Beck's theorem for pseudo-monads

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Abstract

In this work we establish a 2-categorical analogue of Beck's theorem characterizing monadic functors. We show that a 2-functor (a pseudo-functor) U is monadic iff it is a right pseudo-adjoint, it reflects adjoint equivalences and it creates U-absolute pseudo-coequalizers of codescent objects.

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Introduction

One of the most famous theorem in category theory is Beck's theorem characterizing monadic functors. It was proved in 1966 and since then has found an impressive number of applications to one-dimensional categorical algebra. On the other hand, starting from the fundamental papers by Gray [8] and Kelly and Street [13], 2-dimensional categorical algebra has been developped. In this paper we establish a 2-categorical analogue of Beck's theorem, replacing adjunctions and monads by pseudo-adjunctions and pseudo-monads on 2-categories.

Let us recall here the classical theorem : consider an adjunction $F \dashv U\colon \mathcal{A} \to \mathcal{C}$ and the comparison functor $K\colon \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ (where $\mathbf{T}$ is the monad induced by the adjunction $F \dashv U$ and $\mathcal{C}^{\mathbf{T}}$ is the category of $\mathbf{T}$-algebras). The functor K is an equivalence if and only if U is conservative and, given two parallel arrows u and v in $\mathcal{A}$ having a split (and then absolute) coequalizer in $\mathcal{C}$, then u and v have a coequalizer preserved by U. This situation plainly transposes to the 2-categorical setting : any pseudo-adjunction between 2-categories $F \dashv U\colon \mathcal{A} \to \mathcal{C}$ induces a pseudo monad $\mathbf{T}$ on $\mathcal{C}$ and a comparison pseudo-functor $K\colon \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ ($\mathcal{C}^{\mathbf{T}}$ being now the 2-category of pseudo-$\mathbf{T}$-algebras) ; the condition to be conservative becomes, in dimension 2, to reflect adjoint equivalences. It remains, and this is the main point, to understand the appropriate analogue of coequalizer of two parallel arrows. This analogue is provided by the notion of pseudo-coequalizer of a codescent object, introduced by Street in [18, 19].

The paper is organized as follows. In the first section we fix our notations and we establish some preliminary results. Full definitions and basic facts on pseudo-adjunctions and pseudo-monads can be found in [5, 8, 11, 14, 15]. In section 2 we discuss the notion of pseudo-coequalizer of a codescent object and we show its link with pseudo-algebras. Section 3 is devoted to the main results.

In the last section, we specialize our main results to the case where the pseudo-monad induced by a pseudo-adjunction is a KZ-doctrine. For expository reasons, we have stated Beck's theorem for pseudo-monads in terms of absolute pseudo-coequalizers instead of split pseudo-coequalizers. Moreover, for sake of clarity, we have restricted ourselves to pseudo-adjunctions where the pseudo-functors are 2-functors. All the results remain true when one considers the adjoints to be pseudo-functors : the length of the proofs slightly increase, but the techniques employed remain the same. Applications of our results already appear in [1, 2], where the equational hull of some important 2-categories is studied.

1 Notations and preliminary results

In the case of ordinary adjunctions, the right adjoint functor is full (faithful) if and only if the components of the counit are split monomorphisms (are epimorphisms), and dually for left adjoints. A similar analysis can be done for pseudo-adjunctions. We recall the notations for a pseudo-adjunction.

Definition 1.1 *Let $\mathcal{A}$ and $\mathcal{C}$ be two 2-categories and let U and F in the diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "U"'] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd}$$

be 2-functors. F is a left pseudo-adjoint to U if there exists

1. *pseudo-natural transformations $\eta : 1 \to UF$ and $\epsilon : FU \to 1$*
2. *invertible modifications $s : 1_F \Rightarrow (\epsilon F) \circ (F\eta)$ and $t : (U\epsilon) \circ (\eta U) \Rightarrow 1_U$*

such that the following equations hold [8]:

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\overset{\eta_{(\eta)}}{\Leftarrow}"] & UF \arrow[d, "UF\eta"] \arrow[ddr, equal, bend left] & \\
UF \arrow[r, "\eta UF"'] \arrow[drr, equal, bend right] & UFUF \arrow[dr, "U\epsilon F"] \arrow[r, phantom, "\overset{Us}{\Leftarrow}"] \arrow[d, phantom, "\overset{tF}{\Leftarrow}"] & {} \\
& {} & UF
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[d, bend right, "\eta"'] \arrow[d, bend left, "\eta"] \arrow[d, phantom, "="] \\
UF
\end{tikzcd}$$

$$
\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] \arrow[drr, phantom, "\overset{Ft}{\Rightarrow}"] \arrow[ddr, phantom, "\overset{sU}{\Rightarrow}"'] & & & & FU \arrow[dd, bend right, "\epsilon"'] \arrow[dd, bend left, "\epsilon"] \arrow[dd, phantom, "="] \\
& FUFU \arrow[r, "FU\epsilon"] \arrow[d, "\epsilon FU"] \arrow[dr, phantom, "\overset{\epsilon(\epsilon)}{\Rightarrow}"] & FU \arrow[d, "\epsilon"] & = & \\
& FU \arrow[r, "\epsilon"'] & 1 & & 1
\end{tikzcd}
$$

Proposition 1.2 *With the previous notations.*

1. *U is locally full iff for each $A, A' \in \mathcal{A}$ the functor $\mathcal{A}(\epsilon_A, A') : \mathcal{A}(A, A') \to \mathcal{A}(FUA, A')$ is full.*

2. *U is locally faithful iff for each $A, A' \in \mathcal{A}$ the functor $\mathcal{A}(\epsilon_A, A')$ is faithful.*

3. *U is locally essentially surjective on objects iff for each $A, A' \in \mathcal{A}$ the functor $\mathcal{A}(\epsilon_A, A')$ is essentially surjective.*

4. *U is locally an equivalence iff for each $A, A' \in \mathcal{A}$ the functor $\mathcal{A}(\epsilon_A, A')$ is an equivalence.*

Proof. Let $A, A' \in \mathcal{A}$, then the functors

$$\mathcal{A}(A, A') \xrightarrow{U_{A,A'}} \mathcal{C}(UA, UA') \xrightarrow{\chi_{A,A'}} \mathcal{A}(FUA, A')$$

where $\chi_{A,A'}$ is an adjoint equivalence natural in A and A', and

$$\mathcal{A}(A, A') \xrightarrow{\mathcal{A}(\epsilon_A, A')} \mathcal{A}(FUA, A')$$

are isomorphic, the isomorphism being given by $\epsilon_{(f)}$ for $f \in \mathcal{A}(A, A')$. The pseudo-naturality of ϵ ensures that we do get a natural transformation. Hence we get:

- $U_{A,A'}$ is faithful $\Leftrightarrow$ $\chi_{A,A'}U_{A,A'}$ is faithful $\Leftrightarrow$ $\mathcal{A}(\epsilon_A, A')$ is faithfull
- $U_{A,A'}$ is full $\Leftrightarrow$ $\chi_{A,A'}U_{A,A'}$ is full $\Leftrightarrow$ $\mathcal{A}(\epsilon_A, A')$ is full
- $U_{A,A'}$ is e.s.o. $\Leftrightarrow$ $\chi_{A,A'}U_{A,A'}$ is e.s.o. $\Leftrightarrow$ $\mathcal{A}(\epsilon_A, A')$ is e.s.o.
- $U_{A,A'}$ is an equivalence $\Leftrightarrow$ $\chi_{A,A'}U_{A,A'}$ is an equivalence $\Leftrightarrow$ $\mathcal{A}(\epsilon_A, A')$ is an equivalence

where e.s.o. is essentially surjective on objects. $\square$

Observe that for each $A' \in \mathcal{A}$ the hom-functor $\mathcal{A}(f, A') : \mathcal{A}(A, A') \to \mathcal{A}(B, A')$ induced by an arrow $f : B \to A$, is essentially surjective on objects iff f has a pseudo-retraction, i.e., there exists an arrow $f^* : A \to B$ and an invertible 2-cell

$$\begin{tikzcd} & A \arrow[dr, "f^*"] & \\ B \arrow[ur, "f"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\cong"] & & B . \end{tikzcd}$$

Having in mind 2.6 and 2.7 in [7] or Section 5 in [10], we call the arrow f a pseudo-epi if for each $A' \in \mathcal{A}$ the functor $\mathcal{A}(f, A')$ is full and faithful (f.f.), so that we can restate the previous proposition in the following way:

