The Brauer and Brauer-Taylor groups of a symmetric monoidal category

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Résumé. Les groupes de Brauer et de Brauer-Taylor d'une catégorie monoïdale symétrique $\mathbb{C}$ sont définis comme étant les groupes de Picard de catégories monoïdales symétriques convenables construites à partir de $\mathbb{C}$. Si $\mathbb{C}$ est la catégorie des modules sur un anneau commutatif unitaire, on retrouve les groupes usuels. On utilise cette définition pour construire une suite exacte reliant le groupe de Picard et le groupe de Brauer.

Introduction

If $\mathbb{R}$ is a commutative unital ring, the Brauer group $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ is the group of Morita-equivalence classes of Azumaya $\mathbb{R}$-algebras. Several equivalent definitions of Azumaya $\mathbb{R}$-algebra are known. Most of them can be used to define an Azumaya $\mathbb{C}$-monoid, where $\mathbb{C}$ is a symmetric monoidal category satisfying some extra conditions as closure and some kind of completeness. The different Brauer groups which arise in this way are not necessarily isomorphic, but each of them coincides with the Brauer group $\mathcal{B}(\mathbb{R})$ if $\mathbb{C}$ is the category of modules over $\mathbb{R}$.

It is quite surprising that the simplest possible description of $\mathcal{B}(\mathbb{R})$ has been neglected in all the previous categorical approaches to the Brauer group (at least at my knowledge). In fact we can define $\mathcal{B}(\mathbb{R})$ as the Picard group of the monoidal category of unital monoids of $\mathbb{C}$, taking bimodules as arrows.

In section 1 we show that this definition is available in a monoidal category $\mathbb{C}$ satisfying very weak conditions, that is $\mathbb{C}$ must be symmetric and must have stable coequalizers. Then, assuming more on $\mathbb{C}$, we point out that this definition is equivalent to other possible definitions. We close the first section quoting some examples.

In order to illustrate the usefulness of the simple definition of $\mathcal{B}(\mathbb{C})$, in section 2 we obtain, in a quite straightforward way, exact sequences between Brauer groups and Picard groups.

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When $\mathbb{R}$ is a field, $\mathcal{B}(\mathbb{R})$ is isomorphic to the second tale-cohomology group of $\mathbb{R}$. If $\mathbb{R}$ is only a commutative unital ring, $\mathcal{B}(\mathbb{R})$ is the torsion subgroup of the cohomology group. The full cohomology group is then isomorphic to the so-called Brauer-Taylor group of $\mathbb{R}$.

The third and the fourth sections are devoted to a categorical description of the Brauer-Taylor group of a symmetric monoidal category $\mathbb{C}$. It seems to me a nice fact that, even when $\mathbb{C}$ is the category of modules over $\mathbb{R}$, this categorical description is simpler than the classical one.

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1 The Brauer group

Let us fix some notations. In all the work $\mathbb{C} = (\mathbb{C}, \otimes, I, ...)$ is a symmetric monoidal category with stable coequalizers (that is, if

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$$

is a coequalizer, then, for each object Z of $\mathbb{C}$,

$$X \otimes Z \underset{g \otimes 1}{\overset{f \otimes 1}{\rightrightarrows}} Y \otimes Z \xrightarrow{q \otimes 1} Q \otimes Z$$

is again a coequalizer; this condition is clearly satisfied if $\mathbb{C}$ is closed). A monoid $A = (A, m_A \colon A \otimes A \longrightarrow A)$ in $\mathbb{C}$ is always associative. If it is unital, we denote by $e_A \colon I \longrightarrow A$ its unit. A module $M = (M, \mu_M \colon A \otimes M \longrightarrow M)$ in $\mathbb{C}$ is always associative, but not necessarily unital. In all the work $\mathbb{R}$ is a unital commutative ring and $\mathbb{R}$Mod is the category of unital modules over $\mathbb{R}$.

If $\mathcal{M}$ is a unital monoid (in Set), the obvious way to build up a group from $\mathcal{M}$ is to take the set of invertible elements. This is the so called Picard group Pic($\mathcal{M}$) of $\mathcal{M}$ and it is abelian if $\mathcal{M}$ is commutative. If $\underline{\mathcal{M}}$ is a monoidal category, the isomorphism classes of objects form a monoid $(\underline{\mathcal{M}}_0/\simeq)$ (commutative if $\underline{\mathcal{M}}$ is symmetric). The Picard group Pic($\underline{\mathcal{M}}$) of $\underline{\mathcal{M}}$ is, by definition, the Picard group of the monoid $(\underline{\mathcal{M}}_0/\simeq)$.

Now the Brauer group: let $\mathbb{C}$ be a symmetric monoidal category as at the beginning of the section. We can build up a new symmetric monoidal category UMon($\mathbb{C}$) in the following way:

- objects are unital monoids

- arrows are isomorphism classes of unital bimodules
- composition: given two bimodules $M\colon A \longrightarrow B$ and $N\colon B \longrightarrow C$, the composite of the corresponding classes is the class of the tensor product $M \otimes_B N\colon A \longrightarrow C$ (recall that $M \otimes_B N$ is the coequalizer of

$$M \otimes B \otimes N \underset{1 \otimes \mu_N}{\overset{\mu_M \otimes 1}{\rightrightarrows}} M \otimes N \quad)$$

- identities: the identity arrow on a monoid A is the class of A itself
- tensor product: the tensor of $\mathbb{C}$.

The crucial point is that the tensor product of $\mathbb{C}$ gives rise to a functor

$$\mathrm{UMon}(\mathbb{C}) \times \mathrm{UMon}(\mathbb{C}) \longrightarrow \mathrm{UMon}(\mathbb{C})$$

We do not prove this fact here, because it follows from lemma 2.3. Let us only observe that it can also be deduced, using the stability of coequalizers, from the following lemma

Lemma 1.1 *Consider two coequalizers in* $\mathbb{C}$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Z \qquad\qquad X' \underset{g'}{\overset{f'}{\rightrightarrows}} Y' \xrightarrow{q'} Z'$$

if f and g have a common section (that is if there exists $h\colon Y \longrightarrow X$ such that $h \cdot f = 1_Y = h \cdot g$) and if the same holds for f' and g', then

$$X \otimes X' \underset{g \otimes g'}{\overset{f \otimes f'}{\rightrightarrows}} Y \otimes Y' \xrightarrow{q \otimes q'} Z \otimes Z'$$

is a coequalizer. ∎

Let me insist on the fact that the commutativity of a diagram in $\mathrm{UMon}(\mathbb{C})$ is up to isomorphisms in $\mathbb{C}$. For example, two unital monoids A and B are isomorphic in $\mathrm{UMon}(\mathbb{C})$ (we will say *equivalent*) if there exist two unital bimodules $M\colon A \longrightarrow B$ and $N\colon B \longrightarrow A$ such that $M \otimes_B N$ and $N \otimes_A M$ are isomorphic, in $\mathbb{C}$, respectively to A and B.

Now we define the *first Brauer group* $\mathcal{B}_1(\mathbb{C})$ of $\mathbb{C}$ as the Picard group of $\mathrm{UMon}(\mathbb{C})$ (the reason for the "first" will be clear at the end of the third section). In other words, a unital monoid A is *Azumaya* if there exists a unital monoid A^* such that $A \otimes A^*$ is equivalent to I. The first Brauer group is then the group of equivalence classes of Azumaya $\mathbb{C}$-monoids. It is known that, if $\mathbb{C}$ is $\mathbb{R}\mathrm{Mod}$, then $\mathcal{B}_1(\mathbb{C})$ is the usual Brauer group of $\mathbb{R}$ (cf. [35]).

Let us now give a glance at the classical definition of Azumaya $\mathbb{R}$-algebra (cf. [2], [19], [25], [28], [35]). An $\mathbb{R}$-algebra A is Azumaya if it satisfies one of the following equivalent conditions:

- there exists an $\mathbb{R}$-algebra A^* such that $A \otimes A^*$ is Morita-equivalent to $\mathbb{R}$

- $A \otimes A^o$ is Morita-equivalent to $\mathbb{R}$ (where A^o is the opposite algebra of A)

- A is a faithfully projective $\mathbb{R}$-module and the canonical morphism

$$A \otimes A \longrightarrow \mathrm{Lin}_{\mathbb{R}}(A, A)$$

  is an isomorphism (where $\mathrm{Lin}_{\mathbb{R}}(A, A)$ is the $\mathbb{R}$-module of $\mathbb{R}$-linear transformations from A to A)

- A is central and separable.

