B_1 \arrow[r, "b_i" above, "\vdots" below] & B_2 \end{tikzcd}$$

2) *Consider two naturally connected small families of arrows with common codomain (that is $a_i \cdot f = f_i \cdot b_i$ for each i in a small set I)*

$$\begin{tikzcd} A_i \arrow[r, "a_i"] \arrow[d, "f_i"'] & A \arrow[d, "f"] \\ B_i \arrow[r, "b_i"'] & B \end{tikzcd}$$

and the unique factorization $\varphi \colon P \to Q$ between the corresponding small pullbacks ; if f is a mono and each f_i is a regular epimorphism, then φ is a regular epimorphism.

Lemma 3 *Let $\mathbb{C}$ be a category with weak small limits and $\mathbb{B}$ a completely regular category. If a functor $F\colon \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak small products and weak small equalizers, then it is left covering w.r.t. weak small limits.*

Proof I) F is left covering w.r.t. weak small pullbacks : consider a small family of convergent arrows in $\mathbb{C}$ and a weak pullback as described in the first step of Lemma 1

$$E \xrightarrow{e} \prod_I X_i \xrightarrow{\pi_i} X_i \xrightarrow{f_i} X$$

Applying F we obtain

$$\begin{tikzcd}
F(E) \arrow[rr, "Fe"] \arrow[dr, "t"'] & & F(\prod_I X_i) \arrow[rr, "F(\pi_i \cdot f_i)"] \arrow[dd, "p"] & & F(X) \arrow[dd, "1"] \\
& S \arrow[ur, "s"'] \arrow[dl, "q"'] & & & \\
L \arrow[rr] & & \prod_I F(X_i) \arrow[r, "p_i"'] & F(X_i) \arrow[r, "Ff_i"'] & F(X)
\end{tikzcd}$$

where s is the equalizer of the $F(\pi_i \cdot f_i)$'s and the bottom line is the pullback of the $F(f_i)$'s computed using products and equalizers in $\mathbb{B}$. By assumption, p is a regular epi, so that by Lemma 2 also q is a regular epi. Also t is a regular epi by assumption, so that the comparison $t \cdot q\colon F(E) \to L$ is a regular epi.
II) F is left covering w.r.t. weak small limits : consider a functor $\mathcal{L}\colon \mathcal{D} \to \mathbb{C}$ with $\mathcal{D}$ small, and a weak limit as in the second step of Lemma 1

$$E \xrightarrow{\epsilon_d} E_d \xrightarrow{e_d} \prod_{\mathcal{D}} \mathcal{L}(D) \xrightarrow{\pi_D} \mathcal{L}(D) \quad .$$

In the same way we build up the limit of $\mathcal{L} \cdot F\colon \mathcal{D} \to \mathbb{C} \to \mathbb{B}$:

$$L \xrightarrow{\lambda_d} L_d \xrightarrow{l_d} \prod_{\mathcal{D}} F(\mathcal{L}D) \xrightarrow{\tilde{\pi}_D} F(\mathcal{L}D) \quad .$$

Let us call $l = \lambda_d \cdot l_d\colon L \to \prod_{\mathcal{D}} F(\mathcal{L}D)$; it is the limit of

$$\left(\begin{tikzcd} \prod_{\mathcal{D}} F(\mathcal{L}D) \arrow[r, shift left, "\tilde{\pi}_D \cdot F(\mathcal{L}d)"] \arrow[r, shift right, "\tilde{\pi}_{D'}"'] & F(\mathcal{L}D') \end{tikzcd} \right)_d$$

and then it is a mono. Now consider the pullback

$$\begin{tikzcd}
L \arrow[r, "l"] & \prod_{\mathcal{D}} F(\mathcal{L}D) \\
P \arrow[u, "p'"] \arrow[r, "l'"'] & F(\prod_{\mathcal{D}} \mathcal{L}D) \arrow[u, "p"']
\end{tikzcd}$$

By assumption, p is a regular epi and then also p' is a regular epi. Moreover, the mono $l' \colon P \to F(\prod_{\mathcal{D}} \mathcal{L}D)$ is the limit of

$$\left(F(\textstyle\prod_{\mathcal{D}} \mathcal{L}D) \underset{F(\pi_{D'})}{\overset{F(\pi_D)\cdot F(\mathcal{L}d)}{\rightrightarrows}} F(\mathcal{L}D') \right)_d$$

