Localizations of algebraic categories

Enrico M. Vitale
Institut de Mathématique Pure et Appliquée
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium

Abstract: we characterize localizations of monadic categories over $\mathcal{SET}$ using the fact that the category of algebras for a monad over $\mathcal{SET}$ is the exact completion of the full subcategory of free algebras. This also constitutes an unifying argument to characterize reflections and epireflections.

1 Preliminaries

The aim of this work is to characterize localizations of monadic categories over $\mathcal{SET}$. For this, let us consider a monad $\mathbb{T}$ over $\mathcal{SET}$ and write KL($\mathbb{T}$) and EM($\mathbb{T}$) respectively for the Kleisli category of $\mathbb{T}$ and for the Eilenberg-Moore category of $\mathbb{T}$. They can be characterized as follows (all categories are supposed to be locally small):

I) consider a category $\mathbb{C}$; the following conditions are equivalent:

1) $\mathbb{C}$ is equivalent to the category KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$
2) there exists an object G in $\mathbb{C}$ which admits all copowers and such that each object of $\mathbb{C}$ is isomorphic to a copower of G

II) consider a category $\mathbb{A}$; the following conditions are equivalent:

1) $\mathbb{A}$ is equivalent to the category EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$
2) $\mathbb{A}$ is exact and has a regular projective regular generator which admits all copowers.

An easy proof can be found in [8], where it is shown that the basic fact to obtain the second characterization is to observe that KL($\mathbb{T}$) is a *projective cover* of EM($\mathbb{T}$), that is each object of KL($\mathbb{T}$) is regular projective in EM($\mathbb{T}$) and for each object X of EM($\mathbb{T}$) there exists an object P in KL($\mathbb{T}$) together with a regular epimorphism $P \longrightarrow X$.

The fact that KL($\mathbb{T}$) is a projective cover of the exact category EM($\mathbb{T}$) means exactly that EM($\mathbb{T}$) is the *exact completion* of KL($\mathbb{T}$). We recall here some basic features of the exact completion; for more details we refer to [2] and to [7], whereas for an introduction to exact and monadic categories we refer to [1].

Definition 1.1 *Let $F\colon \mathbb{C} \longrightarrow \mathbb{A}$ be a functor between a weakly lex category $\mathbb{C}$ and a lex category $\mathbb{A}$; F is left covering if for each finite diagram $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ and for each (equivalently, for one) weak limit $w\lim \mathcal{L}$, the canonical factorization $p\colon F(w\lim \mathcal{L}) \longrightarrow \lim \mathcal{L} \cdot F$ is a strong epimorphism.*

Proposition 1.2 *For each weakly lex category $\mathbb{C}$ there exist an exact category $\mathbb{C}_{ex}$ and a left covering functor $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$ such that, for each exact category $\mathbb{B}$, composing with Γ induces an equivalence*

$$\Gamma \cdot - \ : \ Ex\,[\mathbb{C}_{ex}, \mathbb{B}] \longrightarrow Lco\,[\mathbb{C}, \mathbb{B}]$$

between the category of exact functors from $\mathbb{C}_{ex}$ to $\mathbb{B}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{B}$. Moreover, if $\mathbb{A}$ is an exact category and $\mathbb{P}$ is a projective cover of $\mathbb{A}$, then $\mathbb{P}$ is weakly lex and $\mathbb{A}$ is equivalent to $\mathbb{P}_{ex}$. $\blacksquare$

In what follows, we use not only the universal property of $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{ex}$, but also an explicit description of $\mathbb{C}_{ex}$:

- an object of $\mathbb{C}_{ex}$ is a pseudo equivalence relation in $\mathbb{C}$, that is a pair of arrows $r_1, r_2\colon R \rightrightarrows X$ (not necessarily monomorphic) such that there exist reflexivity $r_R\colon X \longrightarrow R$, symmetry $s_R\colon R \longrightarrow R$ and transitivity $t_R\colon R \star R \longrightarrow R$ (where $R \star R$ is a weak pullback of r_1 and r_2) satisfying the usual equations
- an arrow in $\mathbb{C}_{ex}$ is a class of equivalence of pairs of arrows $(\overline{f}, f)$ (as in the following diagram) such that $\overline{f} \cdot s_1 = r_1 \cdot f$ and $\overline{f} \cdot s_2 = r_2 \cdot f$

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  two pairs $(\overline{f}, f)$ and $(\overline{g}, g)$ from $(r_1, r_2\colon R \rightrightarrows X)$ to $(s_1, s_2\colon S \rightrightarrows Y)$ are equivalent if there exists an arrow $\Sigma\colon X \longrightarrow S$ such that $\Sigma \cdot s_1 = f$ and $\Sigma \cdot s_2 = g$.

2 Localizations

We are now ready to prove our characterization.

Proposition 2.1 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) $\mathbb{B}$ is equivalent to a localization of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ (that is a reflective subcategory such that the reflector is lex)

2) $\mathbb{B}$ is exact and has a regular generator which admits all copowers.