With this situation in mind, let us look for equivalent definitions of Azumaya $\mathbb{C}$-monoid. We need some notations: the bimodule $\eta_A \colon I \longrightarrow A \otimes A^o$ is A with its natural structure of right $A \otimes A^o$-module, the bimodule $\epsilon_A \colon A^o \otimes A \longrightarrow I$ is A with its natural structure of left $A \otimes A^o$-module.

Proposition 1.2 *Let A be a unital $\mathbb{C}$-monoid; the following conditions are equivalent:*

i - A is Azumaya

ii - $\epsilon_A \colon A^o \otimes A \longrightarrow I$ is an isomorphism in UMon($\mathbb{C}$)

iii - $\eta_A \colon I \longrightarrow A \otimes A^o$ is an isomorphism in UMon($\mathbb{C}$)

Proof: Recall that, in a monoidal category $\underline{\mathcal{M}}$, an object X is left adjoint to an object X^*, $X \dashv X^*$, if there exist two arrows $\eta \colon I \longrightarrow X \otimes X^*$ and $\epsilon \colon X^* \otimes X \longrightarrow I$ such that the following diagrams are commutative

$$\begin{tikzcd} X \simeq I \otimes X \arrow[r, "\eta \otimes 1"] \arrow[dr, "1"'] & X \otimes X^* \otimes X \arrow[d, "1 \otimes \epsilon"] \\ & X \simeq X \otimes I \end{tikzcd} \qquad \begin{tikzcd} X^* \simeq X^* \otimes I \arrow[r, "1 \otimes \eta"] \arrow[dr, "1"'] & X^* \otimes X \otimes X^* \arrow[d, "\epsilon \otimes 1"] \\ & X^* \simeq I \otimes X^* \end{tikzcd}$$

(cf. [18]). Moreover, if $X \otimes Y \simeq I$ and $Y \otimes X \simeq I$, then $X \dashv Y$ and unit and counit are invertible (this is a particular case of a 2-categorical argument: given an equivalence, it is always possible to build up an adjoint equivalence (cf. [17]). In UMon($\mathbb{C}$) we have that $A \dashv A^o$ with unit given by $\eta_A \colon I \longrightarrow A \otimes A^o$ and counit given by $\epsilon_A \colon A^o \otimes A \longrightarrow I$. If A is Azumaya, then, by uniqueness of the adjoint, A^* is equivalent to A^o and η_A and ϵ_A are isomorphisms. The converse implications are obvious. ∎

To say more on the notion of Azumaya $\mathbb{C}$-monoid, we need some facts which are part of Morita theory. The proof can be found in any of the sources quoted in the first remark at the end of the section.

Proposition 1.3 *Let $P: A \longrightarrow B$ be a bimodule; the following conditions are equivalent:*

- *$P: A \longrightarrow B$ is an isomorphism in UMon($\mathbb{C}$)*
- *the functor between module categories $P \otimes_B -: B - mod \longrightarrow A - mod$ induced by P is an equivalence of categories*
- *the right adjoint $P \supset_A -: A - mod \longrightarrow B - mod$ of $P \otimes_B -$ is an equivalence of categories*
- *P is faithfully projective as A-module and B is canonically isomorphic (as monoid of $\mathbb{C}$) to $P \supset_A P$*

$\blacksquare$

(Recall that the right adjoint $P \supset_A -$ certainly exists if $\mathbb{C}$ is closed and has equalizers. Faithfully projective means that the internal compositions

$$P \otimes_{P \supset_A P} (P \supset_A A) \longrightarrow A \quad \text{and} \quad (P \supset_A A) \otimes_A P \longrightarrow (P \supset_A P)$$

are isomorphisms.)

We can apply the previous propositions to the bimodules $\eta_A: I \longrightarrow A \otimes A^o$ and $\epsilon_A: A^o \otimes A \longrightarrow I$. We obtain respectively:

iv - an unital $\mathbb{C}$-monoid A is Azumaya if and only if it is faithfully projective in $\mathbb{C}$ and the canonical arrow $A \otimes A^o \longrightarrow A \supset A$ is an isomorphism

v - an unital $\mathbb{C}$-monoid A is Azumaya if and only if it is faithfully projective in $A^o \otimes A$-mod and the canonical arrow $I \longrightarrow A \supset_{A^o \otimes A} A$ is an isomorphism.

This last characterization needs a comment: when $\mathbb{C}$ is $\mathbb{R}$Mod, it is equivalent to say that A is central and separable. Central because $A \supset_{A^o \otimes A} A$ is the center of A. As far as separability is concerned, recall that A is separable if the multiplication $A \otimes A \longrightarrow A$ admits a section A-linear on the left and on the right. But this is equivalent to say that A is projective in $A^o \otimes A$-mod. Clearly, A is finitely generated as $A^o \otimes A$-module. Finally, Auslander-Goldman theorem asserts that if A is central and separable, then it is a generator for the category $A^o \otimes A$-mod, so that it is faithfully projective in this category (cf. [2]).

Remarks and examples

I - Several categorical approaches to Morita theory are available in literature. Among them, the items [3], [13], [20], [26], [31] in the bibliography. Each of them contains (some variant of) proposition 1.3. This proposition is certainly true, for general enriched category theory reasons (cf. [20]), if $\mathbb{C}$ is a complete and cocomplete symmetric monoidal closed category. But the assumption on $\mathbb{C}$ to develop Morita theory can be weakened. For example, in [26], closure is avoided (but to the detriment of the internal character of the theory) and in [31] symmetry is not required.

II - We have just discussed the equivalence between five possible definitions of Azumaya $\mathbb{C}$-monoid. All of them, with the exception of the first one, have been individually considered in other works in which a categorical approach to Brauer group can be found. They are items [11], [12], [15], [26] in the bibliography. Even the definition of separability via the existence of a Casimir-element has been considered in [11] and [26].

III - In the works quoted in the previous remark, several examples of Brauer groups are discussed from a categorical point of view. This means that a group built up " la Brauer" from a certain gadget is nothing that the Brauer group of a suitable monoidal category. We recall here

- the Brauer group of a commutative ringed space, introduced in [1] and considered in [12] and [15];
- the Brauer-Wall group, introduced in [33] (and generalized in [21]) and considered in [11] and [12];
- the relative Brauer group, introduced in [30] and considered in [23];
- the Brauer group of module algebras for a cocommutative Hopf-algebra, introduced in [22] and considered in [11] and [26].

IV - a quite different example arises if we consider, as base category, the category $\mathcal{SL}$ of sup-lattices instead of the category of abelian groups. A monoid in $\mathcal{SL}$ is a quantale so that it is possible to define the Brauer group of a commutative unital quantale Q as the Brauer group of the monoidal category Q-mod of modules over Q. The corresponding Morita-theory has been studied in [7] and the key notion of faithfully projective Q-module turns out to be the following:

a Q-module P is faithfully projective if and only if there exist two sets X and Y such that P is a retract of Q^X and Q is a retract of P^Y (where Q^X and P^Y are the X-indexed and the Y-indexed powers of Q and P).

By the way, the classical Brauer group $\mathcal{B}(\mathbb{R})$ is a small group. This is because an Azumaya $\mathbb{R}$-algebra is, in particular, a faithfully projective $\mathbb{R}$-module and the category of faithfully projective $\mathbb{R}$-modules is small. This is no longer true for Q-modules. It is an open problem to find conditions on $\mathbb{C}$ such that the Brauer group of $\mathbb{C}$ is small.

V - Another example is provided by a commutative algebraic theory $\mathbb{T}$. The models of $\mathbb{T}$ constitute a complete and cocomplete symmetric monoidal closed category (cf. [6] and [34]). The corresponding Morita-theory has been studied in [8] and [10]. This example requires some more efforts and will be discussed in detail in a separated paper.

2 Exact sequences

In this section we build up an exact sequence between Picard groups and Brauer groups.