But this limit can be obtained in two steps : first taking for each d the equalizer

$$S_d \xrightarrow{s_d} F(\textstyle\prod_{\mathcal{D}} \mathcal{L}D) \underset{F(\pi_{D'})}{\overset{F(\pi_D)\cdot F(\mathcal{L}d)}{\rightrightarrows}} F(\mathcal{L}D')$$

and then taking the pullback $(\sigma_d \colon P \to S_d)_d$ of the family of convergent arrows $(s_d \colon S_d \to F(\prod_{\mathcal{D}} \mathcal{L}D))_d$. By assumption, the factorization $q_d \colon F(E_d) \to S_d$ is a regular epi ; then, by Lemma 2, the factorization $q \colon Q \to P$ is a regular epi, where $(j_d \colon Q \to F(E_d))_d$ is the pullback of the convergent family $(F(e_d) \colon F(E_d) \to F(\prod_{\mathcal{D}} \mathcal{L}D))_d$. Moreover, by the first part of this lemma, the factorization $t \colon F(E) \to Q$ is a regular epi. Finally, the comparison

$$F(E) \xrightarrow{t} Q \xrightarrow{q} P \xrightarrow{p'} L$$

is a regular epi because each part is a regular epi. □

The fact that, if $\mathbb{C}$ has weak small limits, then $\mathbb{C}_{ex}$ is completely regular, is stated without proof in [5]. In the next lemma, we add an explicit description of small products in $\mathbb{C}_{ex}$. This description will be used in the proof of Proposition 6.

Lemma 4 *Let $\mathbb{C}$ be a category with weak small limits. Its exact completion $\mathbb{C}_{ex}$ is completely regular and $\Gamma \colon \mathbb{C} \to \mathbb{C}_{ex}$ is left covering w.r.t. weak small limits.*

Proof The last fact follows from the fact that $\mathbb{C}$ is a projective cover of the complete category $\mathbb{C}_{ex}$. The fact that in $\mathbb{C}_{ex}$ regular epis are product-stable follows from the fact that they are of the form

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow\downarrow & & \downarrow\downarrow \\ X & \xrightarrow[1]{} & X \end{array}$$

and from the description of products in $\mathbb{C}_{ex}$ given hereunder.
Consider a family of objects

$$\left(R_i \underset{r_1^i}{\overset{r_0^i}{\rightrightarrows}} X_i \right)_I$$

in $\mathbb{C}_{ex}$; their product is given by

$$\begin{tikzcd} E \arrow[r, "e_i"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & R_i \arrow[d, shift right, "r_0^i"'] \arrow[d, shift left, "r_1^i"] \\ \prod_I X_i \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$

where $\pi_i \colon \prod_I X_i \to X_i$ is a weak product in $\mathbb{C}$ and E is a weak limit in $\mathbb{C}$ as in the following diagram

$$\begin{tikzcd} & E \arrow[dl, "e_0"'] \arrow[dr, "e_1"] \arrow[dd, "e_i"] & \\ \prod_I X_i \arrow[d, "\pi_i"'] & & \prod_I X_i \arrow[d, "\pi_i"] \\ X_i & R_i \arrow[l, "r_0^i"] \arrow[r, "r_1^i"'] & X_i \end{tikzcd}$$

□

The next lemma generalizes Proposition 20 in [5]. Since the proof runs as in the finitary case, we omit it.

Lemma 5 *Let $F \colon \mathbb{A} \to \mathbb{B}$ be a functor between complete categories. If F is left covering w.r.t. small limits, then it is continuous.*

The next proposition is the main result of this section. It completes the preparatory work to establish the universal property of the infinitary exact completion. In the proof, we give an explicit description of small equalizers in $\mathbb{C}_{ex}$.

Proposition 6 *Let $\mathbb{C}$ be a category with weak small limits, $\mathbb{B}$ an exact and completely regular category and $F \colon \mathbb{C} \to \mathbb{B}$ a functor left covering w.r.t. weak small limits. Then the exact extension $\hat{F} \colon \mathbb{C}_{ex} \to \mathbb{B}$ is continuous.*

Proof Recall that $\hat{F}$ sends an object of $\mathbb{C}_{ex}$, that is a pseudo equivalence relation $r_0, r_1 \colon R \rightrightarrows X$, to the coequalizer of the pair (Fr_0, Fr_1). This coequalizer exists because, by Theorem 26 in [5], the jointly monic part of the regular epi-jointly mono factorization of (Fr_0, Fr_1) is an equivalence relation. $\hat{F}$ extends to morphisms via the universal property of the coequalizer. By Lemma 5, to prove that $\hat{F}$ is continuous is enough to prove that $\hat{F}$ is left covering w.r.t. small limits. By Lemma 3, it suffices to prove that $\hat{F}$ is left covering w.r.t. small products and small equalizers.
I) Consider a small product in $\mathbb{C}_{ex}$

$$\begin{tikzcd} E \arrow[r, "e_i"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & R_i \arrow[d, shift right, "r_0^i"'] \arrow[d, shift left, "r_1^i"] \\ \prod_I X_i \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$