Proposition 1.3 *The following conditions are equivalent:*

1. *U is locally an equivalence,*
2. *for each $A \in \mathcal{A}$, the arrow ϵ_A is a pseudo-epi and has a pseudo-retraction,*
3. *for each $A \in \mathcal{A}$, the arrow ϵ_A is an equivalence.*

Proof. 3) implies 1). If ϵ_A is an equivalence, then $\mathcal{A}(\epsilon_A, A')$ is an equivalence.
1) implies 2). If U is locally an equivalence then $\mathcal{A}(\epsilon_A, A')$ is f.f. and e.s.o., so ϵ_A is a pseudo-epi and has a pseudo-retraction.
2) implies 3). Suppose that ϵ_A is a pseudo-epi and has a pseudo-retraction, then we have an arrow ϵ_A^* and an isomorphism $\alpha : \epsilon_A^* \circ \epsilon_A \cong 1_{FUA}$. As ϵ_A is a pseudo-epi, there exists a unique isomorphism $\beta : \epsilon_A \circ \epsilon_A^* \cong 1_A$ such that $\beta\epsilon_A = \epsilon_A \alpha$. $\square$

As far as the left pseudo-adjoint is concerned, Proposition 1.2 holds when one replaces U by F, the co-unit ϵ by the unit η and $\mathcal{A}(\epsilon_A, A')$ by $\mathcal{C}(C', \eta_C)$ for $C, C' \in \mathcal{C}$. Finally, say that an arrow $g : C \to D$ of $\mathcal{C}$ has a pseudo-section if there exists an arrow $g^* : D \to C$ and an invertible 2-cell $\gamma : g \circ g^* \cong 1_D$. Call g a pseudo-mono if for each $C \in \mathcal{C}$ the functor

$$\mathcal{C}(C', g) : \mathcal{C}(C', C) \to \mathcal{C}(C', D)$$

is full and faithfull. Proposition 1.3 becomes:

Proposition 1.4 *The following conditions are equivalent:*

1. *F is locally an equivalence,*
2. *for each $C \in \mathcal{C}$, the arrow η_C is a pseudo-mono and has a pseudo-section,*
3. *for each $C \in \mathcal{C}$, the arrow η_C is an equivalence.*

We recall now the notations for pseudo-monads and pseudo-algebras.

Definition 1.5 *Let $\mathcal{C}$ be a 2-category. A pseudo-monad on $\mathcal{C}$ is a six-tuple (T, η, μ, l, r, a) where $T : \mathcal{C} \to \mathcal{C}$ is a 2-functor, $\eta : 1 \to T$ and $\mu : T^2 \to T$ are pseudo-natural transformations and l, r, a are the following modifications which are isomorphisms:*

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\overset{l}{\Leftarrow}"] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \arrow[dl, phantom, shift right=3, "\overset{r}{\Leftarrow}"'] \\
& T &
\end{tikzcd}$$

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\overset{a}{\Rightarrow}"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

satisfying two compatibility conditions (equations (1) and (2) on page 95 of [15]).

It is well known that a pseudo-adjunction $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ generates an associated pseudo-monad $\mathbf{T}$ given by $T = UF$, $\eta^{\mathbf{T}} = \eta$, $\mu^{\mathbf{T}} = U\epsilon F$, $l^{\mathbf{T}} = tF$, $r^{\mathbf{T}} = Us$ and $a^{\mathbf{T}} = U\epsilon_{(\epsilon F)}$.

Definition 1.6 *Let $\mathbf{T} = (T, \eta, \mu, l, r, a)$ be a pseudo-monad on the 2-category $\mathcal{C}$. A pseudo-$\mathbf{T}$-algebra is a quadruple $(C, c, c^0, \overline{c})$ where C is an object of $\mathcal{C}$, $c : TC \to C$ is an arrow in $\mathcal{C}$, and c^0 and $\overline{c}$ are the following 2-cell isomorphisms*

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\overset{c^0}{\Leftarrow}"] & TC \arrow[d, "c"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2C \arrow[r, "Tc"] \arrow[d, "\mu_C"'] \arrow[dr, phantom, "\overset{\overline{c}}{\Rightarrow}"] & TC \arrow[d, "c"] \\
TC \arrow[r, "c"'] & C
\end{tikzcd}$$

satisfying two compatibility conditions (equations (6) and (7) on page 96 of [15]).

A morphism of pseudo-$\mathbf{T}$-algebras is a double $(f, \overline{f}) : (C, c, c^0, \overline{c}) \to (D, d, d^0, \overline{d})$ where $f : C \to D$ is an arrow in $\mathcal{C}$ and $\overline{f}$ is the following invertible 2-cell:

$$\begin{tikzcd}
TC \arrow[r, "Tf"] \arrow[d, "c"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & TD \arrow[d, "d"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

satisfying two compatibility conditions (equations (9) and (10) on page 97 of [15]).

A 2-cell $\alpha : (f, \overline{f}) \Rightarrow (g, \overline{g})$ between morphisms of pseudo-$\mathbf{T}$-algebra is simply a 2-cell $\alpha : f \Rightarrow g$ in $\mathcal{C}$ satisfying a compatibility condition (equation (11) on page 97 of [15]).

Let us denote by $\mathcal{C}^{\mathbf{T}}$ the 2-category of pseudo-$\mathbf{T}$-algebras. There is an evident forgetful 2-functor $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ defined by $U^{\mathbf{T}}(\,(C, c, c^0, \overline{c})\,) = C$, $U^{\mathbf{T}}(\,(f, \overline{f})\,) = f$ and $U^{\mathbf{T}}(\alpha) = \alpha$. The 2-functor $U^{\mathbf{T}}$ has a pseudo-adjoint $F^{\mathbf{T}}$ given by $F^{\mathbf{T}}(D) = (TD, \mu_D, a_D, l_D)$, $F^{\mathbf{T}}(h) = (Th, \mu_{(h)})$ and $F^{\mathbf{T}}(\beta) = T\beta$. The unit $\eta^{\mathbf{T}}$ of this pseuso-adjunction is given by the pseudo-natural transformation η, the co-unit $\epsilon^{\mathbf{T}}$ is given on objects by $\epsilon^{\mathbf{T}}_{(C,c,c^0,\overline{c})} = (c, \overline{c})$ and on arrows by $\epsilon^{\mathbf{T}}_{(f,\overline{f})} = \overline{f}$. The modification $s^{\mathbf{T}}$ is defined by $s^{\mathbf{T}}_C = r_C$ and the modification $t^{\mathbf{T}}$ is defined by $t^{\mathbf{T}}_{(C,c,c^0,\overline{c})} = c^0$.

Given a pseudo-adjunction $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ one obtains a comparison 2-functor $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$, where $\mathbf{T}$ is the associated pseudo-monad, such that $U^{\mathbf{T}} \circ K = U$ and $K \circ F = F^{\mathbf{T}}$:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "K"] \arrow[dr, shift right, "U"'] & & \mathcal{C}^{\mathbf{T}} \arrow[dl, shift right, "U^{\mathbf{T}}"'] \\ & \mathcal{C} \arrow[ul, shift right, "F"'] \arrow[ur, shift right, "F^{\mathbf{T}}"'] & \end{tikzcd} .$$

The 2-functor K is given by $K(A) = (UA, U(\epsilon_A), t_A, U(\epsilon_{\epsilon_A}))$ on an object A of $\mathcal{A}$, $K(f) = (Uf, U(\epsilon_f))$ on an arrow f of $\mathcal{A}$, and $K(\alpha) = U\alpha$ on a 2-cell α of $\mathcal{A}$.

From Proposition 1.2, we obtain the following fact:

Proposition 1.7 *The 2-functor K is locally faithful if and only if for each $A, A' \in \mathcal{A}$ the functor $\mathcal{A}(\epsilon_A, A')$ is faithful.*

2 Codescent objects

In the ordinary case, a functor U is monadic if and only if it has a left adjoint, reflects isomorphisms and creates coequalizers of U-split pairs of arrows (see [3, 6, 16]). In order to generalize this theorem in dimension 2, we must replace "pairs of parallel arrows" by codescent objects (see [18, 19]).