Let us consider two symmetric monoidal categories with stable coequalizers $\mathbb{C} = (\mathbb{C}, \otimes, I, \dots)$ and $\mathbb{D} = (\mathbb{D}, \otimes, J, \dots)$. Let $F : \mathbb{C} \longrightarrow \mathbb{D}$ be a monoidal functor such that

i - F preserves coequalizers

ii - F is strict on invertible bimodules and on Azumaya $\mathbb{C}-$monoids (this means that the morphism $FX \otimes FY \longrightarrow F(X \otimes Y)$ is an isomorphism when X and Y are as above)

iii - the morphism $J \longrightarrow F(I)$ is an isomorphism

For such a functor F, it is straightforward to prove the following lemma:

Lemma 2.1

- *if $M : A \longrightarrow B$ is an invertible bimodule between two Azumaya $\mathbb{C}$-monoids, then $FM : FA \longrightarrow FB$ is an invertible bimodule between Azumaya $\mathbb{D}$-monoids*
- *if $A \xrightarrow{M} B \xrightarrow{N} C$ are invertible bimodules between Azumaya $\mathbb{C}$-monoids, then $F(M \otimes_B N)$ is isomorphic to $FM \otimes_{FB} FN$.*

$\blacksquare$

The previous lemma allows us to build up a new symmetric monoidal category $\mathcal{F}$ in the following way:

- objects are invertible bimodules of the form $X : FA \longrightarrow FB$, where A and B are two specified Azumaya $\mathbb{C}$-monoids
- an arrow between two objects $(X : FA \longrightarrow FB)$ and $(Y : FC \longrightarrow FD)$ is a pair of invertible bimodules $M : A \longrightarrow C$ and $N : B \longrightarrow D$ such that the following diagram is commutative

$$\begin{tikzcd} FA \arrow[r, "X"] \arrow[d, "FM"'] & FB \arrow[d, "FN"] \\ FC \arrow[r, "Y"'] & FD \end{tikzcd}$$

- composition and identities are the obvious ones
- the tensor product of $(X\colon FA \longrightarrow FB)$ and $(Y\colon FC \longrightarrow FD)$ is given by $(X \otimes Y\colon F(A \otimes C) \longrightarrow F(B \otimes D))$
- the unit of the tensor is given by $(J\colon FI \longrightarrow FI)$.

Again a notation: if $\underline{\mathcal{M}}$ is a monoidal category, $I(\underline{\mathcal{M}})$ is the monoidal subcategory of invertible objects and isomorphisms. Clearly, $\mathrm{Pic}(\underline{\mathcal{M}})$ and $\mathrm{Pic}(I(\underline{\mathcal{M}}))$ are equal.

Starting from the functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$, we can define the following four functors:

(1) $F\colon I(\mathbb{C}) \longrightarrow I(\mathbb{D})$ which is simply the restriction of $F\colon \mathbb{C} \longrightarrow \mathbb{D}$

(2) $F_1\colon I(\mathbb{D}) \longrightarrow \mathcal{F}$ defined by

$$h\colon X \simeq Y \quad \rightsquigarrow \quad \begin{array}{ccc} FI & \xrightarrow{X} & FI \\ {\scriptstyle FI}\downarrow & & \downarrow{\scriptstyle FI} \\ FI & \xrightarrow[Y]{} & FI \end{array}$$

(observe that an object of the form $(X\colon FI \longrightarrow FI)$ is invertible with respect to the tensor product of $\mathcal{F}$)

(3) $F_2\colon \mathcal{F} \longrightarrow I(\mathrm{UMon}(\mathbb{C}))$ defined by

$$\begin{array}{ccc} FA & \xrightarrow{X} & FB \\ {\scriptstyle FM}\downarrow & & \downarrow{\scriptstyle FN} \\ FC & \longrightarrow & FD \end{array} \quad \rightsquigarrow \quad \begin{array}{c} A \otimes B^o \\ \downarrow{\scriptstyle M \otimes (N^o)^{-1}} \\ C \otimes D^o \end{array}$$

(4) $\overline{F}\colon I(\mathrm{UMon}(\mathbb{C})) \longrightarrow I(\mathrm{UMon}(\mathbb{D}))$ defined by

$$(X\colon A \longrightarrow B) \rightsquigarrow (FX\colon FA \longrightarrow FB)$$

(this definition makes sense by lemma 2.1).

All these functors are strict monoidal functors, so that, passing to the Picard groups, we obtain four group homomorphisms (square brackets are isomorphism classes)

(1) $f\colon \mathrm{Pic}(\mathbb{C}) \longrightarrow \mathrm{Pic}(\mathbb{D}) \quad [X] \rightsquigarrow [FX]$

(2) $f_1\colon \mathrm{Pic}(\mathbb{D}) \longrightarrow \mathrm{Pic}(\mathcal{F})$ $\quad [Y] \rightsquigarrow [Y\colon FI \longrightarrow FI]$

(3) $f_2\colon \mathrm{Pic}(\mathcal{F}) \longrightarrow \mathcal{B}_1(\mathbb{C})$ $\quad [X\colon FA \longrightarrow FB] \rightsquigarrow [A \otimes B^o]$

(4) $\overline{f}\colon \mathcal{B}_1(\mathbb{C}) \longrightarrow \mathcal{B}_1(\mathbb{D})$ $\quad [A] \rightsquigarrow [FA]$

Proposition 2.2 *The sequence*

$$Pic(\mathbb{C}) \xrightarrow{f} Pic(\mathbb{D}) \xrightarrow{f_1} Pic(\mathcal{F}) \xrightarrow{f_2} \mathcal{B}_1(\mathbb{C}) \xrightarrow{\overline{f}} \mathcal{B}_1(\mathbb{D})$$

is a complex; moreover, it is exact in $Pic(\mathbb{D})$ *and in* $\mathcal{B}_1(\mathbb{C})$.

Proof:

- $f \cdot f_1 = 0$ because if X is in $I(\mathbb{C})$, then $(FX\colon FI \longrightarrow FI)$ is isomorphic in $\mathcal{F}$ to $(J\colon FI \longrightarrow FI)$; the isomorphism is given by

$$\begin{tikzcd} FI \arrow[r, "FX"] \arrow[d, "FI"'] & FI \arrow[d, "FX^{-1}"] \\ FI \arrow[r, "J"'] & FI \end{tikzcd}$$

- $f_1 \cdot f_2 = 0$ because $I \otimes I^o$ is isomorphic (in $\mathbb{C}$ and then in $\mathrm{UMon}(\mathbb{C})$) to I

- $f_2 \cdot \overline{f} = 0$ because, if $X\colon FA \longrightarrow FB$ is an invertible bimodule, then $FA \otimes FB^o$ is equivalent to $FB \otimes FB^o$ and then to J (recall that B is Azumaya)

- exactness in $\mathrm{Pic}(\mathbb{D})$: let Y be an object of $I(\mathbb{D})$ and suppose that $[Y] \in \mathrm{Ker} f_1$. This means that $(Y\colon FI \longrightarrow FI)$ and $(J\colon FI \longrightarrow FI)$ are isomorphic in $\mathcal{F}$. Let

$$\begin{tikzcd} FI \arrow[r, "Y"] \arrow[d, "FM"'] & FI \arrow[d, "FN"] \\ FI \arrow[r, "J"'] & FI \end{tikzcd}$$

  be the isormorphism in $\mathcal{F}$. One has that Y is isomorphic in $\mathbb{D}$ to $F(M \otimes N^{-1})$ so that $[Y] = [F(M \otimes N^{-1})] \in \mathrm{Im} f$

- exactness in $\mathcal{B}_1(\mathbb{C})$: let A be an object of $I(\mathrm{UMon}(\mathbb{C}))$ and suppose that $[A] \in \mathrm{Ker}\overline{f}$. This means that there exists an invertible bimodule $X\colon FA \longrightarrow FI$. One has $f_2[X\colon FA \longrightarrow FI] = [A \otimes I^o] = [A]$.