Using F and $\hat{F}$ we obtain the following commutative diagram

$$\begin{tikzcd}
F(\prod_I X_i) \arrow[r, "\lambda"] \arrow[d, "q"'] & \prod_I F(X_i) \arrow[d, "\prod_I q_i"] \\
\hat{F}(e_0, e_1) \arrow[r, "\mu"'] & \prod_I \hat{F}(r_0^i, r_1^i)
\end{tikzcd}$$

Each q_i is a regular epi, so that $\prod_I q_i$ is a regular epi because $\mathbb{B}$ is completely regular ; by assumption λ is a regular epi, so that also μ is a regular epi.
II) Small equalizers in $\mathbb{C}_{ex}$: let

$$\begin{tikzcd}
R \arrow[r, "\overline{f}_i", "\vdots"'] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
X \arrow[r, "f_i", "\vdots"'] & Y
\end{tikzcd}$$

be a small family of parallel arrows in $\mathbb{C}_{ex}$ and consider the diagram which contains all the diagrams of the form

$$\begin{tikzcd}
 & & S \arrow[dl, "s_0"'] \arrow[dd, "s_1"] \\
 & Y & \\
X \arrow[ur, "f_i"] \arrow[rr, "f_j"'] & & Y
\end{tikzcd}$$

for each $i \neq j$, all the subdiagrams being connected by the identity on X. Take now a weak limit E of such diagram

$$\begin{tikzcd}
E \arrow[rr, "\varphi_{i,j}"] \arrow[dd, "e"'] & & S \arrow[dl, "s_0"'] \arrow[dd, "s_1"] \\
 & Y & \\
X \arrow[ur, "f_i"] \arrow[rr, "f_j"'] & & Y
\end{tikzcd}$$

and a weak limit $\underline{R}$ as in the diagram below

$$\begin{tikzcd}
E \arrow[d, "e"'] & \underline{R} \arrow[l, "e_0"'] \arrow[r, "e_1"] \arrow[d, "\overline{e}"] & E \arrow[d, "e"] \\
X & R \arrow[l, "r_0"] \arrow[r, "r_1"'] & X
\end{tikzcd}$$

The equalizer in $\mathbb{C}_{ex}$ is

$$\begin{tikzcd}
\underline{R} \arrow[r, "\overline{e}"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & R \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] \\
E \arrow[r, "e"'] & X
\end{tikzcd}$$

III) Consider a small equalizer in $\mathbb{C}_{ex}$ as in the previous step

$$\begin{tikzcd}
\underline{R} \arrow[r, "\overline{e}"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & R \arrow[r, "\overline{f}_i", "\vdots"'] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
E \arrow[r, "e"'] & X \arrow[r, "f_i", "\vdots"'] & Y
\end{tikzcd}$$

and apply $\hat{F} \colon \mathbb{C}_{ex} \to \mathbb{B}$. We obtain

$$\begin{tikzcd}
F\underline{R} \arrow[r, "F\overline{e}"] \arrow[d, shift right, "Fe_0"'] \arrow[d, shift left, "Fe_1"] & FR \arrow[r, "F\overline{f}_i", "\vdots"'] \arrow[d, shift right, "Fr_0"'] \arrow[d, shift left, "Fr_1"] & FS \arrow[d, shift right, "Fs_0"'] \arrow[d, shift left, "Fs_1"] \arrow[dr, "p"] & \\
FE \arrow[r, "Fe"'] \arrow[d, "q_1"'] & FX \arrow[r, "Ff_i", "\vdots"'] \arrow[d, "q_2"'] & FY \arrow[d, "q_3"] & N(q_3) \arrow[l, shift right, "n_0"'] \arrow[l, shift left, "n_1"] \\
\hat{F}(e_0, e_1) \arrow[r, "\hat{F}[e]"] \arrow[d, "t"'] & \hat{F}(r_0, r_1) \arrow[r, "\hat{F}[f_i]", "\vdots"'] & \hat{F}(s_0, s_1) & \\
L \arrow[ur, "h"'] & & &
\end{tikzcd}$$

where the triangle on the right is the regular epi-jointly mono factorization of (Fs_0, Fs_1) and the triangle on the bottom is the factorization of $\hat{F}[e]$ through the equalizer of the $\hat{F}[f_i]$'s. We have to prove that $t \colon \hat{F}(e_0, e_1) \to L$ is a regular epi. For this, consider the limit A in $\mathbb{B}$ of the following diagram (a subdiagram for each $i \neq j$, all the subdiagrams connected by the identity of $F(X)$)

$$\begin{tikzcd}
A \arrow[rr, "k_{i,j}"] \arrow[dd, "i"'] & & N(q_3) \arrow[dl, "n_0"'] \arrow[dd, "n_1"] \\
& FY & \\
FX \arrow[ur, "Ff_i"] \arrow[rr, "Ff_j"'] & & FY
\end{tikzcd}$$

