Localizations of Maltsev varieties

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Abstract

We give an abstract characterization of categories which are localizations of Maltsev varieties. These results can be applied to characterize localizations of naturally Maltsev varieties.

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Introduction

In this paper we study localizations of Maltsev varieties.

A finitary (one sorted) variety V is called a Maltsev variety if its theory is equipped with a ternary operation $p(x,y,z)$ satisfying the axioms $p(x,y,y)=x$, $p(x,x,y)=y$. The varieties of groups, abelian groups, modules over a fixed ring, crossed modules, rings, commutative rings, associative algebras, Lie algebras, quasi-groups and Heyting algebras are all examples of Maltsev varieties. The classical Maltsev theorem [11] asserts that the existence of such a ternary operation is equivalent to the fact that any reflexive relation in V is an equivalence relation. This semantical formulation of the Maltsev property has been the starting point for a purely categorical approach to Maltsev varieties developed in recent years in [3], [4], [8], [12] and [13].

The second aspect of our work concerns localizations. In 1964 P. Gabriel and N. Popescu proved that Grothendieck categories are exactly localizations of module categories [6]. This theorem has been extended to a non-abelian setting by the second author [16], who has characterized localizations of varieties as exact categories (in the sense of Barr [1]) with a regular generator and exact filtered colimits.

In this work we show that the categorical approach to Maltsev varieties and the exact completion technique developed in [15] and [16] fit together well, giving an abstract characterization of localizations of Maltsev varieties. In the first section we characterize localizations of Maltsev monadic categories, and we then restrict our attention to the finitary case (recall that varieties are exactly monadic categories over Set for a filtered colimit preserving monad). In the second section we specialize our results to localizations of naturally Maltsev monadic categories and naturally Maltsev varieties. These varieties are very closed to module categories, the only difference consisting in the existence of a zero object. In fact, Gabriel-Popescu theorem is an obvious corollary of our result on naturally Maltsev varieties.

1 Maltsev monadic categories

Let $\mathbb{T}$ be a monad in Set: we denote by $KL(\mathbb{T})$ the Kleisli category of $\mathbb{T}$ and by $EM(\mathbb{T})$ the Eilenberg-Moore category of $\mathbb{T}$. The category $KL(\mathbb{T})$ is a projective cover of $EM(\mathbb{T})$, this meaning that each object of $KL(\mathbb{T})$ is regular projective in $EM(\mathbb{T})$ and for each object X in $EM(\mathbb{T})$ there is an object P in $KL(\mathbb{T})$ with a regular epimorphism $P \to X$. The category $EM(\mathbb{T})$ is the exact completion of $KL(\mathbb{T})$ [14]: this fact has a central role in the characterization of the localizations of monadic categories over Set (we use the term "monadic category" for a category equivalent to $EM(\mathbb{T})$ for a monad $\mathbb{T}$ in Set). In the following all categories we consider will be assumed to be locally small.

Theorem 1.1. *[15] For a category $\mathcal{B}$ the following conditions are equivalent:*

1. *$\mathcal{B}$ is equivalent to a localization of a monadic category over Set*
2. *$\mathcal{B}$ is exact and has a regular generator G which admits all copowers*

Let us recall the construction involved in the proof of the previous theorem and fix some notations. Let $\mathcal{B}$ be a category as in condition 2. of the theorem: we denote by $\mathcal{C}$ the full subcategory of $\mathcal{B}$ spanned by copowers of G. If S is a set we write $S \bullet G$ for the S-indexed copower of G. If $\mathcal{C}_{ex}$ is the exact completion of $\mathcal{C}$, we denote by Γ and U the canonical full inclusions as in the diagram

$$\begin{tikzcd} & & \mathcal{C}_{ex} \arrow[d, shift right, "r"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{C} \arrow[urr, "\Gamma"] \arrow[rr, "U"'] & & \mathcal{B}. \arrow[u, shift right, "i"'] \end{tikzcd} \tag{1.}$$