■

Let me now discuss the exactness in $\mathrm{Pic}(\mathcal{F})$. Let $[X\colon FA \longrightarrow FB]$ be in $\mathrm{Ker}f_2$. This means that $A \otimes B^o$ is equivalent to I. Then A is equivalent to B. Let $Y\colon B \longrightarrow A$ be the isomorphism. We have that

$$\begin{tikzcd}
FA \arrow[r, "X"] \arrow[d, "FA"'] & FB \arrow[d, "FY"] \\
FA \arrow[r, "X\otimes_{FB}FY"'] & FA
\end{tikzcd}$$

is an isomorphism in $\mathcal{F}$. Clearly, if $(X\colon FA \longrightarrow FB)$ is invertible with respect to the tensor product of $\mathcal{F}$, also $(X\otimes_{FB}FY\colon FA \longrightarrow FA)$ is invertible. All this means that, as representative of an element in $\mathrm{Ker}f_2$, we can always choose an object of $\mathcal{F}$ of the form $(X\colon FA \longrightarrow FA)$. We wonder if $[X\colon FA \longrightarrow FA]$ is in $\mathrm{Im}f_1$. This is not true in general. What is true is that $[X\colon FA \longrightarrow FA] + [FA^o\colon FA^o \longrightarrow FA^o]$ is in $\mathrm{Im}f_1$. (Observe that $[FA^o\colon FA^o \longrightarrow FA^o]$ is in $\mathrm{Pic}(\mathcal{F})$ because $(FA^o\colon FA^o \longrightarrow FA^o)$ is invertible with respect to the tensor product of $\mathcal{F}$, the inverse is given by $(FA\colon FA \longrightarrow FA)$.)

To prove that $[X \otimes FA^o\colon F(A \otimes A^o) \longrightarrow F(A \otimes A^o)]$ is in $\mathrm{Im}f_1$, we can observe that

$$\begin{tikzcd}
F(A\otimes A^o) \arrow[r, "X \otimes FA^o"] \arrow[d, "F(\eta_A^{-1})"'] & F(A\otimes A^o) \arrow[dr, "F(\eta_A^{-1})"] & \\
FI \arrow[rr, "F(\eta_A)\otimes_{F(A\otimes A^o)}(X \otimes FA^o)\otimes_{F(A\otimes A^o)}F(\eta_A^{-1})"'] & & FI
\end{tikzcd}$$

is an isomorphism in $\mathcal{F}$, so that

$$[X \otimes FA^o\colon F(A \otimes A^o) \longrightarrow F(A \otimes A^o)]$$

is equal to

$$f_1[F(\eta_A)\otimes_{F(A\otimes A^o)}(X \otimes FA^o)\otimes_{F(A\otimes A^o)}F(\eta_A^{-1})]$$

It only remains to observe that the elements of the form $[FA\colon FA \longrightarrow FA]$ constitute a subgroup of $\mathrm{Pic}(\mathcal{F})$, say $\mathcal{N}$, and that they are contained in the kernel of f_2 (exactly because A is Azumaya).

Let us consider the canonical projection $\pi\colon \mathrm{Pic}(\mathcal{F}) \longrightarrow \mathrm{Pic}(\mathcal{F})/\mathcal{N}$ and let us call $f_2'\colon \mathrm{Pic}(\mathcal{F})/\mathcal{N} \longrightarrow \mathcal{B}_1(\mathbb{C})$ the unique factorization of f_2 through π. We are ready to prove the following proposition.

Proposition 2.3 *The sequence*

$$Pic(\mathbb{C}) \xrightarrow{f} Pic(\mathbb{D}) \xrightarrow{f_1 \cdot \pi} Pic(\mathcal{F})/\mathcal{N} \xrightarrow{f_2'} \mathcal{B}_1(\mathbb{C}) \xrightarrow{\overline{f}} \mathcal{B}_1(\mathbb{D})$$

is exact.

Proof: The fact that the sequence is a complex as well as the exactness in $\mathcal{B}_1(\mathbb{C})$ come from proposition 2.2 because π is an epimorphism.

- Exactness in Pic($\mathbb{D}$) : the argument is the same as in proposition 2.2, but replacing $J\colon FI \longrightarrow FI$ by $FA\colon FA \longrightarrow FA$ and $M \otimes N^{-1}$ by $M \otimes_A N^{-1}$.
- Exactness in Pic($\mathcal{F}$)/$\mathcal{N}$: it follows from the previous discussion since $[X\colon FA \longrightarrow FA]$ and $[X\colon FA \longrightarrow FA] + [FA^o\colon FA^o \longrightarrow FA^o]$ are equal in Pic($\mathcal{F}$)/$\mathcal{N}$.

■

We have just constructed, in a very elementary way, an exact sequence between Picard groups and Brauer groups. Another way, classically used to build up such a sequence, is based on standard K-theoretical arguments (cf. [5] and [16]). This is possible also in our categorical framework, via the following proposition. For the definitions of the Grothendieck group K_0 and of the Whitehead group K_1, the reader can consult [5].

Proposition 2.4

i - the Grothendieck group $K_0(I(UMon(\mathbb{C})))$ *is isomorphic to* $\mathcal{B}_1(\mathbb{C})$

ii - the Whitehead group $K_1(I(UMon(\mathbb{C})))$ *is isomorphic to* $Pic(\mathbb{C})$

Proof: i - obvious because I(UMon($\mathbb{C}$)) is a monoidal groupoid in which each object is invertible;
ii - consider the category $\Omega(I(\text{UMon}(\mathbb{C})))$ defined as follows

- objects are endomorphisms $X\colon A \longrightarrow A$ in I(UMon($\mathbb{C}$))
- an arrow is a commutative diagram in I(UMon($\mathbb{C}$))

$$\begin{tikzcd} A \arrow[r, "X"] \arrow[d, "M"'] & A \arrow[d, "M"] \\ B \arrow[r, "Y"'] & B \end{tikzcd}$$

- composition and identities are the obvious ones.

In this category we can define a tensor product of $X\colon A \longrightarrow A$ and $Y\colon B \longrightarrow B$ by $X\otimes Y\colon A\otimes B \longrightarrow A\otimes B$ and a composition of $X\colon A \longrightarrow A$ and $Y\colon A \longrightarrow A$ by $X\otimes_A Y\colon A \longrightarrow A$.

To obtain an isomorphism $K_1(I(\mathrm{UMon}(\mathbb{C}))) \longrightarrow \mathrm{Pic}(\mathbb{C})$, we need a map on objects

$$\gamma\colon \Omega(I(\mathrm{UMon}(\mathbb{C}))) \longrightarrow I(\mathbb{C})$$

such that

1) γ sends isomorphic objects into isomorphic objects
2) γ preserves the tensor product
3) γ sends composition into tensor product
4) γ is essentially surjective
5) if $\gamma(X\colon A \longrightarrow A)$ is isomorphic to I in $I(\mathbb{C})$, then $X\colon A \longrightarrow A$ is isomorphic to $A\colon A \longrightarrow A$ in $\Omega(I(\mathrm{UMon}(\mathbb{C})))$

Conditions 1), 2) and 3) imply the existence of a group homomorphism

$$\overline{\gamma}\colon K_1(I(\mathrm{UMon}(\mathbb{C}))) \longrightarrow \mathrm{Pic}(\mathbb{C})$$

which is surjective by condition 4). Condition 5), together with lemma 2.5, implies the injectivity of $\overline{\gamma}$. We define $\gamma\colon \Omega(I(\mathrm{UMon}(\mathbb{C}))) \longrightarrow I(\mathbb{C})$ by

$$(X\colon A \to A) \quad \rightsquigarrow \quad I \xrightarrow{\ \eta_A\ } A\otimes A^o \xrightarrow{X\otimes A^o} A\otimes A^o \xrightarrow{\ \eta_A^{-1}\ } I$$

Let us now verify the five conditions.