Using the morphisms $\varphi_{i,j} \colon E \to S$, we obtain a factorization $\sigma \colon FE \to A$ making commutative the following (solid) diagram

$$\begin{tikzcd}
FE \arrow[r, "\sigma"] \arrow[d, "q_1"'] & A \arrow[d, "i"] \arrow[ddl, dotted, "\tau"] \\
\hat{F}(e_0, e_1) \arrow[d, "t"'] & FX \arrow[d, "q_2"] \\
L \arrow[r, "h"'] & \hat{F}(r_0, r_1)
\end{tikzcd}$$

Observe that h is a mono, and assume that σ is a regular epi (we will prove this later), so that there exists $\tau\colon A \to L$ making commutative the two triangles. If we can show that τ is a regular epi, we have that also t is a regular epi, as we need. To show that τ is a regular epi, we can check that it is the pullback of q_3 along h. This can be done using that (n_0, n_1) is the kernel pair of its coequalizer q_3 (because $\mathbb{B}$ is exact) and i is a mono (because (n_0, n_1) are jointly monic). It remains to prove that $\sigma\colon FE \to A$ is a regular epi. For this, consider the limit $\underline{A}$ in $\mathbb{B}$ of the following diagram (with the usual conventions)

$$\begin{tikzcd}
\underline{A} \arrow[rr, "K_{i,j}"] \arrow[dd, "\underline{i}"'] & & FS \arrow[dl, "Fs_0"'] \arrow[dd, "Fs_1"] \\
& FY & \\
FX \arrow[ur, "Ff_i"] \arrow[rr, "Ff_j"'] & & FY
\end{tikzcd}$$

Since $F\colon \mathbb{C} \to \mathbb{B}$ is left covering, the unique factorization $\alpha\colon FE \to \underline{A}$ such that $\alpha \cdot \underline{i} = Fe$ and $\alpha \cdot K_{i,j} = F\varphi_{i,j}$ is a regular epi. Moreover, comparing the construction of A and of $\underline{A}$, we see that the unique factorization $m\colon \underline{A} \to A$ such that $m \cdot i = \underline{i}$ and $m \cdot k_{i,j} = K_{i,j} \cdot p$, is a regular epi (use part 2 of Lemma 2). Finally, composing with the mono $i\colon A \to FX$, we check that $\sigma = \alpha \cdot m$. $\square$

Corollary 7 *Let $\mathbb{C}$ be a category with weak small limits and consider its exact completion*

$$\Gamma\colon \mathbb{C} \to \mathbb{C}_{ex} .$$

For each exact and completely regular category $\mathbb{B}$, the composition with Γ induces an equivalence between the category of exact and continuous functors from $\mathbb{C}_{ex}$ to $\mathbb{B}$ and the category of functors from $\mathbb{C}$ to $\mathbb{B}$ left covering w.r.t. small limits.

Bearing in mind the fact that a left exact functor which preserves small products is continuous, and that a functor between complete categories is left covering w.r.t. small limits iff it is continuous (Lemma 5), one could think that a left exact functor which is left covering w.r.t. small products is indeed continuous. This is wrong. In fact, let $\mathbb{C}$ and $\mathbb{B}$ be as in Corollary 7 and assume $F: \mathbb{C} \to \mathbb{B}$ be left covering w.r.t. weak finite limits and weak small products. A glance at the proof of Proposition 6 shows that the exact extension $\hat{F}: \mathbb{C}_{ex} \to \mathbb{B}$ is (left exact and) left covering w.r.t. small products. Now, if $\hat{F}$ is continuous, then F is left covering w.r.t. weak small limits. The following example shows that this is not always true.

Example 8 Let $\mathcal{C}$ be a small category and let $i: \mathbb{B} \to [\mathcal{C}^{op}, Set]$ be a persistent localization of the presheaf category $[\mathcal{C}^{op}, Set]$, i.e. the left adjoint $F: [\mathcal{C}^{op}, Set] \to \mathbb{B}$ is left exact and units are regular epis (see [7]). First of all, observe that $\mathbb{B}$, being a Grothendick topos, is exact and extensive and has a family of regular generators. Moreover, since units are regular epis, the generators are regular projective and indecomposable objects (see [9]). Then $\mathbb{B}$ is equivalent to a presheaf category and then (as any exact and complete category with enough regular projective objects) it is completely regular. Let us now prove that the reflector is left covering w.r.t. small products. Consider a family of objects $(X_j)_J$ in $[\mathcal{C}^{op}, Set]$ and their product in $[\mathcal{C}^{op}, Set]$ and in $\mathbb{B}$. Consider also the comparison $e: F(\prod_J X_j) \to \prod_J F(X_j)$; we have to show that e is a regular epi. Consider the following commutative diagram