Definition 2.1 *Let $\mathcal{X}$ be the 2-category generated by the following truncated bi-cosimplicial diagram*

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_2"'] & X_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & X_0 \arrow[l, "\iota" description] \end{tikzcd}$$

with the following invertible 2-cells

$$\begin{aligned} \sigma_{ij} &: \delta_i\delta_j \cong \delta_{j-1}\delta_i \quad i < j \\ n_0 &: \delta_0\iota \cong 1 \\ n_1 &: 1 \cong \delta_1\iota. \end{aligned}$$

A codescent object in a 2-category $\mathcal{A}$ is a 2-functor $S : \mathcal{X} \to \mathcal{A}$. A morphism of codescent objects is a pseudo-natural transformation $\phi : S \to S' : \mathcal{X} \to \mathcal{A}$, and a 2-cell between morphisms of codescent objects is a modification.

The appropriate notion of "coequalizer" of codescent objects is given by a pseudo-colimit which we shall now describe. The weight that is necessary is given by a 2-functor $J : \mathcal{X}^{op} \to Cat$ for which $J(X_0) = *$ (the terminal category), $J(X_1) = 0 \cong 1$, $J(X_2) = 0 \cong 1 \cong 2$, the image of $\delta_i : X_1 \to X_0$ under J is the functor $* \mapsto i$ for $i = 0, 1$, the image of $\delta_i : X_2 \to X_1$ by J is the functor having as image in $J(X_2)$ the arrow $0 \cong 1$, $0 \cong 2$, $1 \cong 2$ for $i = 0, 1, 2$ respectively, and all two cells in $\mathcal{X}$ are sent to identity 2-cells in Cat.

Given a codescent object S in a 2-category $\mathcal{A}$, the pseudo-coequalizer of S is given by the pseudo-colimit $J * S$ where

$$\mathcal{A}(J * S, A) \simeq Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, A))$$

is natural in A. The category $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, A))$ has pseudo-natural transformations from J to $\mathcal{A}(S-, A)$ as objects and modifications as arrows. Explicitly, an object of $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, A))$ is a pair $(a\colon S(X_0) \to A, \alpha\colon a \circ S(\delta_1) \Rightarrow a \circ S(\delta_0))$ with α an invertible 2-cell such that

$$\begin{tikzcd}
 & & S(X_0) \arrow[dr, "a"] & \\
S(X_0) \arrow[urr, bend left, equal] \arrow[drr, bend right, equal] \arrow[r, "S(\iota)"] \arrow[urr, phantom, "S(n_0)\Uparrow"] \arrow[drr, phantom, "S(n_1)\Uparrow"'] & S(X_1) \arrow[ur, "S(\delta_0)"'] \arrow[dr, "S(\delta_1)"] \arrow[rr, phantom, "\alpha\Uparrow"] & & A \\
 & & S(X_0) \arrow[ur, "a"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S(X_0) \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\|"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
S(X_2) \arrow[rr, "S(\delta_0)"] \arrow[dr, "S(\delta_1)"] \arrow[dd, "S(\delta_2)"'] & & S(X_1) \arrow[dr, "S(\delta_0)"] \arrow[dl, phantom, "S(\sigma_{01})\Rightarrow"] & \\
 & S(X_1) \arrow[rr, "S(\delta_0)"] \arrow[dd, "S(\delta_1)"] \arrow[dl, phantom, "S(\sigma_{12})\Rightarrow"] \arrow[ddrr, phantom, "\alpha\Rightarrow"] & & S(X_0) \arrow[dd, "a"] \\
S(X_1) \arrow[dr, "S(\delta_1)"'] & & & \\
 & S(X_0) \arrow[rr, "a"'] & & A
\end{tikzcd}
=
\begin{tikzcd}
S(X_2) \arrow[rr, "S(\delta_0)"] \arrow[dd, "S(\delta_2)"'] \arrow[ddrr, phantom, "S(\sigma_{02})\Rightarrow"] & & S(X_1) \arrow[dr, "S(\delta_0)"] \arrow[dd, "S(\delta_1)"'] & \\
 & & & S(X_0) \arrow[dd, "a"] \arrow[dl, phantom, "\alpha\Rightarrow"] \\
S(X_1) \arrow[rr, "S(\delta_0)"'] \arrow[dr, "S(\delta_1)"'] & & S(X_0) \arrow[dr, "a"] \arrow[dl, phantom, "\alpha\Rightarrow"] & \\
 & S(X_0) \arrow[rr, "a"'] & & A\,;
\end{tikzcd}$$

an arrow $\lambda\colon (a\colon S(X_0) \to A, \alpha\colon a \circ S(\delta_1) \Rightarrow a \circ S(\delta_0)) \Rightarrow (a'\colon S(X_0) \to A, \alpha'\colon a' \circ S(\delta_1) \Rightarrow a' \circ S(\delta_0))$ in $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, A))$ is a 2-cell $\lambda\colon a \Rightarrow a'$ such that

$$
\begin{tikzcd}
S(X_1) \arrow[r, "S(\delta_0)"] \arrow[d, "S(\delta_1)"'] \arrow[dr, phantom, "\overset{\alpha'}{\Rightarrow}"] & S(X_0) \arrow[d, "a'"] \\
S(X_0) \arrow[r, bend left, "a'"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\overset{\lambda}{\Uparrow}"] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S(X_1) \arrow[r, "S(\delta_0)"] \arrow[d, "S(\delta_1)"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & S(X_0) \arrow[d, bend right, "a"'] \arrow[d, bend left, "a'"] \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] \\
S(X_0) \arrow[r, "a"'] & A \, .
\end{tikzcd}
$$

A pseudo-coequalizer of $S : \mathcal{X} \to \mathcal{A}$ is an object $(q\colon S(X_0) \to Q, \varphi\colon q \circ S(\delta_1) \Rightarrow q \circ S(\delta_0))$ in $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, Q))$ such that for each object A of $\mathcal{A}$ the obvious functor

$$\kappa_A : \mathcal{A}(Q, A) \to Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-, A)$$

is an equivalence of categories.

Remark 2.2

The following fact is an obvious consequence of the universal property of a pseudo-coequalizer. We observe it explicitly for future references. Let $S' : \mathcal{X} \to \mathcal{A}$ be a second codescent object in $\mathcal{A}$ and let $(q'\colon S'(X_0) \to Q', \varphi'\colon q' \circ S'(\delta_1) \Rightarrow q' \circ S'(\delta_0))$ be an object in $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S'-, Q'))$. Consider a pseudo-natural transformation $(x_-, y_-)\colon S \Rightarrow S'$ with

$$
\begin{tikzcd}
S(X_i) \arrow[r, "S(f)"] \arrow[d, "x_i"'] \arrow[dr, phantom, "\overset{y_f}{\Rightarrow}"] & S(X_j) \arrow[d, "x_j"] \\
S'(X_i) \arrow[r, "S'(f)"'] & S'(X_j) \, .
\end{tikzcd}
$$

By the universal property of the coequalizer of $S : \mathcal{X} \to \mathcal{A}$ we obtain an arrow $x\colon Q \to Q'$ and an invertible 2-cell $y\colon q' \circ x_0 \Rightarrow x \circ q$ such that

$$
\begin{tikzcd}
S(X_1) \arrow[rr, "S(\delta_0)"] \arrow[dr, "S(\delta_1)"] \arrow[ddd, "x_1"'] & \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & S(X_0) \arrow[dr, "q"] & \\
 & S(X_0) \arrow[rr, "q"] \arrow[dd, "x_0"] \arrow[dl, phantom, "\overset{y_{\delta_1}}{\Rightarrow}"] & \arrow[d, phantom, "\overset{y}{\Rightarrow}"] & Q \arrow[dd, "x"] \\
 S'(X_1) \arrow[dr, "S'(\delta_1)"'] & & {} & \\
 & S'(X_0) \arrow[rr, "q'"'] & & Q'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S(X_1) \arrow[rr, "S(\delta_0)"] \arrow[dd, "x_1"'] \arrow[ddrr, phantom, "\overset{y_{\delta_0}}{\Rightarrow}"] & & S(X_0) \arrow[dr, "q"] \arrow[dd, "x_0"] & \\
 & & & Q \arrow[dd, "x"] \arrow[dl, phantom, "\overset{y}{\Rightarrow}"] \\
S'(X_1) \arrow[rr, "S'(\delta_0)"'] \arrow[dr, "S'(\delta_0)"'] & \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & S'(X_0) \arrow[dr, "q'"] & \\
 & S'(X_0) \arrow[rr, "q'"'] & & Q' \, .
\end{tikzcd}
$$