1) Let

$$\begin{tikzcd} A \arrow[r, "X"] \arrow[d, "M"'] & A \arrow[d, "M"] \\ B \arrow[r, "Y"'] & B \end{tikzcd}$$

be an isomorphism in $\Omega(I(\mathrm{UMon}(\mathbb{C})))$. Up to isomorphism in $\mathbb{C}$, $\gamma(Y)$ is

$$\begin{tikzcd} I \arrow[d, "\eta_B"'] & & & I \\ B\otimes B^o \arrow[r, "M^{-1}\otimes B^o"] & A\otimes B^o \arrow[r, "X\otimes B^o"] & A\otimes B^o \arrow[r, "M\otimes B^o"] & B\otimes B^o \arrow[u, "\eta_B^{-1}"'] \end{tikzcd}$$

Now observe that, since M is invertible,

$$I \xrightarrow{\eta_B} B \otimes B^o \xrightarrow{M^{-1} \otimes M^o} A \otimes A^o$$

is a unit for the adjunction $A \dashv A^o$ in UMon($\mathbb{C}$). This implies that there exists a unique invertible bimodule $\alpha\colon A^o \longrightarrow A^o$ such that the following diagram commutes

$$\begin{tikzcd}
I \arrow[r, "\eta_B"] \arrow[d, "\eta_A"'] & B \otimes B^o \arrow[d, "M^{-1} \otimes M^o"] \\
A \otimes A^o \arrow[r, "A \otimes \alpha"'] & A \otimes A^o
\end{tikzcd}$$

Finally, the following diagram is commutative in each part, so that $\gamma(X)$ is isomorphic in $\mathbb{C}$ to $\gamma(Y)$.

$$\begin{tikzcd}
B \otimes B^o \arrow[r, "M^{-1} \otimes B^o"] & A \otimes B^o \arrow[r, "X \otimes B^o"] \arrow[d, "A \otimes M^o"'] & A \otimes B^o \arrow[r, "M \otimes B^o"] & B \otimes B^o \arrow[d, "\eta_B^{-1}"] \\
I \arrow[u, "\eta_B"] \arrow[dr, "\eta_A"'] & A \otimes A^o \arrow[d, "A \otimes \alpha^{-1}"'] & A \otimes A^o \arrow[u, "A \otimes (M^o)^{-1}"'] & I \\
& A \otimes A^o \arrow[r, "X \otimes A^o"'] & A \otimes A^o \arrow[u, "A \otimes \alpha"'] \arrow[ur, "\eta_A^{-1}"'] &
\end{tikzcd}$$

2) straightforward, using the fact that, given two unital monoids A and B, $\eta_{A \otimes B}$ is (isomorphic to) $\eta_A \otimes \eta_B$

3) let $A \xrightarrow{X} A \xrightarrow{Y} A$ be two composable objects of $\Omega(I(\mathrm{UMon}(\mathbb{C})))$. The following diagram is commutative

$$\begin{tikzcd}
A \otimes A^o \arrow[rr, "(X \otimes_A Y) \otimes A^o"] \arrow[d, "X \otimes A^o"'] & & A \otimes A^o \\
A \otimes A^o \arrow[r, "\eta_A^{-1}"'] & I \arrow[r, "\eta_A"'] & A \otimes A^o \arrow[u, "Y \otimes A^o"']
\end{tikzcd}$$

so that $\gamma(X \otimes_A Y)$ is isomorphic to $\gamma(X) \otimes_I \gamma(Y)$, that is to $\gamma(X) \otimes \gamma(Y)$

4) let X be an invertible object of $\mathbb{C}$ and let A be any Azumaya $\mathbb{C}$-monoid. Then γ of $X \otimes A \colon A \simeq I \otimes A \longrightarrow I \otimes A \simeq A$ is isomorphic to X

5) consider an object $X \colon A \longrightarrow A$ in $\Omega(I(\mathrm{UMon}(\mathbb{C})))$ such that the following diagram commutes

$$\begin{tikzcd}
A \otimes A^o \arrow[r, "X \otimes A^o"] & A \otimes A^o \arrow[d, "\eta_A^{-1}"] \\
I \arrow[u, "\eta_A"] \arrow[r, "I"'] & I
\end{tikzcd}$$

that is such that $\gamma(X)$ is isomorphic to I. An isomorphism

$$\begin{tikzcd}
A \arrow[r, "X"] \arrow[d, "M"'] & A \arrow[d, "M"] \\
A \arrow[r, "A"'] & A
\end{tikzcd}$$

in $\Omega(I(\mathrm{UMon}(\mathbb{C})))$ can be obtained taking as M the composition

$$A \simeq I \otimes A \xrightarrow{\eta_A \otimes 1} A \otimes A^o \otimes A \xrightarrow{1 \otimes \epsilon_A} A \otimes I \simeq A$$

■

Lemma 2.5 *Let $\underline{\mathcal{M}}$ be a monoidal groupoid with unital object I and let A be any object of $\underline{\mathcal{M}}$; in $K_0(\Omega(\underline{\mathcal{M}}))$ one has that the class of $\mathrm{id}_A \colon A \longrightarrow A$ is equal to the class of $\mathrm{id}_I \colon I \longrightarrow I$, which is the zero of $K_0(\Omega(\underline{\mathcal{M}}))$.*

Proof: Easy from 1.2 in [4]. ■

To obtain an exact sequence between Picard groups and Brauer groups, it only remains to observe that, if $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ is a functor as at the beginning of the section, then the induced functor

$$\overline{F}\colon I(\mathrm{UMon}(\mathbb{C})) \longrightarrow I(\mathrm{UMon}(\mathbb{D}))$$

is cofinal (cf.[4]). In fact, if A is an object in $I(\mathrm{UMon}(\mathbb{D}))$, then there exists an object B in $I(\mathrm{UMon}(\mathbb{D}))$ and an object C in $I(\mathrm{UMon}(\mathbb{C}))$ such that $\overline{F}(C)$ is equivalent to $A \otimes B$. It suffices to take A^o as B and I as C.

Remarks

I - The existence of an exact sequence between Picard groups and Brauer groups has been established also in [15], using K-theoretical arguments and working on categories which satisfy an analogue of proposition 1.3. If this proposition holds, then two Azumaya $\mathbb{C}$-monoids A and B are isomorphic in $\mathrm{UMon}(\mathbb{C})$ if and only if there exist two faithfully projective objects P and Q such that $A \otimes (P \supset P)$ and $B \otimes (Q \supset Q)$ are isomorphic as monoids in $\mathbb{C}$. In [15] this fact is extensively used. This makes explicit calculations quite different from those presented in this section.

II - As far as the assumptions on $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ are concerned, we refer to [15], where these assumptions are discussed in some relevant examples. Let us only observe here that an easy example is provided by the left adjoint of the change of base functor induced by a morphism of unital commutative rings. This left adjoint is, in fact, a strict monoidal functor.

III - A careful analysis of the proof of proposition 2.4, as well as the simple proof of proposition 1.2, suggests that a further generalization of the theory can be obtained taking as primitive the bicategory $\mathcal{B} = \mathrm{UMon}(\mathbb{C})$ and defining $\mathbb{C}$ as $\mathcal{B}(I, I)$. Such a theory could be so general to contain, as an example, the *categorical Brauer group*, taking as $\mathcal{B}$ the bicategory of small $\mathbb{C}$-categories and $\mathbb{C}$-distributors. The categorical Brauer group has, for elements, classes of Azumaya $\mathbb{C}$-categories and has been studied, for $\mathbb{C} = \mathbb{R}\mathrm{Mod}$, in [24]. For a more general $\mathbb{C}$, some elementary observations on the categorical Brauer group are contained in [32], but much more remains to do in this direction.

3 The Brauer-Taylor group

The Picard group of $\mathbb{R}$ is isomorphic to the first tale-cohomology group of $\mathbb{R}$. On the contrary, the Brauer group is only the torsion subgroup of the second tale-cohomology group (cf. [14]). A purely algebraic description of the whole second cohomology group is provided by the Brauer-Taylor group (cf. [29], [27] and [9]). In this and in the next section we look for a categorical description of the Brauer-Taylor group. We will work as at the beginning of the first section. To start, we need some definitions: we want monoids and modules not necessarily unital, but such that some points of Morita theory hold (cf. lemma 4.2).

Definition 3.1

a) a monoid A is regular if the canonical map $A \otimes_A A \longrightarrow A$ is an isomorphism

b) a monoid A is splitting if there exists $\varphi_A : A \longrightarrow A \otimes A$ such that the following diagrams are commutative

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "\varphi_A"'] & A \\ A \otimes A \arrow[ur, "m_A"'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "m_A"] \arrow[d, "\varphi_A \otimes 1"'] & A \arrow[d, "\varphi_A"] \\ A \otimes A \otimes A \arrow[r, "1 \otimes m_A"'] & A \otimes A \end{tikzcd}$$

c) a module M is regular if the canonical map $A \otimes_A M \longrightarrow M$ is an isomorphism

d) if A is a splitting monoid (with section φ_A), a module M is splitting (with respect to φ_A) if there exists $\varphi_M : M \longrightarrow A \otimes M$ such that the following diagrams are commutative

$$\begin{tikzcd} M \arrow[r, "1"] \arrow[d, "\varphi_M"'] & M \\ A \otimes M \arrow[ur, "\mu_M"'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes M \arrow[r, "\mu_M"] \arrow[d, "\varphi_A \otimes 1"'] & M \arrow[d, "\varphi_M"] \\ A \otimes A \otimes M \arrow[r, "1 \otimes \mu_M"'] & A \otimes M \end{tikzcd}$$

In the next proposition we compare the notions of unital, splitting and regular modules.