$$\begin{tikzcd} i(F(\prod_J X_j)) \arrow[rr, "i(e)"] & & i(\prod_J F(X_j)) \simeq \prod_J i(F(X_j)) \\ & \prod_J X_j \arrow[ul, "\eta_{\prod_J X_j}"] \arrow[ur, "\prod_J \eta_{X_j}"'] & \end{tikzcd}$$

(where $\eta_X: X \to i(F(X))$ is the unit at the point X). Since units are regular epis and $[\mathcal{C}^{op}, Set]$ is completely regular, $i(e)$ is a regular epi. But, since i is full and faithful and left exact, it reflects regular epis, and then e is a regular epi. To finish the argument, we need an example of a persistent localization of a presheaf category such that the reflector is not continuous. Such an example is provided by Example 2.6 in [7] : let M be a semilattice with a non-principal ideal I and take as $\mathbb{B}$ the full subcategory of the category M-Set of those objects on which the elements of I act trivially. $\mathbb{B}$ is a persistent localization, but the reflector does not preserve small products, as communicated to me by P.T. Johnstone.

2 Essential localizations of varieties

Let $\mathbb{B}$ be a complete and cocomplete category and consider the following conditions:

(LM) $\mathbb{B}$ is exact and has a small regular generator $\mathcal{G}$. This means that $\mathcal{G} = (G_j)_J$ is a small family of objects in $\mathbb{B}$ such that, for each object X in $\mathbb{B}$, the canonical arrow x is a regular epimorphism

$$x: \coprod_{j \in J, \mathbb{B}(G_j, X)} G_j \longrightarrow X \ ;$$

(AB4*) Regular epimorphisms are product-stable ;

(AB5) Filtered colimits are exact, i.e. commute with finite limits ;

(AB6) Given a small family of functors $(H_i \colon \mathcal{A}_i \to \mathbb{B})_I$ defined on small filtered categories, the canonical comparison τ is an isomorphism

$$\tau \colon colim(\prod_I \mathcal{A}_i \to \prod_I \mathbb{B} \to \mathbb{B}) \longrightarrow \prod_I (colim H_i) \ ;$$

(WAB6) With the same notations as in (AB6), the comparison morphism τ is a regular epimorphism.

It is known that

i - Condition (LM) characterizes localizations of monadic categories over a power of *Set* [14] ;

ii - Conditions (LM) and (AB4*) characterize continuous localizations of monadic categories over a power of *Set* [1] (continuous localization means that the reflector is continuous) ;

iii - Conditions (LM) and (AB5) characterize localizations of (multisorted, finitary) varieties [15] ;

iv - Conditions (LM), (AB4*), (AB5) and (AB6) characterize essential localizations of varieties [1] (essential localization means that the reflector has a left adjoint).

The aim of this section is to give a new proof of the last mentioned characterization using the theory developed in the first section, and replacing condition (AB6) with the weaker version (WAB6).

Proposition 9 *A complete and cocomplete category $\mathbb{B}$ is equivalent to an essential localization of a variety if and only if it satisfies conditions (LM), (AB4*), (AB5) and (WAB6).*

Proof To avoid heavy notations, let us write the proof in the one-sorted case, i.e. when the small regular generator $\mathcal{G}$ is reduced to a single regular generator G. Following [15], we consider the subcategory $\mathbb{C}$ of $\mathbb{B}$ having as objects copowers $S \bullet G$ of G, for S varying in *Set*. An arrow $G \to S \bullet G$ is in $\mathbb{C}$ if it factors through the canonical arrow $S' \bullet G \to S \bullet G$ for some finite subset S' of S. There is a finitary monad over *Set* given by

$$\mathbb{T} \colon \ Set \xrightarrow{-\bullet G} \mathbb{C} \xrightarrow{\mathbb{C}(G,-)} Set$$

The non-full inclusion $F \colon \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak finite limits, so that it has an exact extension $\hat{F} \colon \mathbb{C}_{ex} \to \mathbb{B}$. Moreover, $\hat{F}$ has a full and faithful right adjoint, so that $\mathbb{B}$ is a localization of $\mathbb{C}_{ex}$. But $\mathbb{C}$ is equivalent to the category of free $\mathbb{T}$-algebras, so that $\mathbb{C}_{ex}$ is equivalent to the variety of $\mathbb{T}$-algebras. Since $\mathbb{C}_{ex}$ is locally finitely presentable, $\mathbb{B}$ is locally presentable [3]. Then to prove that $\hat{F}$ has a left adjoint it suffices to prove that it is continuous [2]. According to Proposition 6 and Lemma 3, we have to show that F is left covering w.r.t. weak small products and weak small equalizers.