The pair (x, y) is unique in the following sense : if $(\overline{x}\colon Q \to Q', \overline{y}\colon q' \circ x_0 \Rightarrow \overline{x} \circ q)$ (with $\overline{y}$ invertible) satisfies the analogous equation, then there exists a unique invertible 2-cell $z\colon x \Rightarrow \overline{x}$ such that

$$\begin{tikzcd}
S(X_0) \arrow[r, "q"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\overset{y}{\Rightarrow}"] & Q \arrow[d, bend right, "x"'] \arrow[d, bend left, "\overline{x}"] \arrow[d, phantom, "\overset{z}{\Rightarrow}"] \\
S'(X_0) \arrow[r, "q'"'] & Q'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S(X_0) \arrow[r, "q"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\overset{\overline{y}}{\Rightarrow}"] & S(X_0) \arrow[d, "\overline{x}"] \\
S'(X_0) \arrow[r, "q'"'] & Q' .
\end{tikzcd}$$

Moreover, if $(x'_-, y'_-)\colon S \Rightarrow S'$ is another pseudo-natural transformation inducing a pair $(x'\colon Q \to Q', y'\colon q' \circ x'_0 \Rightarrow x' \circ q)$ and if $\theta_-\colon (x_-, y_-) \Rightarrow (x'_-, y'_-)\colon S \Rightarrow S'$ is a modification with

$$\begin{tikzcd}
S(X_i) \arrow[r, bend left, "x_i"] \arrow[r, bend right, "x'_i"'] \arrow[r, phantom, "\overset{\theta_i}{\Downarrow}"] & S'(X_i) ,
\end{tikzcd}$$

then there exists a unique 2-cell $\theta\colon x \Rightarrow x'$ such that

$$\begin{tikzcd}
S(X_0) \arrow[r, "q"] \arrow[d, "x_0"'] \arrow[dr, phantom, "\overset{y}{\Rightarrow}"] & Q \arrow[d, bend right, "x"'] \arrow[d, bend left, "x'"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\
S'(X_0) \arrow[r, "q'"'] & Q'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S(X_0) \arrow[r, "q"'] \arrow[d, bend right, "x_0"'] \arrow[d, bend left, "x'_0"] \arrow[d, phantom, "\overset{\theta_0}{\Rightarrow}"] \arrow[dr, phantom, "\overset{y'}{\Rightarrow}"] & Q \arrow[d, "x'"] \\
S'(X_0) \arrow[r, "q'"'] & Q' .
\end{tikzcd}$$

Lemma 2.3 *Let $\mathbf{T} = (T, \eta, \mu, l, r, a)$ be a pseudo-monad on a 2-category $\mathcal{C}$. Let $(C, c, c^0, \overline{c})$ be a pseudo-$\mathbf{T}$-algebra. The canonical diagram*

$$\begin{tikzcd}
T^3C \arrow[r, shift left=2, "T^2c"] \arrow[r, "T\mu_C" description] \arrow[r, shift right=2, "\mu_{TC}"'] & T^2C \arrow[r, shift left=2, "Tc"] \arrow[r, shift right=2, "\mu_C"'] & TC, \arrow[l, "T\eta_C" description]
\end{tikzcd}$$

with the appropriate 2-cells, is a codescent object in $\mathcal{C}$, and the arrow $c : TC \to C$ with the invertible 2-cell $\overline{c} : c \circ \mu_C \cong c \circ Tc$ is an absolute pseudo-coequalizer of this codescent object.

Proof. It is straightforward to prove that the above diagram is a codescent object, which we will call $S : \mathcal{X} \to \mathcal{C}$, and that $(c : TC \to C, \overline{c} : c \circ \mu_C \cong c \circ Tc)$ is an object in $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{C}(S-, C))$. We must show that for each $A \in \mathcal{C}$ the canonical map

$$\kappa_A : \mathcal{C}(C, A) \to Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{C}(S-, A))$$

is an equivalence, i.e., fully faithful and essentially surjective on objects. We shall show explicitly that the above cocone satisfies the required universal property using the “splitting” given by the arrows η_C, η_{TC}, η_{T^2C} and the associated canonical 2-cells.
κ_A is essentially surjective on objects: Let $(a : TC \to A, \alpha : a \circ \mu_C \Rightarrow a \circ Tc)$ be an object in $Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{C}(S-, A))$. The factorization arrow is given by $k = a \circ \eta_C$ and the invertible 2-cell $\pi\colon a \Rightarrow k \circ c$, given by the following pasting diagram, is an isomorphism $\pi\colon (a, \alpha) \to \kappa_A(k)$:

$$\begin{tikzcd}
TC \arrow[rr, "c"] \arrow[dd, equal] \arrow[dr, "\eta_{TC}"] & \arrow[d, phantom, "\overset{\eta_{(c)}}{\Rightarrow}"] & C \arrow[dr, "\eta_C"] & \\
\arrow[r, phantom, "\overset{\iota_C^{-1}}{\Rightarrow}"] & T^2C \arrow[rr, "Tc"] \arrow[dl, "\mu_C"] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & TC \arrow[dl, "a"] \\
TC \arrow[rr, "a"] & & A\,. &
\end{tikzcd}$$

κ_A is full: Let $k'\colon C \to A$ be an arrow in $\mathcal{C}$ and consider an arrow $\pi'\colon (a, \alpha) \to \kappa_A(k')$. The following pasting diagram gives us a 2-cell $\psi\colon k \Rightarrow k'$

$$\begin{tikzcd}
C \arrow[d, "\eta_C"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\overset{c^0}{\Rightarrow}"] & \\
TC \arrow[r, "c"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\pi'}{\Rightarrow}"] & C \arrow[dl, "k'"] \\
A\,, &
\end{tikzcd}$$

and one checks that

$$\begin{tikzcd}
TC \arrow[r, "c"] \arrow[dr, "a"'] \arrow[r, phantom, shift right=6, "\overset{\pi}{\Rightarrow}"] & C \arrow[d, bend right=40, "k"'] \arrow[d, bend left=40, "k'"] \arrow[d, phantom, "\overset{\psi}{\Rightarrow}"] \\
& A
\end{tikzcd}
=
\begin{tikzcd}
TC \arrow[r, "c"] \arrow[dr, "a"'] \arrow[r, phantom, shift right=6, "\overset{\pi'}{\Rightarrow}"] & C \arrow[d, "k'"] \\
& A\,.
\end{tikzcd}$$

κ_A is faithful: The arrow c has a pseudo-section as $c^0 : 1_C \Rightarrow c \circ \eta_C$ is an isomorphism. This easily implies that κ_A is faithful.
Since the splitting of the pseudo-coequalizer is given by equations on the 2-cells, it is automatically preserved by any 2-functor. □

3 Beck’s theorem

Definition 3.1 *Let $U : \mathcal{A} \to \mathcal{C}$ be a 2-functor. A codescent object $S : \mathcal{X} \to \mathcal{A}$ is said to be U-absolute if the codescent object $U \circ S$ admits an absolute pseudo-*

coequalizer in $\mathcal{C}$.

Proposition 3.2 *Let $\mathbf{T} = (T, \eta, \mu, l, r, a)$ be a pseudo-monad on a 2-category $\mathcal{C}$. The forgetful functor $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ creates pseudo-coequalizers of $U^{\mathbf{T}}$-absolute codescent objects in $\mathcal{C}^{\mathbf{T}}$.*

Proof. Let $\mathbf{S} : \mathcal{X} \to \mathcal{C}^{\mathbf{T}}$ be a $U^{\mathbf{T}}$-absolute codescent object. Let us write S for the codescent object $U^{\mathbf{T}} \circ \mathbf{S}$ in $\mathcal{C}$. Thus we may define

- $\mathbf{S}(X_i) = (S(X_i), x_i, x_i^0, \overline{x_i})$,
- $\mathbf{S}(\delta_i) = (S(\delta_i), \overline{S(\delta_i)})$,
- $\mathbf{S}(n_i) = (S(n_i), \overline{S(n_i)})$.

Let $(q\colon S(X_0) \to C, \varphi\colon q \circ S(\delta_1) \Rightarrow q \circ S(\delta_0))$ be an absolute pseudo-coequalizer of S. Thus applying T to the pseudo-coequalizer we get a new absolute pseudo-coequalizer. The actions x_0, x_1 and x_2 constitute a 2-cell (or a pseudo-natural transformation) $x\colon T \circ S \Rightarrow S$ in the 2-category of codescent objects. Omitting the 2-cells, we have the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
T(S(X_2)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "x_2"'] & T(S(X_1)) \arrow[r, shift left=2] \arrow[l] \arrow[r, shift right=2] \arrow[d, "x_1"'] & T(S(X_0)) \arrow[r, "Tq"] \arrow[d, "x_0"'] & TC \\
S(X_2) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & S(X_1) \arrow[r, shift left=2] \arrow[l] \arrow[r, shift right=2] & S(X_0) \arrow[r, "q"] & C.
\end{tikzcd}$$

Since the top line is a pseudo-coequalizer, we can apply Remark 2.2. In this way we get a unique (up to isomorphisms) arrow $c : TC \to C$ and an invertible 2-cell $\overline{q}\colon q \circ x_0 \Rightarrow c \circ Tq$ which is compatible with the rest of the diagram.