Proposition 3.2

1) if A is unital and M is unital, then M is splitting (with φ_M given by $e_A \otimes 1 : M \simeq I \otimes M \longrightarrow A \otimes M$; in particular, if A is unital, then A is splitting

2) if A is splitting and M is splitting, then M is regular; in particular, if A is splitting, then A is regular

3) if A is splitting and M is regular, then M is splitting

4) if A is unital and M is splitting, then M is unital.

Proof: 1): obvious. 2): the associativity of M, the first condition of splitness on A and the splitness of M say exactly that the following diagram is a splitting fork (with sections given by $\varphi_M \colon M \longrightarrow A \otimes M$ and $\varphi_A \otimes 1 \colon A \otimes M \longrightarrow A \otimes A \otimes M$) and then μ_M is an absolute coequalizer

$$A \otimes A \otimes M \underset{1 \otimes \mu_M}{\overset{m_A \otimes 1}{\rightrightarrows}} A \otimes M \xrightarrow{\mu_M} M$$

3): if $t \colon A \otimes_A M \longrightarrow M$ is the canonical isomorphism, it suffices to define the section $\varphi_M \colon M \longrightarrow A \otimes M$ as

$$M \xrightarrow{t^{-1}} A \otimes_A M \xrightarrow{\varphi_A \otimes_A 1} A \otimes A \otimes_A M \xrightarrow{1 \otimes t} A \otimes M$$

(which is possible because the second condition of splitness on A says that φ_A is A-linear on the right). 4): since A is unital, $\varphi_A \colon A \longrightarrow A \otimes A$ is given by $e_A \otimes 1 \colon A \simeq I \otimes A \longrightarrow A \otimes A$. Then, using the fact that μ_M is an epimorphism and the second condition of splitness on M, one can prove that $\varphi_M \colon M \longrightarrow A \otimes M$ is given by $e_A \otimes 1 \colon M \simeq I \otimes M \longrightarrow A \otimes M$. Now the first condition of splitness on M means that M is unital. ■

In definition 3.1, modules are left modules and the second condition on φ_A means that it is right linear. Let us say that in this case A is right splitting. One can consider left splitting monoids and the analogue of proposition 3.2 holds for right modules. We call a monoid A *bisplitting* if its multiplication $m_A \colon A \otimes A \longrightarrow A$ admits a right linear section and a left linear section (not necessarily equal). The fundamental example of bisplitting monoids are the *elementary algebras* which will be studied in the next section.

Now the Brauer-Taylor group. Given the symmetric monoidal category $\mathbb{C}$, we can build up a new symmetric monoidal category $\mathrm{SMon}(\mathbb{C})$ in the following way:

- objects are bisplitting monoids
- arrows are isomorphism classes of regular bimudules
- composition: given two bimodules $M \colon A \longrightarrow B$ and $N \colon B \longrightarrow C$, the composite of the corresponding classes is the class of $M \otimes_B N \colon A \longrightarrow C$
- identities: the identity arrow on a monoid A is the class of A itself

- tensor product: the tensor of $\mathbb{C}$.

Once again, the crucial point is that the tensor product of $\mathbb{C}$ induces a tensor product in SMon($\mathbb{C}$). We give the proof of this fact in the following lemma.

Lemma 3.3 *The functor* $\otimes\colon \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ *induces a functor*

$$\mathit{SMon}(\mathbb{C}) \times \mathit{SMon}(\mathbb{C}) \longrightarrow \mathit{SMon}(\mathbb{C})$$

Proof: consider the bimodules $A \xrightarrow{M} B \xrightarrow{N} C$ and $D \xrightarrow{X} E \xrightarrow{Y} F$. The actions involved in the proof are $\mu\colon M\otimes B \longrightarrow M, \eta\colon B\otimes N \longrightarrow N, \varphi\colon\ X\otimes E \longrightarrow X$ and $\psi\colon E\otimes Y \longrightarrow Y$. We need to prove that

$$(M\otimes_B N)\otimes(X\otimes_E Y) \simeq (M\otimes X)\otimes_{B\otimes E}(N\otimes Y)$$

The second object is the codomain of the coequalizer q of the following pair of parallel arrows

$$M\otimes B\otimes N\otimes X\otimes E\otimes Y \underset{1\otimes\eta\otimes 1\otimes\psi}{\overset{\mu\otimes 1\otimes\varphi\otimes 1}{\rightrightarrows}} \qquad M\otimes N\otimes X\otimes Y$$

Consider also the two coequalizers

$$M\otimes B\otimes N \underset{1\otimes\eta}{\overset{\mu\otimes 1}{\rightrightarrows}} M\otimes N \xrightarrow{\quad q_1 \quad} M\otimes_B N$$

$$X\otimes E\otimes Y \underset{1\otimes\psi}{\overset{\varphi\otimes 1}{\rightrightarrows}} X\otimes Y \xrightarrow{\quad q_2 \quad} X\otimes_E Y$$

(At this point if A and E are unital, one could use lemma 1.1 taking as common sections the units and the proof would be achieved.) Clearly, $q_1\otimes q_2$ coequalizes $\mu\otimes 1\otimes\varphi\otimes 1$ and $1\otimes\eta\otimes 1\otimes\psi$, so that there exists a unique arrow $r\colon (M\otimes X)\otimes_{B\otimes E}(N\otimes Y) \longrightarrow (M\otimes_B N)\otimes(X\otimes_E Y)$ such that $q\cdot r = q_1\otimes q_2$. Now the problem is to show that q coequalizes the two following pairs of parallel arrows

$$M\otimes B\otimes N\otimes X\otimes Y \qquad \underset{1\otimes\eta\otimes 1\otimes 1}{\overset{\mu\otimes 1\otimes 1\otimes 1}{\rightrightarrows}} \qquad M\otimes N\otimes X\otimes Y \qquad (1)$$

$$M\otimes N\otimes X\otimes E\otimes Y \qquad \underset{1\otimes 1\otimes 1\otimes\psi}{\overset{1\otimes 1\otimes\varphi\otimes 1}{\rightrightarrows}} \qquad M\otimes N\otimes X\otimes Y \qquad (2)$$

Once this is done, the proof runs as follows: since q coequalizes (2), there exists a unique arrow p_2 such that the following diagram commutes

$$\begin{tikzcd}
M \otimes N \otimes X \otimes Y \arrow[r, "q"] \arrow[d, "1 \otimes 1 \otimes q_2"'] & (M \otimes X) \otimes_{B \otimes E} (N \otimes Y) \\
M \otimes N \otimes (X \otimes_E Y) \arrow[ur, "p_2"'] &
\end{tikzcd}$$

From the previous equation and since q coequalizes (1), an easy diagram chasing shows that there exists a unique arrow p_1 such that the following diagram commutes

$$\begin{tikzcd}
M \otimes N \otimes (X \otimes_E Y) \arrow[r, "p_2"] \arrow[d, "q_1 \otimes 1 \otimes_E 1"'] & (M \otimes X) \otimes_{B \otimes E} (N \otimes Y) \\
(M \otimes_B N) \otimes (X \otimes_E Y) \arrow[ur, "p_1"'] &
\end{tikzcd}$$

The arrows p_1 and r give the required isomorphism.