I) $F\colon \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak small products. Consider a small family of sets $(S_i)_I$, the corresponding objects $S_i \bullet G$ in $\mathbb{C}$ and their product $(\pi_i\colon \prod_I (S_i \bullet G) \to S_i \bullet G)_I$ in $\mathbb{B}$. There is a canonical arrow

$$\Sigma\colon \mathbb{C}(G, \prod_I (S_i \bullet G)) \bullet G \longrightarrow \prod_I (S_i \bullet G)$$

$$\text{where} \quad \mathbb{C}(G, \prod_I (S_i \bullet G)) = \{x\colon G \to \prod_I (S_i \bullet G) \text{ s.t. } \forall i \in I \;\; x \cdot \pi_i \text{ is in } \mathbb{C}\}.$$

Since a weak product in $\mathbb{C}$ of the $S_i \bullet G$'s is obtained precomposing the π_i's with Σ, to prove that F is left covering w.r.t. small products means to prove that Σ is a regular epi. If each S_i is finite, then $\mathbb{C}(G, \prod_I (S_i \bullet G)) = \mathbb{B}(G, \prod_I (S_i \bullet G))$, so that Σ is a regular epi because G is a regular generator. In general, each S_i is the filtered colimit of its finite subsets,

$$S_i = colim_{\mathcal{P}_f(S_i)} S_i', \quad \text{and then} \quad S_i \bullet G = colim_{\mathcal{P}_f(S_i)} (S_i' \bullet G)\,.$$

For each choice $(S_i') \in \prod_I \mathcal{P}_f(S_i)$, there is a canonical arrow

$$\lambda\colon \mathbb{C}(G, \prod_I (S_i' \bullet G)) \bullet G \longrightarrow \prod_I (S_i' \bullet G)$$

which is a regular epi because each S_i' is finite ; taking the colimit, we obtain a regular epi Λ. Moreover, a diagram chase shows that there is an arrow μ making commutative the following diagram

$$\begin{tikzcd}
\mathbb{C}(G, \prod_I (S_i \bullet G)) \bullet G \arrow[r, "\Sigma"] & \prod_I (S_i \bullet G) = \prod_I (colim_{\mathcal{P}_f(S_i)} S_i' \bullet G) \\
colim_{\prod_I \mathcal{P}_f(S_i)} (\mathbb{C}(G, \prod_I (S_i' \bullet G)) \bullet G) \arrow[u, "\mu"] \arrow[r, "\Lambda"'] & colim_{\prod_I \mathcal{P}_f(S_i)} (\prod_I (S_i' \bullet G)) \arrow[u, "\tau"']
\end{tikzcd}$$

Since Λ is a regular epi and, by condition (WAB6), τ is a regular epi, it follows that Σ is a regular epi.

II) $F\colon \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak small equalizers. Consider a small family of parallel arrows in $\mathbb{C}$

$$\left(S \bullet G \underset{\vdots}{\xrightarrow{f_i}} R \bullet G \right)_I$$

with $S, R \in Set$, and their equalizer $e\colon E \to S \bullet G$ in $\mathbb{B}$. There is a canonical arrow $\Sigma\colon \mathbb{C}(G, e) \bullet G \to E$, where

$$\mathbb{C}(G, e) = \{x\colon G \to E \text{ s.t. } x \cdot e \text{ is in } \mathbb{C}\}$$

Since a weak equalizer in $\mathbb{C}$ of the f_i's is given by $\Sigma \cdot e \colon \mathbb{C}(G,e) \bullet G \to S \bullet G$, to prove that F is left covering w.r.t. small equalizers means to prove that Σ is a regular epi. If S is finite, then $\mathbb{C}(G,e) = \mathbb{B}(G,E)$ so that Σ is a regular epi because G is a regular generator. In general,

$$S = colim_{\mathcal{P}_f(S)} S', \quad \text{and then} \quad \sigma_{S'} \colon S' \bullet G \to S \bullet G = colim_{\mathcal{P}_f(S)}(S' \bullet G)$$

is a filtered colimit. For each $S' \in \mathcal{P}_f(S)$, consider the following small equalizer

$$E_{S'} \xrightarrow{e_{S'}} S' \bullet G \xrightarrow[\vdots]{\sigma_{S'} \cdot f_i} R \bullet G \ ;$$