The 2-cells x_0^0, x_1^0, x_2^0 constitute a modification $x^0 : x \circ \eta \to 1_S : S \Rightarrow S$. By the universal property of pseudo-coequalizers, we have a unique induced invertible 2-cell $c^0 : c \circ \eta_C \Rightarrow 1_C$ which is compatible with the diagram.

Similarly the 2-cells $\overline{x_0}$, $\overline{x_1}$, $\overline{x_2}$ constitute a modification $\overline{x} : x \circ \mu_S \to x \circ Tx$ which induces a unique invertible 2-cell $\overline{c} : c \circ \mu_C \to c \circ Tc$ as $(T^2(q), T^2(\varphi))$ is a pseudo-coequalizer of $T^2 \circ S$.

Since $(S, x, x^0, \overline{x})$ satisfies component-wise the equations of a $\mathbf{T}$-algebra and (q, φ), $(T(q), T(\varphi))$, $(T^2(q), T^2(\varphi))$, $(T^3(q), T^3(\varphi))$ are the (absolute) pseudo-coequalizers of the codescent objects S, TS, T^2S, T^3S respectively, the quadruple $(C, c, c^0, \overline{c})$ is forced to satisfy the equations of a $\mathbf{T}$-algebra, $(q, \overline{q})$ is a morphism of $\mathbf{T}$-algebras and φ is a 2-cell in $\mathcal{C}^{\mathbf{T}}$. Furthermore the couple $((q, \overline{q}), \varphi)$ is a pseudo-coequalizer of the codescent object $\mathbf{S}$. $\square$

Corollary 3.3 *Each pseudo-$\mathbf{T}$-algebra is a pseudo-coequalizer of free pseudo-$\mathbf{T}$-algebras.*

Proposition 3.4 *Let $\mathbf{T} = (T, \eta, \mu, l, r, a)$ be a pseudo-monad on a 2-category $\mathcal{C}$. The 2-functor $U^{\mathbf{T}}$ reflects adjoint equivalences.*

Proof. Let $(f, \overline{f}) : (C, c, c^0, \overline{c}) \to (D, d, d^0, \overline{d})$ be a morphism in $\mathcal{C}^{\mathbf{T}}$ such that f forms part of an adjoint equivalence in $\mathcal{C}$, i.e., there exists a morphism $g : D \to C$ and invertible 2-cells $\alpha : 1 \Rightarrow gf$, $\beta : fg \Rightarrow 1$ satisfying the triangle equations. We have to show that g, α and β lift to the 2-category $\mathcal{C}^{\mathbf{T}}$. For this, it suffices to define the 2-cell $\overline{g}$ by the following diagram:

$$
\begin{tikzcd}
TD \arrow[r, "Tg"] \arrow[d, equal] \arrow[dr, phantom, "\overset{T\beta}{\Rightarrow}" near start] & TC \arrow[dl, "Tf" description] \arrow[d, "c"] \\
TD \arrow[d, "d"'] \arrow[r, phantom, "\overset{\overline{f}^{-1}}{\Rightarrow}"] & C \arrow[dl, "f" description] \arrow[d, equal] \arrow[dl, phantom, "\overset{\alpha}{\Rightarrow}" near end] \\
D. \arrow[r, "g"'] & C\,.
\end{tikzcd}
$$

$\square$

Theorem 3.5 *Let $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ be a pseudo-adjunction. The comparison functor $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ is locally an equivalence iff for each A, the canonical diagram, equipped with the obvious 2-cells,*

$$
\begin{tikzcd}[column sep=large]
FUFUFU(A) \arrow[r, shift left=2, "FUFU(\epsilon_A)"] \arrow[r, "FU(\epsilon_{FU(A)})" description] \arrow[r, shift right=2, "\epsilon_{FUFU(A)}"'] & FUFU(A) \arrow[r, shift left=2, "FU(\epsilon_A)"] \arrow[r, "\epsilon_{FU(A)}"', shift right=2] & FU(A) \arrow[l, "F(\eta_{U(A)})" description] \arrow[r, "\epsilon_A"] & A
\end{tikzcd}
$$

is a pseudo-coequalizer in $\mathcal{A}$.

Proof. Suppose that the comparison functor K is locally an equivalence. Applying the 2-functor K to the above diagram we obtain

$$
\begin{tikzcd}[column sep=large]
K(FU)^3A \arrow[r, shift left=2, "KFUFU\epsilon_A"] \arrow[r, "KFU(\epsilon_{FU(A)})" description] \arrow[r, shift right=2, "K\epsilon_{FUFU(A)}"'] & K(FU)^2A \arrow[r, shift left=2, "KFU(\epsilon_A)"] \arrow[r, "K\epsilon_{FU(A)}"', shift right=2] & KFU(A) \arrow[l, "KF(\eta_{U(A)})" description] \arrow[r, "K\epsilon_A"] & KA\,.
\end{tikzcd}
$$

By Lemma 2.3, its image by $U^{\mathbf{T}}$ is an absolute pseudo-coequalizer. Then, by Proposition 3.2 and Proposition 3.4, it is a pseudo-coequalizer in $\mathcal{C}^{\mathbf{T}}$. Let $S : \mathcal{X} \to \mathcal{A}$ be the codescent object, with the appropriate 2-cells,

$$
\begin{tikzcd}[column sep=large]
FUFUFU(A) \arrow[r, shift left=2, "FUFU(\epsilon_A)"] \arrow[r, "FU(\epsilon_{FU(A)})" description] \arrow[r, shift right=2, "\epsilon_{FUFU(A)}"'] & FUFU(A) \arrow[r, shift left=2, "FU(\epsilon_A)"] \arrow[r, "\epsilon_{FU(A)}"', shift right=2] & FU(A) \arrow[l, "F(\eta_{U(A)})" description]
\end{tikzcd}
$$

and $J : \mathcal{X}^{op} \to Cat$ be the weight used to define pseudo-coequalizers. Hence for any $B \in \mathcal{A}$ we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[rr] \arrow[d, "K_{A,B}"'] & & Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{A}(S-,B)) \arrow[d, "Psd[\mathcal{X}^{op},Cat](J-,K_{S-,B})"] \\
\mathcal{C}^{\mathbf{T}}(KA,KB) \arrow[rr, "\simeq"] & & Psd[\mathcal{X}^{op}, Cat](J-, \mathcal{C}^{\mathbf{T}}(S-,B))\ ,
\end{tikzcd}$$

where the top and bottom horizontal lines are canonical arrows defined by the composition with the cocones $(\epsilon_A : FUA \to A, \epsilon_{\epsilon_A})$ and $(K\epsilon_A : KFUA \to KA, \epsilon_{\epsilon_A})$ respectively. The bottom line is an equivalence of categories since the latter cocone is a pseudo-coequalizer of the codescent object KS. The vertical lines are also equivalences since K is locally an equivalence. Hence it follows that the top horizontal line is also an equivalence, i.e., that $(\epsilon_A : FUA \to A, \epsilon_{\epsilon_A})$ is pseudo-coequalizer of S.

Suppose now that for each $A \in \mathcal{A}$ the canonical diagram, with the appropriate 2-cells,

$$\begin{tikzcd}
(FU)^2A \arrow[r, shift left=2, "FUFU(\epsilon_A)"] \arrow[r, "FU(\epsilon_{FU(A)})" description] \arrow[r, shift right=2, "\epsilon_{FUFU(A)}"'] & (FU)^2A \arrow[r, shift left=2, "FU(\epsilon_A)"] \arrow[r, shift right=2, "\epsilon_{FU(A)}"'] & FU(A) \arrow[l, "F(\eta_{U(A)})" description] \arrow[r, "\epsilon_A"] & A
\end{tikzcd}$$

is a pseudo-coequalizer in $\mathcal{A}$. Let us show that K is locally an equivalence.