It remains to show that q coequalizes (1) and (2). We do the first verification, the second is analogue. Since B is splitting and M is regular, we know from proposition 3.2 that there exists a section $\beta\colon M \longrightarrow M \otimes B$ for the action $\mu\colon M \otimes B \longrightarrow M$. Analogously, there exists a section $\epsilon\colon X \longrightarrow X \otimes E$ for the action $\varphi\colon X \otimes E \longrightarrow E$. Now to conclude the proof it suffices to write in diagrammatic terms the following "elementary" argument: if $m \in M, n \in N, x \in X, y \in Y, e \in E$ and if $\beta(m) = m' \otimes b', \epsilon(x) = x' \otimes e'$ with $m' \in M, b' \in B, x' \in X, e' \in E$, then $m \otimes n \otimes x \cdot e \otimes y = m' \cdot b' \otimes n \otimes (x' \cdot e') \cdot e \otimes y = m' \cdot b' \otimes n \otimes x' \cdot (e' \cdot e) \otimes y = m' \otimes b' \cdot n \otimes x' \otimes (e' \cdot e) \cdot y = m' \otimes b' \cdot n \otimes x' \otimes e' \cdot (e \cdot y) = m' \cdot b' \otimes n \otimes x' \cdot e' \otimes e \cdot y = m \otimes n \otimes x \otimes e \cdot y$. ■

Now we can define the *first Brauer-Taylor group* $\mathcal{BT}_1(\mathbb{C})$ of $\mathbb{C}$ as the Picard group of SMon($\mathbb{C}$).

By proposition 3.2, we have that a bimodule between unital monoids is regular if and only if it is unital. This means that UMon($\mathbb{C}$) is a full monoidal subcategory of SMon($\mathbb{C}$). This implies that $\mathcal{B}_1(\mathbb{C})$ is a subgroup of $\mathcal{BT}_1(\mathbb{C})$.

If we take as $\mathbb{C}$ the category $\mathbb{R}$Mod, $\mathcal{BT}_1(\mathbb{C})$ is in general bigger than the Brauer-Taylor group of $\mathbb{R}$ as defined in [29]. To obtain exactly the Brauer-Taylor group of $\mathbb{R}$ we need a further condition on the objects of SMon($\mathbb{C}$), that is we consider the full monoidal subcategory RSMon($\mathbb{C}$) of SMon($\mathbb{C}$) whose objects are bisplitting monoids which contain I as a retract in $\mathbb{C}$. Now we define the *second Brauer-Taylor group* $\mathcal{BT}_2(\mathbb{C})$ of $\mathbb{C}$ as the Picard group of RSMon($\mathbb{C}$). Clearly we can add the retract condition also in UMon($\mathbb{C}$) and define in this way the *second Brauer group* of $\mathbb{C}$. The following two diagrams summarize the situation: in the left one, arrows are full monoidal inclusions; in the right one, arrows are inclusions of subgroups.

$$\begin{tikzcd} \mathrm{UMon}(\mathbb{C}) \arrow[r] & \mathrm{SMon}(\mathbb{C}) \\ \mathrm{RUMon}(\mathbb{C}) \arrow[r] \arrow[u] & \mathrm{RSMon}(\mathbb{C}) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_1(\mathbb{C}) \arrow[r] & \mathcal{BT}_1(\mathbb{C}) \\ \mathcal{B}_2(\mathbb{C}) \arrow[r] \arrow[u] & \mathcal{BT}_2(\mathbb{C}) \arrow[u] \end{tikzcd}$$

If $\mathbb{C}$ is $\mathbb{R}$Mod, both $\mathcal{B}_1(\mathbb{C})$ and $\mathcal{B}_2(\mathbb{C})$ coincide with the classical Brauer group of $\mathbb{R}$. As far as $\mathcal{B}_2(\mathbb{C})$ is concerned, recall that, if a unital $\mathbb{R}$-algebra A is faithfully projective as $\mathbb{R}$-module, then A contains $\mathbb{R}$ as a retract; in other words, the invertible objects of UMon($\mathbb{C}$) are contained in RUMon($\mathbb{C}$) if we take as $\mathbb{C}$ the category of modules over $\mathbb{R}$. (The same argument holds when $\mathbb{C}$ is the category of modules over a commutative unital quantale.) Moreover, $\mathcal{BT}_2(\mathbb{C})$ coincides with the usual Brauer-Taylor group of $\mathbb{R}$. This will be proved in the next section.

4 Elementary algebras

Given a morphism $\lambda\colon Y\otimes X \longrightarrow I$ in $\mathbb{C}$ which admits a section $s\colon I \longrightarrow Y\otimes X$, we can build up a monoid $E_\lambda = (X\otimes Y, m_\lambda)$ with multiplication m_λ given by $1\otimes\lambda\otimes 1\colon X\otimes Y\otimes X\otimes Y \longrightarrow X\otimes I\otimes Y \simeq X\otimes Y$. We call E_λ the *elementary algebra* associated with λ. It is a bisplitting monoid. And even more: it is a separable monoid, that is m_λ admits a section $\varphi_\lambda\colon E_\lambda \longrightarrow E_\lambda\otimes E_\lambda$ which is at the same time left and right linear. For this, it suffices to define φ_λ as $1\otimes s\otimes 1\colon X\otimes Y \simeq X\otimes I\otimes Y \longrightarrow X\otimes Y\otimes X\otimes Y$. Clearly, E_λ contains I as a retract.

If $\mathbb{C}$ is closed and I is regular projective, an example of (not necessarily unital) elementary algebra can be obtained taking as Y a generator of $\mathbb{C}$, as X the dual of Y and as λ the evaluation (generator means that the internal composition $Y\otimes_{Y\supset Y}(Y\supset I) \longrightarrow I$ is an isomorphism). In particular, if $\mathbb{C}$ is $\mathbb{R}$Mod, as Y one can take any (not necessarily finite) power of $\mathbb{C}$. Elementary algebras with unit will be considered in proposition 4.7.

The next proposition gives an equivalent description of the first and the second Brauer-Taylor group of $\mathbb{C}$.

Proposition 4.1

1) E_λ is isomorphic to I in SMon($\mathbb{C}$)

2) *if I is regular projective in $\mathbb{C}$ and if E is isomorphic to I in SMon($\mathbb{C}$), then E is isomorphic (as monoid of $\mathbb{C}$) to a suitable elementary algebra E_λ (here regular projective means that each regular epimorphism with codomain I has a section).*

Since the additional condition on I is clearly satisfied if we choose as $\mathbb{C}$ the category $\mathbb{R}$Mod, we can use the previous proposition to prove that in this case the second Brauer-Taylor group of $\mathbb{C}$ is the usual Brauer-Taylor group of $\mathbb{R}$. As far as the proof of proposition 4.1 is concerned, we need some points of terminology and some lemmas.

Definition 4.2

- *a set of pre-equivalence data (or Morita context) is a pair of arrows $P\colon A \longrightarrow B$ and $Q\colon B \longrightarrow A$ in SMon($\mathbb{C}$) together with two bimodule homomorphisms $f\colon P \otimes_B Q \longrightarrow A$ and $g\colon Q \otimes_A P \longrightarrow B$*
- *a set of equivalence data (or strict Morita context) is a set of pre-equivalence data (A, B, P, Q, f, g) such that f and g are bimodule isomorphisms.*

In other words, (A, B, P, Q, f, g) is a set of equivalence data exactly when the regular bimodules $P\colon A \longrightarrow B$ and $Q\colon B \longrightarrow A$ give an isomorphism in SMon($\mathbb{C}$).

Lemma 4.3 *If (A, B, P, Q, f, g) is a set of equivalence data, then*

$$- \otimes_A P\colon \textit{mod-}A \longrightarrow \textit{mod-}B \qquad - \otimes_B Q\colon \textit{mod-}B \longrightarrow \textit{mod-}A$$

constitute an equivalence of categories (where mod-A is the category of regular right A-modules and mod-B is the category of regular right B-modules). In particular, $- \otimes_A P$ is full and faithful. ■

The previous lemma is the trivial part of Morita theory, which clearly holds also for splitting monoids.