there is a canonical arrow $\lambda_{S'} \colon \mathbb{C}(G, e_{S'}) \bullet G \to E_{S'}$ which is a regular epi because S' is finite. Taking the colimit, we obtain a regular epi Λ. Moreover, the various $\sigma_{S'}$'s give rise to a canonical arrow η, and a diagram chase shows that there is an arrow μ making commutative the following diagram

$$\begin{tikzcd}
\mathbb{C}(G,e) \bullet G \arrow[r, "\Sigma"] & E \\
colim_{\mathcal{P}_f(S)}(\mathbb{C}(G,e_{S'}) \bullet G) \arrow[u, "\mu"] \arrow[r, "\Lambda"'] & colim_{\mathcal{P}_f(S)} E_{S'} \arrow[u, "\eta"']
\end{tikzcd}$$

To end the proof, we will show that η is an isomorphism. For this, we apply to the diagram

$$\left(S \bullet G \xrightarrow[\vdots]{f_i} R \bullet G \right)_I$$

the usual formula to compute a limit as an equalizer of a *pair* of arrows between two small products, and we obtain

$$E \longrightarrow S \bullet G \times R \bullet G \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \textstyle\prod_I (R \bullet G) \times S \bullet G \times R \bullet G$$

We do the same with the diagram

$$\left(S' \bullet G \xrightarrow[\vdots]{\sigma_{S'} \cdot f_i} R \bullet G \right)_I$$

and we obtain

$$E_{S'} \xrightarrow{\epsilon_{S'}} S' \bullet G \times R \bullet G \underset{b_{S'}}{\overset{a_{S'}}{\rightrightarrows}} \textstyle\prod_I (R \bullet G) \times S' \bullet G \times R \bullet G$$

Taking the colimit, we obtain the following commutative diagram

$$\begin{tikzcd}
L \arrow[rd, "n"] \arrow[dd, bend right=60, "l"'] & \\
colim_{\mathcal{P}_f(S)} E_{S'} \arrow[u, "m"'] \arrow[r, "\eta"] \arrow[d, "\epsilon"] & E \arrow[d] \\
colim_{\mathcal{P}_f(S)}(S' \bullet G \times R \bullet G) \arrow[r, "\gamma"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & S \bullet G \times R \bullet G \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\
colim_{\mathcal{P}_f(S)}(\prod_I (R \bullet G) \times S' \bullet G \times R \bullet G) \arrow[r, "\delta"'] & \prod_I (R \bullet G) \times S \bullet G \times R \bullet G
\end{tikzcd}$$

where l is the equalizer of a and b. Since in $\mathbb{B}$ filtered colimits commute with finite limits, the arrows m, γ and δ are isomorphisms, and then also n is an isomorphism. This implies that η is an isomorphism. $\square$

3 Essential localizations of presheaf categories

In [10, 11] and [1], the following theorem is proved : a category is equivalent to an essential localization of a presheaf category if and only if it satisfies conditions (LM) and (AB4*), it is infinitary extensive [4] (i.e. small sums are disjoint and universal) and the following complete distributivity holds

(CD) let I be a small set and, for each $i \in I$, consider a small family $(A_{i,j})_{j \in J_i}$ of objects. The canonical comparison τ is an isomorphism

$$\tau\colon \coprod_{f \in \prod J_i} (\prod_{i \in I} A_{i,f(i)}) \longrightarrow \prod_{i \in I} (\coprod_{j \in J_i} A_{i,j}) \ .$$

Similarly to what we have done for varieties, the aim of this section is to prove the above characterization replacing the last condition by its weaker version (WCD) in which the comparison τ is assumed to be a regular epimorphism.

Proposition 10 *A complete and cocomplete category $\mathbb{B}$ is equivalent to an essential localization of a presheaf category if and only if it is infinitary extensive and satisfies conditions (LM), (AB4*) and (WCD).*

Proof Let $\mathcal{G} = (G_j)_J$ be the small regular generator, which can be seen as a full subcategory of $\mathbb{B}$. The classical Giraud's theorem shows that $\mathbb{B}$ is a localization of $[\mathcal{G}^{op}, Set]$ (see [5] for a proof of Giraud's theorem using the (finitary) exact completion). The presheaf category $[\mathcal{G}^{op}, Set]$ is equivalent to the exact completion of $Fam\mathcal{G}$, the sum-completion of $\mathcal{G}$ (this is because the subcategory spanned by sums of representables is a projective cover of $[\mathcal{G}^{op}, Set]$). Moreover,