The universal property of $\mathcal{C}_{ex}$ gives a unique exact (left exact and regular epimorphism preserving) functor $r\colon \mathcal{C}_{ex} \to \mathcal{B}$ with $r \circ \Gamma = U$. Moreover, there is a fully faithful functor $i\colon \mathcal{B} \to \mathcal{C}_{ex}$ (right adjoint to r) defined as follows: if $A \in \mathcal{B}$, one considers its canonical cover $a\colon \mathcal{B}(G, A) \bullet G \to A$, then the kernel pair of a

$$a_1, a_2 \colon N(a) \rightrightarrows \mathcal{B}(G, A) \bullet G$$

and finally its canonical cover $n\colon \mathcal{B}(G, N(a)) \to N(a)$. The object $i(A)$ in $\mathcal{C}_{ex}$ is given by the pseudo equivalence relation

$$a_1 \circ n, a_2 \circ n \colon \mathcal{B}(G, N(a)) \bullet G \rightrightarrows \mathcal{B}(G, A) \bullet G \ ,$$

with a similar definition on arrows.

The next simple result is stated separately for future references; it can be proved by using the explicit description of the exact completion of a category with finite weak limits [5].

Lemma 1.1. *Let $\mathcal{B}$ be an exact category with a regular generator G which admits all copowers. Then, with the notations as above, one has that $\Gamma = i \circ U$ (up to natural isomorphisms).*

We are interested in localizations of Maltsev monadic categories over Set. In a category with finite limits there are several equivalent definitions of the Maltsev property [3]. We adopt the simplest one: *a category $\mathcal{C}$ is Maltsev if any reflexive relation in $\mathcal{C}$ is symmetric.*

In [13] Pedicchio proved that, for a monadic category, the Maltsev property is completely determined by the behaviour of the regular projective regular generator $P = F(1)$ (the free algebra on one generator). In order to state her result we recall that an object X in a category with finite sums is an internal Maltsev coalgebra if there exists an arrow $t\colon X \to X+X+X$ such that the following diagram commutes

$$
\begin{tikzcd}
 & X \arrow[dl, "i_1"'] \arrow[d, "t"] \arrow[dr, "i_2"] & \\
X+X & X+X+X \arrow[l, "\nabla+1"] \arrow[r, "1+\nabla"'] & X+X,
\end{tikzcd}
$$

where i_1 and i_2 are the canonical injections in the sum and ∇ is the codiagonal.

Theorem 1.2. *[13] The following conditions are equivalent for a category $\mathcal{B}$:*

1. *$\mathcal{B}$ is a Maltsev monadic category over Set*
2. *$\mathcal{B}$ is exact with a regular projective regular generator P which admits all copowers and P is an internal Maltsev coalgebra*

It is then natural to ask whether the Maltsev property can be determined by a condition on the regular generator in the case of localizations of Maltsev monadic categories as well. A positive answer to this question is given in the proposition 1.1 below; for this, the following lemma, which is a special case of lemma 7. in [12], will be needed.

Lemma 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be finitely complete categories and let $R \underset{c}{\overset{d}{\rightrightarrows}} A$ be a relation in $\mathcal{A}$. If $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact and conservative functor, then $R \underset{c}{\overset{d}{\rightrightarrows}} A$ is symmetric precisely when $F(R) \underset{F(c)}{\overset{F(d)}{\rightrightarrows}} F(A)$ is symmetric.*

Let us introduce a point of terminology:

Definition 1.1. *Let $\mathcal{A}$ be a regular category. A graph $X_1 \underset{x_1}{\overset{x_0}{\rightrightarrows}} X_0$ in $\mathcal{A}$ is pseudo-symmetric if its image factorisation $I \underset{\overline{x}_1}{\overset{\overline{x}_0}{\rightrightarrows}} X_0$ in $\mathcal{A}$ is a symmetric relation.*

We can now prove our characterization:

Proposition 1.1. *The following conditions are equivalent:*

1. *$\mathcal{B}$ is equivalent to a localization of a Maltsev monadic category over Set*
2. *$\mathcal{B}$ is exact with a regular generator G which admits all copowers and G is an internal Maltsev coalgebra*
3. *$\mathcal{B}$ is exact with a regular generator G which admits all copowers and the functor $\mathcal{B}(G, -)$ sends reflexive graphs to pseudo-symmetric graphs*

Proof: By theorem 1.1 the category $\mathcal{B}$ is equivalent to a localization of a monadic category over Set if and only if it is exact with a regular generator G which admits all copowers.
1. $\Rightarrow$ 2. By assumption $\mathcal{B}$ is a localization of a monadic Maltsev category $\mathcal{A}$ as in the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right, "r"'] \end{tikzcd},$$

where r preserves finite limits. Let P be the regular projective regular generator which is an internal Maltsev coalgebra in $\mathcal{A}$ by theorem 1.2. The regular generator $G = r(P)$ is then an internal Maltsev coalgebra in $\mathcal{B}$, since r preserves sums.