Lemma 4.4 *If (A, B, P, Q, f, g) is a set of equivalence data, with no lost of generality we can suppose f and g associative, that is such that the following diagrams are commutative*

$$\begin{tikzcd}
P \otimes_B Q \otimes_A P \arrow[r, "f \otimes_A 1"] \arrow[d, "1 \otimes_B g"'] & A \otimes_A P \arrow[d, "\simeq"] \\
P \otimes_B B \arrow[r, "\simeq"'] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \otimes_A P \otimes_B Q \arrow[r, "g \otimes_B 1"] \arrow[d, "1 \otimes_A f"'] & B \otimes_B Q \arrow[d, "\simeq"] \\
Q \otimes_A A \arrow[r, "\simeq"'] & Q
\end{tikzcd}$$

Proof: we generalize proposition 3.1 (3) in [5] avoiding units. Let us call $t: A \otimes_A P \longrightarrow P$ and $t': P \otimes_B B \longrightarrow P$ the canonical isomorphisms, which are bimodule isomorphisms. Consider the bimodule isomorphism $u: A \otimes_A P \longrightarrow A \otimes_A P$ given by $(f^{-1} \otimes_A 1_P) \cdot (1_P \otimes_B g) \cdot t' \cdot t^{-1}$. Since $- \otimes_A P$: mod-$A \longrightarrow$ mod-B is full and faithful, u is of the form $u' \otimes_A 1$ for an isomorphism $u': A \longrightarrow A$ in mod-A. Since u is A-linear on the left and $- \otimes_A P$ is faithful, also u' is A-linear on the left. If we replace f by $f \cdot u'$, we obtain the first associativity. The second associativity follows from the first one, using once again that $- \otimes_A P$ is faithful.

■

Lemma 4.5 *If (A, B, P, Q, f, g) is a set of pre-equivalence data with f and g associative and if f has a left linear section, then f is an isomorphism of bimodules.*

Proof: let x be the section of f. Using the two associativities of f and g, a diagram chasing argument shows that the following diagram is commutatve

$$\begin{tikzcd}
P \otimes_B Q \arrow[rr, "f"] \arrow[d, "\simeq"'] & & A \arrow[rr, "x"] & & P \otimes_B Q \arrow[d, "\simeq"] \\
P \otimes_B Q \otimes_A A \arrow[rr, "1 \otimes_B 1 \otimes_A x"'] & & P \otimes_B Q \otimes_A P \otimes_B Q \arrow[rr, "1 \otimes_B 1 \otimes_A f"'] & & P \otimes_B Q \otimes_A A
\end{tikzcd}$$

■

We can now come back to proposition 4.1.

Proof of proposition 4.1: 1): consider an arrow $\lambda: Y \otimes X \longrightarrow I$ with section $s: I \longrightarrow Y \otimes X$ and consider the elementary algebra $E_\lambda = (X \otimes Y, m_\lambda)$ defined at the beginning of the section. To show that E_λ is equivalent to I, we build up a set of equivalence data $(I, E_\lambda, Y, X, f, g)$ in the following way. The actions are $\lambda \otimes 1: Y \otimes E_\lambda = Y \otimes X \otimes Y \longrightarrow I \otimes Y \simeq Y$ and $1 \otimes \lambda: E_\lambda \otimes X = X \otimes Y \otimes X \longrightarrow X \otimes I \simeq X$. This two modules are splitting (the sections are $s \otimes 1$ and $1 \otimes s$) and then, by proposition 3.2, they are regular. As $g: X \otimes_I Y \longrightarrow E_\lambda$ we take the identity. The definition of f is given by the following diagram, where q is the coequalizer,

$$\begin{tikzcd}
Y \otimes X \otimes Y \otimes X \arrow[r, shift left, "\lambda \otimes 1 \otimes 1"] \arrow[r, shift right, "1 \otimes 1 \otimes \lambda"'] & Y \otimes X \arrow[r, "q"] \arrow[d, "\lambda"'] & Y \otimes_E X \arrow[dl, "f"] \\
 & I &
\end{tikzcd}$$

It is easy to show that f and g are associative. Moreover, f has a section (given by $s \cdot q \colon I \longrightarrow Y \otimes X \longrightarrow Y \otimes_E X$) so that, by lemma 4.5, it is an isomorphism.
2): let (E, I, P, Q, f, g) be a set of equivalence data giving the isomorphism between E and I in SMon($\mathbb{C}$). Consider the coequalizer

$$Q \otimes E \otimes P \rightrightarrows Q \otimes P \xrightarrow{q} Q \otimes_E P$$

The composite $\lambda = q \cdot g \colon Q \otimes P \longrightarrow Q \otimes_E P \xrightarrow{\simeq} I$ is a regular epimorphism, and then, since I is regular projective, it has a section $s \colon I \longrightarrow Q \otimes P$. Clearly, the bimodule isomorphism $f \colon P \otimes Q \longrightarrow E$ gives an isomorphism in $\mathbb{C}$ between the elementary algebra E_λ and E. The fact that f is indeed an isomorphism of monoids follows from the associativity of f and g, which is guaranteed by lemma 4.4. ■

As a consequence of proposition 4.1, we obtain the following description of the first Brauer-Taylor group of $\mathbb{C}$: the group $\mathcal{BT}_1(\mathbb{C})$ is the (possibly large) set of equivalence classes of bisplitting monoids A such that there exists a bisplitting monoid B and an elementary algebra E_λ isomorphic (as monoid of $\mathbb{C}$) to $A \otimes B$; two monoids A and A' of this kind are equivalent if there exist two elementary algebras E_λ and $E_{\lambda'}$ such that $A \otimes E_\lambda$ and $A' \otimes E_{\lambda'}$ are isomorphic (as monoids of $\mathbb{C}$). An analogous description holds for the second Brauer-Taylor group, adding on monoids the condition to contain I as a retract.

A warning: to verify in details, via proposition 4.1, that the previous description of the Brauer-Taylor groups of $\mathbb{C}$ is equivalent to that given in section 3, one uses everywhere that, if A and B are isomorphic as bisplitting monoids of $\mathbb{C}$, then they are isomorphic in SMon($\mathbb{C}$). This can be proved as follows: if $f \colon A \longrightarrow B$ is an isomorphism of monoids and A is right splitting, we can provide B with the structure of a left A-module via $A \otimes B \xrightarrow{f \otimes 1} B \otimes B \xrightarrow{m_B} B$; to prove that B is regular, we can take as section $B \xrightarrow{f^{-1}} A \xrightarrow{\varphi_A} A \otimes A \xrightarrow{1 \otimes f} A \otimes B$ (where φ_A is the section of m_A).

Finally, we can consider the classical case.

Corollary 4.6 *If $\mathbb{C}$ is the category $\mathbb{R}$Mod of unital modules over an unital commutative ring $\mathbb{R}$, then $\mathcal{BT}_2(\mathbb{C})$ is the Brauer-Taylor group of $\mathbb{R}$.*

Proof: let us recall that, in any abelian category, an object X contains an object Y as a direct summand if and only if Y is a retract of X. Now, using our proposition 4.1, the statement follows from proposition 2.2, proposition 3.10 (and its proof) and proposition 4.4 in [29]. ■

In general, in the situation of the previous corollary, $\mathcal{BT}_1(\mathbb{C})$ is bigger than the Brauer-Taylor group of $\mathbb{R}$. This is because, unlike the case of unital monoids, if A and B are splitting monoids such that $A \otimes B$ is an elementary algebra (and then contains I as a retract), one cannot deduce that A (and B) contains I as a retract.

To end the work, let us come back to the Brauer group. Recall that, if $\mathbb{C}$ is closed, an object Y is faithfully projective (or a progenerator) if the internal compositions

$$(I \supset Y) \otimes_{Y \supset Y} (Y \supset I) \longrightarrow (I \supset I) \qquad (Y \supset I) \otimes (I \supset Y) \longrightarrow (Y \supset Y)$$

are isomorphisms.

Proposition 4.7 *Let $\mathbb{C}$ be a closed symmetric monoidal category with coequalizers and suppose I is regular projective. Elementary algebras with unit are exactly monoids of the form $Y \supset Y$ with Y faithfully projective.*

Proof: from our proposition 4.1 and proposition 7.3 in [31], using once again only the trivial part of Morita theory. ■

From propositions 4.1 and 4.7, we obtain an equivalent description of the Brauer groups of $\mathbb{C}$ in terms of equivalence classes of unital monoids. It is like the description given for Brauer-Taylor groups, but we can replace elementary algebras by monoids of the form $Y \supset Y$ with Y faithfully projective. If we specialize this description with $\mathbb{R}$Mod for $\mathbb{C}$, we obtain the description of $\mathcal{B}(\mathbb{R})$ originally given by Auslander and Goldman in [2].

5 *

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