$Fam\mathcal{G}$ can be described as the following non-full subcategory $\mathbb{C}$ of $\mathbb{B}$: an object of $\mathbb{C}$ is a sum of generators, that is an object of the form $\coprod_{s\in S} G_{f(s)}$ for $f\colon S \to J$ a map in Set ; an arrow $\coprod_S G_{f(s)} \to \coprod_R G_{g(r)}$ in $\mathbb{C}$ is an arrow in $\mathbb{B}$ such that, for each $s \in S$, there is an $r_s \in R$ and an arrow $G_{f(s)} \to G_{g(r_s)}$ making commutative the following diagram

$$\begin{tikzcd}
\coprod_S G_{f(s)} \arrow[r] & \coprod_R G_{g(r)} \\
G_{f(s)} \arrow[u, "\sigma_s"] \arrow[r] & G_{g(r_s)} \arrow[u, "\sigma_{r_s}"']
\end{tikzcd}$$

Up to the equivalence $[\mathcal{G}^{op}, Set] \simeq \mathbb{C}_{ex}$, the left exact reflector $[\mathcal{G}^{op}, Set] \to \mathbb{B}$ is given by the exact extension $\hat{F}\colon \mathbb{C}_{ex} \to \mathbb{B}$ of the inclusion $F\colon \mathbb{C} \to \mathbb{B}$. As in the case of varieties, to prove that $\hat{F}$ has a left adjoint it is enough to prove that it is continuous. For this, we have to show that F is left covering w.r.t. weak small products and weak small equalizers.
I) $F\colon \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak small products. Observe that, given a small category $\mathcal{D}$ and a functor $\mathcal{L}\colon \mathcal{D} \to \mathbb{C}$, a weak limit of $\mathcal{L}$ in $\mathbb{C}$ can be computed in the following way : let $(\pi_D\colon L \to F(\mathcal{L}D))_{\mathcal{D}}$ be the limit in $\mathbb{B}$ of $\mathcal{L} \cdot F$ and consider the canonical morphism

$$\lambda\colon \coprod_{j\in J, \mathbb{C}(G_j, L)} G_j \longrightarrow L$$

$$\text{where} \quad \mathbb{C}(G_j, L) = \{f\colon G_j \to L \text{ s.t. } \forall D \in \mathcal{D} \;\; f \cdot \pi_D \text{ is in } \mathbb{C}\}$$

A weak limit of $\mathcal{L}$ is obtained precomposing the π_D's with λ. To prove that F is left covering w.r.t. small products then means to prove that λ is a regular epi when $\mathcal{D}$ is discrete, say $\mathcal{D} = I$ for a set I. If, for each $i \in I$, $\mathcal{L}(i)$ is in $\mathcal{G}$, then $\mathbb{C}(G_j, L) = \mathbb{B}(G_j, L)$ and λ is a regular epi because $\mathcal{G}$ is a regular generator. In general, let us write $X_i = \mathcal{L}(i)$ for $i \in I$. Each X_i is a sum of generators, say $X_i = \coprod_{j\in J_i} A_{i,j}$ with $A_{i,j} \in \mathcal{G}$, and we can consider the following diagram

$$\begin{tikzcd}
F(\coprod_{f\in\prod J_i}(\prod_{i\in I} A_{i,f(i)})) \arrow[r, "a"] & F(\prod_I X_i) \arrow[d, "\lambda"] \\
 & \prod_I FX_i \simeq \prod_{i\in I}(\coprod_{j\in J_i} FA_{i,j}) \\
\coprod_{f\in\prod J_i}(F(\prod_{i\in I} A_{i,f(i)})) \arrow[uu, "\simeq"] \arrow[r, "c"'] & \coprod_{f\in\prod J_i}(\prod_{i\in I} FA_{i,f(i)}) \arrow[u, "b"']
\end{tikzcd}$$

The arrow a is the image under F of the canonical morphism $\coprod(\prod A_{i,j}) \to \prod X_i$ in $\mathbb{C}$; the two isomorphisms depend on the fact that sums in $\mathbb{C}$ are computed in $\mathbb{B}$; the arrow b is a regular epi by condition (WCD) ; the arrow c is a regular epi because it is sum of regular epis, each $F(\prod_I A_{i,j}) \to \prod_I FA_{i,j}$ being a regular

epi for the particular case previously discussed. Finally, since the diagram is commutative, λ is a regular epi.
II) $F : \mathbb{C} \to \mathbb{B}$ is left covering w.r.t. weak small equalizers. Once again, the proof is very similar to that for varieties and we omit details. Let us only point out that, imitating the constructions done in step II of the proof of Proposition 9, one arrives at a diagram in which two equalizers (of *pairs* of parallel arrows) are compared, and one has to prove that they are connected by two isomorphisms. This is the case because of the universality of sums in $\mathbb{B}$ (in the varietal case we used the exactness of filtered colimits). $\square$

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