• K is locally faithful:

Let $\alpha, \alpha' : f \Rightarrow g : A \to B$ be a pair of 2-cells in $\mathcal{A}$ such that $K(\alpha) = K(\alpha')$. Since $U^{\mathbf{T}}K = U$, it follows that $U(\alpha) = U(\alpha')$. It follows that, in the diagram

$$\begin{tikzcd}
FU(A) \arrow[rr, "\epsilon_A"] \arrow[d, bend right=40, "FU(f)"'] \arrow[d, bend left=40, "FU(g)"] \arrow[d, phantom, "\overset{FU(\alpha)}{\Rightarrow}"] \arrow[drr, phantom, "\overset{\epsilon_{(g)}}{\Rightarrow}\ \overset{\epsilon_{(f)}}{\Rightarrow}"] & & A \arrow[d, bend right=40, "f"'] \arrow[d, bend left=40, "g"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}\ \overset{\alpha'}{\Rightarrow}"] \\
FU(B) \arrow[rr, "\epsilon_B"'] & & B\ ,
\end{tikzcd}$$

the two possible cylinders commute. In the following diagram the three 2-cells $(FU)^3\alpha$, $(FU)^2\alpha$, $FU\alpha$ constitute a modification or a 2-cell in the 2-category of codescent objects in $\mathcal{A}$.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
(FU)^3A \arrow[r, shift left=2, "(FU)^2\epsilon_A"] \arrow[r, "FU\epsilon_{FUA}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2A}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{(FU)^3\alpha}{\Rightarrow}"] & (FU)^2A \arrow[r, shift left=2, "FU\epsilon_A"] \arrow[r, shift right=2, "\epsilon_{FUA}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{(FU)^2\alpha}{\Rightarrow}"] & FUA \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_A"] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{FU\alpha}{\Rightarrow}"] & A \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}\ \overset{\alpha'}{\Rightarrow}"] \\
(FU)^3B \arrow[r, shift left=2, "(FU)^2\epsilon_B"] \arrow[r, "FU\epsilon_{FUB}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2B}"'] & (FU)^2B \arrow[r, shift left=2, "FU\epsilon_B"] \arrow[r, shift right=2, "\epsilon_{FUB}"'] & FUB \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_B"] & B
\end{tikzcd}
$$

We know that the cocone $(U\epsilon_A, U\epsilon_{\epsilon_A})$ is a pseudo-coequalizer of the top codescent object. By the universal property of pseudo-coequalizers α and α' must be equal.

• K is locally full:

Let $f, g : A \to B$ be a pair of arrows in $\mathcal{A}$. Let $\beta : Kf \Rightarrow Kg$ be a 2-cell in the category of algebras $\mathcal{C}^{\mathbf{T}}$, i.e., the following diagram commutes

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
UFU(A) \arrow[rr, "U(\epsilon_A)"] \arrow[d, bend right, "UFU(f)"'] \arrow[d, bend left, "UFU(g)"] \arrow[d, phantom, "\overset{UF(\beta)}{\Rightarrow}"] & \overset{U(\epsilon_{(g)})}{\Rightarrow} \\
& \overset{U(\epsilon_{(f)})}{\Rightarrow} & U(A) \arrow[d, bend right, "U(f)"'] \arrow[d, bend left, "U(g)"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\
UFU(B) \arrow[rr, "U(\epsilon_B)"'] & & U(B).
\end{tikzcd}
$$

Thus, in the diagram

$$
\begin{tikzcd}[column sep=large, row sep=huge]
(FU)^3A \arrow[r, shift left=2, "(FU)^2\epsilon_A"] \arrow[r, "FU\epsilon_{FUA}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2A}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{(FU)^2F\beta}{\Rightarrow}"] & (FU)^2A \arrow[r, shift left=2, "FU\epsilon_A"] \arrow[r, shift right=2, "\epsilon_{FUA}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{FUF\beta}{\Rightarrow}"] & FUA \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_A"] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{F\beta}{\Rightarrow}"] & A \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
(FU)^3B \arrow[r, shift left=2, "(FU)^2\epsilon_B"] \arrow[r, "FU\epsilon_{FUB}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2B}"'] & (FU)^2B \arrow[r, shift left=2, "FU\epsilon_B"] \arrow[r, shift right=2, "\epsilon_{FUB}"'] & FUB \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_B"] & B\,,
\end{tikzcd}
$$

the 2-cells $(FU)^2F\beta$, $FUF\beta$, $F\beta$ constitute a modification which, by the universal property of the pseudo-coequalizer $(U\epsilon_A, U\epsilon_{\epsilon_A})$, induces a compatible 2-cell $\alpha : f \Rightarrow g$. We must now show that $K\alpha = \beta$ or equivalently that $U\alpha = \beta$.

Applying U to the above diagram we get

$$\begin{tikzcd}
U(FU)^3A \arrow[r, shift left=2, "U(FU)^2\epsilon_A"] \arrow[r, "UFU\epsilon_{FUA}" description] \arrow[r, shift right=2, "U\epsilon_{(FU)^2A}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{U(FU)^2F\beta}{\Rightarrow}"] & U(FU)^2A \arrow[r, shift left=2, "UFU\epsilon_A"] \arrow[r, shift right=2, "U\epsilon_{FUA}"'] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{UFUF\beta}{\Rightarrow}"] & UFUA \arrow[l, "UF\eta_{UA}" description] \arrow[r, "U\epsilon_A"] \arrow[d, bend right] \arrow[d, bend left, "Ug"] \arrow[d, phantom, "\overset{UF\beta}{\Rightarrow}"] & UA \arrow[d, bend right] \arrow[d, bend left, "Uf"] \arrow[d, phantom, "\overset{U\alpha}{\Rightarrow}\ \overset{\beta}{\Rightarrow}"] \\
U(FU)^3B \arrow[r, shift left=2, "U(FU)^2\epsilon_B"] \arrow[r, "UFU\epsilon_{FUB}" description] \arrow[r, shift right=2, "U\epsilon_{(FU)^2B}"'] & U(FU)^2B \arrow[r, shift left=2, "UFU\epsilon_B"] \arrow[r, shift right=2, "U\epsilon_{FUB}"'] & UFUB \arrow[l, "UF\eta_{UA}" description] \arrow[r, "U\epsilon_B"] & UB
\end{tikzcd}$$

where the horinzontal diagrams are pseudo-coequalizers. Since the three 2-cells $U(FU)^2F\beta$, $UFUF\beta$, $UF\beta$ constitute a modification and the 2-cells $U\alpha$, β are both compatible with this modification, it follows that they are equal.

- K is locally essentially surjective on objects:

Let A, B be objects of $\mathcal{A}$ and $(h, \overline{h}) : K(A) \to K(B)$ a morphism in $\mathcal{C}^{\mathbf{T}}$. The triple $(F(h), FUF(h), FUFUF(h))$, with the appropriate invertible 2-cells, constitutes a morphism between codescent objects, hence there exists an essentially unique factorization $(f, \overline{f})$ as $(\epsilon_A, \epsilon_{\epsilon_A})$ is a pseudo-coequalizer:

$$\begin{tikzcd}
(FU)^3A \arrow[r, shift left=2, "(FU)^2\epsilon_A"] \arrow[r, "FU\epsilon_{FUA}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2A}"'] \arrow[d, "FUFUF(h)"'] & (FU)^2A \arrow[r, shift left=2, "FU\epsilon_A"] \arrow[r, shift right=2, "\epsilon_{FUA}"'] \arrow[d, "FUF(h)"'] & FUA \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_A"] \arrow[d, "F(h)"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Rightarrow}"] & A \arrow[d, dotted, "f"] \\
(FU)^3B \arrow[r, shift left=2, "(FU)^2\epsilon_B"] \arrow[r, "FU\epsilon_{FUB}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2B}"'] & (FU)^2B \arrow[r, shift left=2, "FU\epsilon_B"] \arrow[r, shift right=2, "\epsilon_{FUB}"'] & FUB \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_B"] & B\,.
\end{tikzcd}$$