2. $\Rightarrow$ 3. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{B}$ spanned by copowers of the regular generator G. The category $\mathcal{B}$ is equivalent to a localization of the exact completion $\mathcal{C}_{ex}$ of the category $\mathcal{C}$ (see proposition 2.1 in [15] for a proof).

$$\begin{tikzcd} & \mathcal{C}_{ex} \arrow[d, shift right, "r"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{C} \arrow[ur, "\Gamma"] \arrow[r, "U"'] & \mathcal{B} \arrow[u, shift right, "i"'] \end{tikzcd} \tag{1.}$$

The Maltsev coalgebra structure $t\colon G \to G+G+G$ on the generator G induces a Maltsev coalgebra structure on $\Gamma(G)$, simply because Γ preserves all sums that exist in $\mathcal{C}$ [5]. By theorem 1.2 the category $\mathcal{C}_{ex}$ is accordingly Maltsev: indeed, $\mathcal{C}$ is equivalent to $KL(\mathbb{T})$ for a monad $\mathbb{T}$ over Set and $\mathcal{C}_{ex}$ is then equivalent to $EM(\mathbb{T})$ (by the way, this proves the implication 2. $\Rightarrow$ 1.). Let us then consider a reflexive graph

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X_0 \arrow[l, "e"'] \end{tikzcd}$$

in $\mathcal{B}$. The reflexive graph $i(X_1) \rightrightarrows i(X_0)$ can be factorised in $\mathcal{C}_{ex}$ as in the diagram

$$\begin{tikzcd} i(X_1) \arrow[r, shift left, "i(c)"] \arrow[r, shift right, "i(d)"'] \arrow[dr, "p"'] & i(X_0) \\ & I, \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

where p is a regular epi and $I \rightrightarrows i(X_0)$ is a reflexive relation in $\mathcal{C}_{ex}$. $\mathcal{C}_{ex}$ is Maltsev so that $I \rightrightarrows i(X_0)$ is also symmetric: since $\Gamma(G)$ is regular projective, it follows that the image factorisation in Set of the reflexive graph $\mathcal{C}_{ex}(\Gamma(G), i(X_1)) \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), i(X_0))$ is a symmetric relation. By lemma 1.1 one has that the graph

$$\mathcal{C}_{ex}(\Gamma(G), i(X_1)) \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), i(X_0))$$

is isomorphic to the graph

$$\mathcal{C}_{ex}(i(G), i(X_1)) \rightrightarrows \mathcal{C}_{ex}(i(G), i(X_0)) .$$

The functor i is fully faithful: it follows that the graph $\mathcal{B}(G, X_1) \rightrightarrows \mathcal{B}(G, X_0)$ is a pseudo-symmetric graph in Set.

3. $\Rightarrow$ 1. Let $X_1 \underset{d}{\overset{c}{\rightrightarrows}} X_0$ be a reflexive relation in $\mathcal{C}_{ex}$ (with the same notations as above) and we are going to prove that it is symmetric. Recall that there is a commutative diagram of functors

$$\begin{tikzcd} Set^{\mathbb{T}} = \mathcal{C}_{ex} \arrow[r, "V"] & Set \arrow[rr, "F"] \arrow[dr, "- \bullet G"'] & & \mathcal{C}_{ex} = Set^{\mathbb{T}} \\ & & \mathcal{C} \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

where V is the forgetful functor and F is the "free algebra" functor. The reflexive relation above can be covered by a regular epi between reflexive graphs, as in the diagram

$$\begin{tikzcd} \Gamma(X_1') \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "\epsilon_{X_1}"'] & \Gamma(X_0') \arrow[l] \arrow[d, "\epsilon_{X_0}"] \\ X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0, \arrow[l] \end{tikzcd}$$

where the functor $(\,)'$ is the composite $(- \bullet G) \circ V$ and the vertical arrows are the counit components of the adjunction $F \dashv V$. By applying the functor $\mathcal{C}_{ex}(\Gamma(G), -)$ to the commutative diagram above and by taking the regular image factorisation J of the reflexive graph on the top one gets the diagram

$$\begin{tikzcd} \mathcal{C}_{ex}(\Gamma(G), \Gamma({X_1}')) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "p"] \arrow[dd] & & \mathcal{C}_{ex}(\Gamma(G), \Gamma({X_0}')) \arrow[dd] \\ & J \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dl, dotted, "\alpha"] & \\ \mathcal{C}_{ex}(\Gamma(G), X_1) \arrow[rr, shift left] \arrow[rr, shift right] & & \mathcal{C}_{ex}(\Gamma(G), X_0). \end{tikzcd}$$