Applying U to the above diagram we get

$$\begin{tikzcd}
U(FU)^3A \arrow[r, shift left=2, "U(FU)^2\epsilon_A"] \arrow[r, "UFU\epsilon_{FUA}" description] \arrow[r, shift right=2, "U\epsilon_{(FU)^2A}"'] \arrow[d, "(UF)^3h"'] & U(FU)^2A \arrow[r, shift left=2, "UFU\epsilon_A"] \arrow[r, shift right=2, "U\epsilon_{FUA}"'] \arrow[d, "(UF)^2h"'] & UFUA \arrow[l, "UF\eta_{UA}" description] \arrow[r, "U\epsilon_A"] \arrow[d, "UF(h)"'] \arrow[dr, phantom, "\overset{(\overline{h})^{-1}}{\Rightarrow}\ \overset{U\overline{f}}{\Rightarrow}"] & UA \arrow[d, bend right, "h"'] \arrow[d, bend left, "Uf"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
U(FU)^3B \arrow[r, shift left=2, "U(FU)^2\epsilon_B"] \arrow[r, "UFU\epsilon_{FUB}" description] \arrow[r, shift right=2, "U\epsilon_{(FU)^2B}"'] & U(FU)^2B \arrow[r, shift left=2, "UFU\epsilon_B"] \arrow[r, shift right=2, "U\epsilon_{FUB}"'] & UFUB \arrow[l, "UF\eta_{UA}" description] \arrow[r, "U\epsilon_B"] & UB\,,
\end{tikzcd}$$

where $\alpha : h \Rightarrow U(f)$ is the unique invertible 2-cell compatible with two factorizations $(h, (\overline{h})^{-1})$, $(f, \overline{f})$ as $(U(\epsilon_A), U(\epsilon_{\epsilon_A}))$ is a pseudo-coequalizer, which is in fact absolute. Finally, one checks that α is an invertible 2-cell in the category of pseudo-$\mathbf{T}$-algebras. $\square$

Theorem 3.6 *Let $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ be a pseudo-adjunction. The comparison functor $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ is a bi-equivalence iff U reflects adjoint equivalences, $\mathcal{A}$ has pseudo-coequalizers of U-absolute codescent objects and U preserves them.*

Proof. ($\Rightarrow$): We know that $U = U^{\mathbf{T}}K$. It is easily shown that bi-equivalences reflect adjoint equivalences. By Proposition 3.4, $U^{\mathbf{T}}$ reflects adjoint equivalences, hence U reflects adjoint equivalences.

Let $S : \mathcal{X} \to \mathcal{A}$ be a U-absolute codescent object, i.e, the pseudo-coequalizer $J * US$ exists and is absolute, then KS is a $U^{\mathbf{T}}$ codescent object. Hence the pseudo-coequalizer $J * KS$ exists in $\mathcal{C}^{\mathbf{T}}$. As K is a bi-equivalence the pseudo-coequalizer $J * S$ of the codescent object S exists. Since $U = U^{\mathbf{T}}K$, we have

$$U(J * S) \simeq U^{\mathbf{T}}(J * KS) \simeq J * (U^{\mathbf{T}}KS) = J * US.$$

($\Leftarrow$): Let A be an object of $\mathcal{A}$. The image of the diagram (omitting the evident 2-cells)

$$\begin{tikzcd} (FU)^3A \arrow[r, shift left=2, "(FU)^2\epsilon_A"] \arrow[r, "FU\epsilon_{FUA}" description] \arrow[r, shift right=2, "\epsilon_{(FU)^2A}"'] & (FU)^2A \arrow[r, shift left=2, "FU\epsilon_A"] \arrow[r, shift right=2, "\epsilon_{FUA}"'] & FUA \arrow[l, "F\eta_{UA}" description] \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

by U is an absolute pseudo-coequalizer in $\mathcal{C}$. As U reflects adjoint equivalences the cocone $(\epsilon_A, \epsilon_{(\epsilon_A)})$ is a pseudo-coequalizer. By Theorem 3.5, the comparison 2-functor is locally an equivalence.

Let $\mathbf{C} = (C, c, c^0, \overline{c})$ be a pseudo-$\mathbf{T}$-algebra. Consider the codescent object (omitting the 2-cells), constructed from $\mathbf{C}$:

$$\begin{tikzcd} (FU)^2FC \arrow[r, shift left=2, "FUFc"] \arrow[r, "FU\epsilon_{FC}" description] \arrow[r, shift right=2, "\epsilon_{FUFC}"'] & FUFC \arrow[r, shift left=2, "Fc"] \arrow[r, shift right=2, "\epsilon_{FC}"'] & FC \arrow[l, "F\eta_C" description] \end{tikzcd} .$$

Its image by U has an absolute pseudo-coequalizer (see Lemma 2.3), namely

$$\begin{tikzcd} T^3C \arrow[r, shift left=2, "T^2c"] \arrow[r, "T\mu_C" description] \arrow[r, shift right=2, "\mu_{TC}"'] & T^2C \arrow[r, shift left=2, "Tc"] \arrow[r, shift right=2, "\mu_C"'] & TC \arrow[l, "T\eta_C" description] \arrow[r, "c"] & C \end{tikzcd} .$$

By assumption, there exists a pseudo-coequalizer in $\mathcal{A}$ $(f : FC \to A, \alpha)$ of the above codescent object. Applying the 2-functor K to this pseudo-coequalizer we get a cocone in $\mathcal{C}^{\mathbf{T}}$ which has, as image by $U^{\mathbf{T}}$, the above absolute pseudo-coequalizer in $\mathcal{C}$. As $U^{\mathbf{T}}$ creates pseudo-coequalizers of $U^{\mathbf{T}}$-absolute codescent objects and reflects adjoint equivalences, the cocone $(Kf, K\alpha)$ is a pseudo-coequalizer of the diagram (omitting 2-cells)

$$\begin{tikzcd} (KFU)^2FC \arrow[r, shift left=2, "KFUFc"] \arrow[r, "KFU\epsilon_{FC}" description] \arrow[r, shift right=2, "K\epsilon_{FUFC}"'] & KFUFC \arrow[r, shift left=2, "KFc"] \arrow[r, shift right=2, "K\epsilon_{FC}"'] & KFC \arrow[l, "KF\eta_C" description] \end{tikzcd} .$$

However, the pseudo-$\mathbf{T}$-algebra $(C, c, c^0, \overline{c})$ is also a pseudo-coequalizer of this codescent object. It follows that $KA \simeq (C, c, c^0, \overline{c})$, i.e., K is essentially surjective on objects up to equivalence. $\square$

Remark 3.7

A careful analysis of the proof of Lemma 2.3 leads to the definition of "split pseudo-coequalizer", which is the natural generalization of the classical notion of split coequalizer (Definition 4.4.2 in [6]). Lemma 2.3 can be restated saying that the diagram

$$\begin{tikzcd} T^3C \arrow[r, shift left=2, "T^2c"] \arrow[r, "T\mu_C" description] \arrow[r, shift right=2, "\mu_{TC}"'] & T^2C \arrow[r, shift left=2, "Tc"] \arrow[r, shift right=2, "\mu_C"'] & TC, \arrow[l, "T\eta_C" description] \end{tikzcd}$$

with the appropriate 2-cells, has a split pseudo-coequalizer, and that each split pseudo-coequalizer is an absolute pseudo-coequalizer. Once this done, also Theorem 3.6 can be restated with U-split instead of U-absolute.

4 KZ-doctrines

A much easier kind of pseudo-colimits, namely pseudo-coinverters, are needed in the case where the pseudo-monad induced by a pseudo-adjunction turns out to be a KZ-doctrine. We pursue such a case in this section relying on definitions and results in [14].

Definition 4.1 *A diagram of the form*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & B \arrow[r, "h"] & C, \end{tikzcd}$$

with $h\xi$ invertible, is called a split pseudo-coinverter if there exist adjunctions (supscripts are unit and counit respectively) $g \overset{\alpha,\beta}{\dashv} t \overset{\gamma,\delta}{\dashv} f \overset{\sigma,\pi}{\dashv} u$; $h \overset{\varphi,\psi}{\dashv} s$, with $t, u : B \to A$, and $s : C \to B$ arrows and with β and ψ invertible, together with an invertible 2-cell

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\theta"] & A \arrow[d, "g"] \\ C \arrow[r, "s"'] & B, \end{tikzcd}$$

that satisfy the following conditions:

- *ξ equals the 2-cell induced by $g \overset{\alpha,\beta}{\dashv} t \overset{\gamma,\delta}{\dashv} f$.*
- *If we call $\zeta : t \to u$ the 2-cell induced by the adjunction $t \overset{\gamma,\delta}{\dashv} f \overset{\sigma,\pi}{\dashv} u$, then*

the following equation holds:

$$
\begin{tikzcd}
B \arrow[r, bend left, "t"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\zeta"] \arrow[d, "h"'] & A \arrow[d, "g"] \arrow[dl, phantom, "\swArrow_{\theta}"] \\
C \arrow[r, "s"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\beta"] & \\
B \arrow[ur, "t"] \arrow[rr] \arrow[dr, "h"'] & {} \arrow[d, phantom, "\Downarrow\varphi"] & B. \\
 & C \arrow[ur, "s"'] &
\end{tikzcd}
$$

Lemma 4.2 *Assume we have a split pseudo-coinverter as above and an arrow $l : B \to D$. If l coinverts ξ, then l coinverts φ.*

Proof. Since $f\zeta$ is invertible, we have that l coinverts

$$
\begin{tikzcd}
B \arrow[r, bend left, "t"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\zeta"] \arrow[d, "h"'] & A \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\overset{\Leftarrow}{\xi}"] \arrow[dl, phantom, "\swArrow_{\theta}"] \\
C \arrow[r, "s"'] & B.
\end{tikzcd}
$$

This means that

$$
\begin{tikzcd}
 & A \arrow[dr, "g"'] \arrow[dr, bend left, "f"] \arrow[d, phantom, "\Downarrow\beta"] \arrow[dr, phantom, "\swArrow\xi"] & & \\
B \arrow[ur, "t"] \arrow[rr] \arrow[dr, "h"'] & {} \arrow[d, phantom, "\Downarrow\varphi"] & B \arrow[r, "l"] & D \\
 & C \arrow[ur, "s"'] & &
\end{tikzcd}
$$

is invertible. Since β and $l\xi$ are invertible, we obtain $l\varphi$ invertible. $\square$

Now the following proposition is easy to prove:

Proposition 4.3 *If* $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & B \arrow[r, "h"] & C \end{tikzcd}$ *is a split pseudo-coinverter, then it is an absolute pseudo-coinverter.*

Proof. For every object D, denote $\mathcal{C}_\xi(B, D)$ the full subcategory of $\mathcal{C}(B, D)$ consisting of those arrows that coinvert ξ. Now the equivalence is given by precomposing with h in the direction $\mathcal{C}(C, D) \to \mathcal{C}_\xi(B, D)$ and precomposing with s in the opposite direction. The previous lemma is to insure that the composition $\mathcal{C}_\xi(B, D) \to \mathcal{C}(C, D) \to \mathcal{C}_\xi(B, D)$ is isomorphic to the identity. $\square$

We recall the definitions of KZ-doctrine and algebras for a KZ-doctrine:

Definition 4.4 *A KZ-doctrine* $\mathbf{T}$ *in a 2-category* $\mathcal{C}$ *consist of a 2-functor* $T : \mathcal{C} \to \mathcal{C}$ *together with pseudo-natural transformations* $\eta : 1 \to T$ *and* $\mu : T^2 \to T$ *and a fully faithful adjoint string* $T\eta \overset{k,l}{\dashv} \mu \overset{r,p}{\dashv} \eta T$ *satisfying one compatibility condition (equation (1) on page 26 in [14]).*

Definition 4.5 *Let* $\mathbf{T} = (T, \eta, \mu, k, l, r, p)$ *be a KZ-doctrine on the 2-category* $\mathcal{C}$. *A pseudo-*$\mathbf{T}$*-algebra consists of an object* C *together with an adjunction* $c \overset{c_0, c^0}{\dashv} \eta_C$, *with* c^0 *invertible.*

As a corollary to Proposition 4.3, we obtain the following lemma:

Lemma 4.6 *Given a pseudo-*$\mathbf{T}$*-algebra* $c \overset{c_0, c^0}{\dashv} \eta_C$, *let* $m : \mu_C \to Tc$ *be the 2-cell induced by the co-fully-faithful adjoint string* $Tc \dashv T\eta_C \dashv \mu_C$. *Then*

$$\begin{tikzcd} T^2C \arrow[r, bend left, "\mu_C"] \arrow[r, bend right, "Tc"'] \arrow[r, phantom, "\Downarrow m"] & TC \arrow[r, "c"] & C \end{tikzcd}$$

is an absolute pseudo-coinverter.

Proof. The given diagram is a split pseudo-coinverter with $T\eta_C$, η_{TC} and η_C as arrows going back (see (5) in [14]). $\square$

The following proposition has a similar proof as Proposition 3.2:

Proposition 4.7 *Let* $\mathbf{T}$ *be a KZ-doctrine. The forgetful functor* $U^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ *creates pseudo-coinverters of* $U^{\mathbf{T}}$*-absolute pseudo-coinverters.*

Assume we have a pseudo-adjunction $(F, U, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ whose induced pseudo-monad $\mathbf{T}$ turns out to be a KZ-doctrine.

This in particular means that $UsF \dashv \eta UF$ with counit tF and $UF\eta \dashv UsF$ with unit Us. Denote by $u : UF\eta \to \eta FU$ the arrow induced by the adjoint string $UF\eta \dashv UsF \dashv \eta UF$. With u we can construct a 2-cell $v : \epsilon FU \to FU\epsilon$ as the pasting:

$$\begin{tikzcd}[column sep=large] & FU \arrow[rr, "1"] \arrow[dr, "F\eta U"] \arrow[d, phantom, "\Downarrow \epsilon_{F\eta U}"] & \arrow[d, phantom, "\Downarrow Ft^{-1}"] & FU \\ FUFU \arrow[ur, "\epsilon FU"] \arrow[r, bend left, "FUF\eta U"] \arrow[r, bend right, "F\eta UFU"'] \arrow[r, phantom, "\Downarrow FuU"] \arrow[rr, bend right=60, "1"'] & FUFUFU \arrow[r, "\epsilon FUFU"'] \arrow[d, phantom, "\Downarrow sUFU^{-1}"] & FUFU \arrow[ur, "FU\epsilon"'] & \\ & {} & & \end{tikzcd}$$

We will need a couple of properties of v:

Lemma 4.8 Uv *is equal to the 2-cell induced by the adjunction* $UF\eta \dashv U\epsilon F \dashv \eta UF$.

Lemma 4.9 *Given* A, B *in* A *and* $h : KA \to KB$ *in* $\mathcal{C}^{\mathbf{T}}$, *where* $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ *denotes the comparison functor, we have that the following equality holds:*

$$\begin{tikzcd} FUFUA \arrow[r, "FUFh"] \arrow[d, "FU\epsilon_A"'] \arrow[dr, phantom, "\Swarrow Fr"] & FUFUB \arrow[d, bend right=50, "FU\epsilon_B"'] \arrow[d, bend left=50, "\epsilon_{FUB}"] \arrow[d, phantom, "\overset{\Leftarrow}{v_B}"] \\ FUA \arrow[r, "Fh"'] & FUB \end{tikzcd} \quad = \quad \begin{tikzcd} FUFUA \arrow[r, "FUFh"] \arrow[d, bend right=50, "FU\epsilon_A"'] \arrow[d, bend left=50, "\epsilon_{FUA}"] \arrow[d, phantom, "\overset{\Leftarrow}{v_A}"] \arrow[dr, phantom, "\Swarrow \epsilon_{Fh}^{-1}"] & FUFUB \arrow[d, "\epsilon_{FUB}"] \\ FUA \arrow[r, "Fh"'] & FUB, \end{tikzcd}$$

where r denotes the invertible 2-cell induced by h being a morphism of **T**-*algebras.*

Proof. The key equation for the proof is

$$(t_B \circ h) \cdot (U\epsilon_B \circ \eta_h) = (h \circ t_A) \cdot (r \circ \eta_{UA}),$$

which corresponds to (14) of [14] applied to the morphism h. $\square$

The proof of the following two theorems follow the proofs of Theorems 3.5 and 3.6 respectively, using the previous two lemmas.

Theorem 4.10 *Assume that the pseudo-adjunction $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ induces a KZ-doctrine* **T**. *The comparison functor $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ is locally an equivalence iff for each A the diagram*

$$\begin{tikzcd} FUFU(A) \arrow[r, bend left, "\epsilon FU"] \arrow[r, bend right, "FU\epsilon"'] \arrow[r, phantom, "\Downarrow v_A"] & FU(A) \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

is a pseudo-coinverter.

Theorem 4.11 *Assume that the pseudo-adjunction $(U, F, \eta, \epsilon, s, t) : \mathcal{A} \to \mathcal{C}$ induces a KZ-doctrine* **T**. *The comparison functor $K : \mathcal{A} \to \mathcal{C}^{\mathbf{T}}$ is a biequivalence iff U reflects adjoint equivalences and $\mathcal{A}$ has and U preserves pseudo-coinverters of U-absolute pseudo-coinverters in $\mathcal{A}$.*

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