The factorization α making commutative the diagram above comes from the fact that

$$\mathcal{C}_{ex}(\Gamma(G), X_1) \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), X_0)$$

is a relation; the arrow α is surjective because ϵ_{X_1} is a regular epi and $\Gamma(G)$ is regular projective. Since Γ and U are full and faithful, the reflexive graph

$$\mathcal{C}_{ex}(\Gamma(G), \Gamma({X_1}')) \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), \Gamma({X_0}'))$$

is isomorphic to the reflexive graph

$$\mathcal{B}(G, {X_1}') \rightrightarrows \mathcal{B}(G, {X_0}')\ .$$

By assumption this latter is a pseudo-symmetric graph, so that the relation $J \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), \Gamma(X_0'))$ is symmetric. In the category of sets one can easily check that the symmetry of this relation induces a symmetry on the relation

$$\mathcal{C}_{ex}(\Gamma(G), X_1) \rightrightarrows \mathcal{C}_{ex}(\Gamma(G), X_0),$$

by using the fact that the regular epimorphism α splits. The functor $\mathcal{C}_{ex}(\Gamma(G), -)$ is left exact and conservative (it is isomorphic to the forgetful functor V), so that by lemma 1.2 the proof is complete.

□

We remark that the condition 3. in the proposition 1.1 implies in particular that $\mathcal{B}$ is Maltsev. However, this property on reflexive graphs does not seem to be equivalent to the Maltsev property, since the functor $\mathcal{B}(G, -)$ does not preserve regular epimorphisms. In other words, we don't know if an exact Maltsev category with a regular generator (with copowers) is necessarily a localization of a Maltsev monadic category.

Corollary 1.1. *The following conditions are equivalent:*

1. *$\mathcal{B}$ is a Maltsev monadic category over Set*
2. *$\mathcal{B}$ is exact with a regular projective regular generator G which admits all copowers and the functor $\mathcal{B}(G, -)$ sends reflexive graphs to pseudo-symmetric graphs*

Proof: With the same notations as above, this follows from the fact that when $G \in \mathcal{B}$ is regular projective, then the adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_{ex} \arrow[l, shift right, "r"'] \end{tikzcd}$$

is an equivalence.

□

In the following we shall use the term variety for a finitary variety in the sense of universal algebra. An application of our previous result can be given to characterize localizations of Maltsev varieties.

Proposition 1.2. *The following conditions are equivalent:*

1. *$\mathcal{B}$ is a localization of a Maltsev variety*
2. *$\mathcal{B}$ is exact with a regular generator G which admits all copowers, G is an internal Maltsev coalgebra and filtered colimits commute with finite limits in $\mathcal{B}$*
3. *$\mathcal{B}$ is exact with a regular generator G which admits all copowers, the functor $\mathcal{B}(G, -)$ sends reflexive graphs to pseudo-symmetric graphs and filtered colimits commute with finite limits in $\mathcal{B}$*

Proof: The implications 1. $\Rightarrow$ 2. and 1. $\Rightarrow$ 3. follow from proposition 1.1 above and corollary 1.2 in [16]. To see that 2. $\Rightarrow$ 1., we first fix some notations. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ (as in proposition 1.1) with same objects as $\mathcal{C}$ and arrows $f: R \bullet G \to S \bullet G$ the arrows in $\mathcal{C}$ such that, for each $r \in R$, $f \circ \sigma_r : G \to R \bullet G \to S \bullet G$ factorises through $S' \bullet G$ for some finite subset S' of S, where $\sigma_r : G \to R \bullet G$ is the rth injection in the coproduct. We denote by ${\mathcal{C}'}_{ex}$ the exact completion of the category $\mathcal{C}'$ and by $\Gamma' \colon \mathcal{C}' \to {\mathcal{C}'}_{ex}$ the canonical embedding. We know that ${\mathcal{C}'}_{ex}$ is the category of algebras for the finitary part $\mathbb{T}'$ of the monad $\mathbb{T}$ which has $\mathcal{C}_{ex}$ as category of algebras.

$$\begin{tikzcd}
 & {\mathcal{C}'}_{ex} \arrow[rr, shift right, "q"'] & & \mathcal{C}_{ex} \arrow[ll, shift right, "j"'] \arrow[ll, phantom, "\top"] \arrow[d, shift right, "r"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C}' \arrow[ur, "\Gamma'"] \arrow[rr, "U'"'] & & \mathcal{C} \arrow[ur, "\Gamma"] \arrow[r, "U"'] & \mathcal{B} \arrow[u, shift right, "i"']
\end{tikzcd} \tag{2.}$$

In the diagram above the forgetful functor U' gives rise to a morphism from $\mathbb{T}'$ to $\mathbb{T}$, inducing an adjunction $q \dashv j \colon \mathcal{C}_{ex} \to {\mathcal{C}'}_{ex}$. Observe that $\mathcal{B}$ is a localization of the Maltsev monadic category $\mathcal{C}_{ex}$ and it is also a localization of the variety ${\mathcal{C}'}_{ex}$ by corollary 1.2 in [16]. The regular generator G has an internal Maltsev coalgebra operation $t \colon G \to G+G+G$, and this t is an arrow that belongs to $\mathcal{C}'$. It follows that $\Gamma'(t) \colon \Gamma'(G) \to \Gamma'(G) + \Gamma'(G) + \Gamma'(G)$ is an internal Maltsev coalgebra operation in ${\mathcal{C}'}_{ex}$, proving that ${\mathcal{C}'}_{ex}$ is Maltsev.
3. $\Rightarrow$ 2. By proposition 1.1 $\mathcal{C}_{ex}$ is a Maltsev monadic category. Accordingly, the object $r(P) = r(\Gamma(G)) = r(i(G)) = G$ (using lemma 1.1) is an internal Maltsev coalgebra.

□

2 Naturally Maltsev varieties

We first recall the notion of naturally Maltsev category introduced by Johnstone in [9]:

Definition 2.1. *A finitely complete category $\mathcal{C}$ is naturally Maltsev if there exists a natural transformation μ from the functor $A \mapsto A \times A \times A$ to the identity functor on $\mathcal{C}$, such that $\mu_A \colon A \times A \times A \to A$ is a Maltsev operation on A, for any $A \in \mathcal{C}$.*

Let us denote by $Grpd(\mathcal{C})$ and by $RG(\mathcal{C})$ the categories of internal groupoids and internal reflexive graphs in a finitely complete category $\mathcal{C}$. We shall need the following results:

Theorem 2.1. *[9] For a finitely complete category $\mathcal{C}$ the following conditions are equivalent:*

1. *$\mathcal{C}$ is naturally Maltsev*
2. *the forgetful functor $U \colon Grpd(\mathcal{C}) \to RG(\mathcal{C})$ is an isomorphism*

Theorem 2.2. *[7] If $\mathcal{C}$ is an exact Maltsev category, then the category $Grpd(\mathcal{C})$ is closed in $RG(\mathcal{C})$ under quotients.*

We can now prove our result on localizations of naturally Maltsev monadic categories. Observe that, unlike the Maltsev case, the condition is now given on the subcategory $\mathcal{B}$ and not on the regular generator G.

Proposition 2.1. *The following conditions are equivalent:*

1. *$\mathcal{B}$ is equivalent to a localization of a naturally Maltsev monadic category over Set*
2. *$\mathcal{B}$ is exact naturally Maltsev with a regular generator G which admits all copowers*

Proof: 1. $\Rightarrow$ 2. Trivial.
2. $\Rightarrow$ 1. In the proof we shall refer to the diagram (1.). First remark that any reflexive graph in $\mathcal{B}$ is symmetric by theorem 2.1. Accordingly, the functor $\mathcal{B}(G,-)$ sends reflexive graphs to pseudo-symmetric graphs in Set so that the category $\mathcal{C}_{ex}$ is Maltsev by proposition 1.1. Let us then prove that any reflexive graph

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X_0 \arrow[l, "e"'] \end{tikzcd}$$

in $\mathcal{C}_{ex}$ is an internal groupoid. As in the proof of proposition 1.1, one can apply the functor $(\,)' = (-\bullet G)\circ V$ to this reflexive graph and the counit components of the adjunction $F \dashv V$ give a regular epimorphism of reflexive graphs

$$\begin{tikzcd} \Gamma(X_1') \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "\epsilon_{X_1}"'] & \Gamma(X_0') \arrow[l] \arrow[d, "\epsilon_{X_0}"] \\ X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0. \arrow[l] \end{tikzcd}$$

By theorem 2.2 to complete the proof it suffices to show that

$$-(X_1') \rightrightarrows -(X_0')$$

is an internal groupoid in $\mathcal{C}_{ex}$. The reflexive graph

$$U(X_1') \rightrightarrows U(X_0')$$

is an internal groupoid in $\mathcal{B}$, the category $\mathcal{B}$ being naturally Maltsev by assumption. It follows that

$$iU(X_1') \rightrightarrows iU(X_0')$$

is an internal groupoid in $\mathcal{C}_{ex}$, and this latter is isomorphic to the graph

$$-(X_1') \rightrightarrows -(X_0')$$

by lemma 1.1.

□

Proposition 2.2. *The following conditions are equivalent:*

1. *$\mathcal{B}$ is equivalent to a localization of a naturally Maltsev variety*
2. *$\mathcal{B}$ is exact naturally Maltsev with a regular generator G which admits all copowers and filtered colimits commute with finite limits*

Proof: The non trivial implication is 2. $\Rightarrow$ 1. We use the same notations as in the proof of proposition 1.2 and we first consider the diagram (2.): the category $\mathcal{C}_{ex}$ is naturally Maltsev by proposition 2.1 and $\mathcal{C}'_{ex}$ is Maltsev by proposition 1.2. The rest of the proof runs as in proposition 2.1, once one has proved that there exists a natural transformation from the functor $j\circ i\circ U\circ U'$ to the functor Γ' such that any component $\alpha_A\colon jiUU'(A)\to\Gamma'(A)$ is a split epimorphism. We leave this verification to the reader (hint: use the explicit description of the composite functor $\mathcal{B} \xrightarrow{i} \mathcal{C}_{ex} \xrightarrow{j} \mathcal{C'}_{ex}$ given in [16]). Any reflexive graph

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X_0 \arrow[l, "e"' description] \end{tikzcd}$$

in $\mathcal{C'}_{ex}$ is accordingly covered by a regular epi between reflexive graphs as the vertical composite in the diagram

$$\begin{tikzcd}
jiUU'(X'_1) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "\alpha_{X'_1}"'] & jiUU'(X'_0) \arrow[l] \arrow[d, "\alpha_{X'_0}"] \\
\Gamma'(X'_1) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "\epsilon_{X_1}"'] & \Gamma'(X'_0) \arrow[l] \arrow[d, "\epsilon_{X_0}"] \\
X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0. \arrow[l]
\end{tikzcd}$$

The reflexive graph

$$\begin{tikzcd} jiUU'(X_1{}') \arrow[r, shift left=2] \arrow[r, shift right=2] & jiUU'(X'_0) \arrow[l] \end{tikzcd}$$

is an internal groupoid in the exact Maltsev category $\mathcal{C'}_{ex}$ and by theorem 2.2 the proof is complete.

□

As a corollary we get the classical Gabriel-Popescu characterization of Grothendieck categories [6]. Recall that by the Tierney theorem [2] a category is abelian if and only if it is exact and additive, while Johnstone proved that a category with products is additive if and only if it is naturally Maltsev and has a zero object [9]. With the same notations as above, the result follows by proposition 2.2 and the fact that $\mathcal{B}$ has a zero object precisely when $\mathcal{C'}_{ex}$ has one. Indeed, proposition 2.2 states that $\mathcal{C'}_{ex}$ is an abelian variety and then, by Lawvere theorem in [10], it is equivalent to the category of modules over a unital ring. We have then proved the following result:

Corollary 2.1. *[6] The following conditions are equivalent:*

1. *$\mathcal{B}$ is equivalent to a localization of a module category*
2. *$\mathcal{B}$ is abelian with a regular generator G which admits all copowers and filtered colimits commute with finite limits*

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