Proof: the implication 1⇒2 is quite obvious, so let us look at the implication 2⇒1. Let G be a regular generator as in condition 2 and let us fix some notations: if S is a set, $S \bullet G$ is the S-indexed copower of G and $i_s\colon G \longrightarrow S \bullet G$ is the s'th canonical injection ($s \in S$); if $\alpha\colon S \longrightarrow T$ is in $\mathcal{SET}$, $\alpha'\colon S \bullet G \longrightarrow T \bullet G$ is the arrow in $\mathbb{B}$ defined by $i_s \cdot \alpha' = i_{\alpha(s)}$

$$\begin{tikzcd} S \bullet G \arrow[rr, "\alpha'"] & & T \bullet G \\ & G \arrow[ul, "i_s"] \arrow[ur, "i_{\alpha(s)}"'] & \end{tikzcd}$$

Given an object A in $\mathbb{B}$, the canonical cover of A by G is the unique arrow $a\colon \mathbb{B}(G,A) \bullet G \longrightarrow A$ such that for each $f\colon G \longrightarrow A$ the following diagram commutes

$$\begin{tikzcd} \mathbb{B}(G,A) \bullet G \arrow[rr, "a"] & & A \\ & G \arrow[ul, "i_f"] \arrow[ur, "f"'] & \end{tikzcd}$$

The fact that G is a regular generator means exactly that, for each object A of $\mathbb{B}$, such cover is a regular epimorphism. Now consider the full subcategory $\mathbb{C}$ of $\mathbb{B}$ spanned by copowers of G.

First step: the full inclusion $F\colon \mathbb{C} \longrightarrow \mathbb{B}$ is a left covering functor. Consider a finite diagram $\mathcal{L}\colon \mathcal{D} \longrightarrow \mathbb{C}$ and its limit in $\mathbb{B}$

$$\lim \mathcal{L} \cdot F = (\pi_D\colon L \longrightarrow \mathcal{L}(D))_{D \in \mathcal{D}_0}$$

We obtain a weak limit in $\mathbb{C}$ precomposing each projection with the canonical cover by G, $l\colon \mathbb{B}(G,L) \bullet G \longrightarrow L$. By assumption, l is a regular epimorphism. This means exactly that F is left covering (recall that, in a regular category, strong epimorphisms coincide with regular epimorphisms). By the universal property of the exact completion $\Gamma\colon \mathbb{C} \longrightarrow \mathbb{C}_{\mathrm{ex}}$, this implies that there exists an exact functor $\hat{F}\colon \mathbb{C}_{\mathrm{ex}} \longrightarrow \mathbb{B}$ such that F and $\Gamma \cdot \hat{F}$ are naturally isomorphic. Let us recall that, if

$$\begin{tikzcd} R \bullet G \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \bullet G \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \bullet G \arrow[r, "f"'] & Y \bullet G \end{tikzcd}$$

is an arrow in $\mathbb{C}_{\mathrm{ex}}$, $\hat{F}[\overline{f}, f]$ is the unique extension to the quotient as in the following diagram

$$\begin{tikzcd} R \bullet G \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\overline{f}"'] & X \bullet G \arrow[r, "q_1"] \arrow[d, "f"] & A \arrow[d, "{\hat{F}[\overline{f}, f]}"] \\ S \bullet G \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y \bullet G \arrow[r, "q_2"] & B \end{tikzcd}$$

Second step: embedding of $\mathbb{B}$ in $\mathbb{C}_{\text{ex}}$.
Given an object A in $\mathbb{B}$, consider its canonical cover by G, $a\colon \mathbb{B}(G,A) \bullet G \longrightarrow A$, the kernel pair of a

$$a_1, a_2\colon N(a) \rightrightarrows \mathbb{B}(G,A) \bullet G$$

and again the canonical cover by G, $n\colon \mathbb{B}(G,N(a)) \bullet G \longrightarrow N(a)$. It is only a straightforward calculation to prove that the pair of arrows

$$n \cdot a_1, n \cdot a_2\colon \mathbb{B}(G,N(a)) \bullet G \rightrightarrows \mathbb{B}(G,A) \bullet G$$

is an object of $\mathbb{C}_{\text{ex}}$. Consider now an arrow $\varphi\colon A \longrightarrow B$ in $\mathbb{B}$; we can build up the following diagram, commutative in each part

$$\begin{tikzcd}
\mathbb{B}(G,N(a)) \bullet G \arrow[r, "n"] \arrow[d, "\overline{\alpha}'"'] & N(a) \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "t"'] & \mathbb{B}(G,A) \bullet G \arrow[r, "a"] \arrow[d, "\alpha'"] & A \arrow[d, "\varphi"] \\
\mathbb{B}(G,N(b)) \bullet G \arrow[r, "m"] & N(b) \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \mathbb{B}(G,B) \bullet G \arrow[r, "b"] & B
\end{tikzcd}$$

The construction of the horizontal lines has just been explained; as far as the columns are concerned, α' is induced by $\alpha\colon \mathbb{B}(G,A) \longrightarrow \mathbb{B}(G,B)$ which sends $h\colon G \longrightarrow A$ into $h \cdot \varphi$; the existence of a unique t such that $t \cdot b_1 = a_1 \cdot \alpha'$ and $t \cdot b_2 = a_2 \cdot \alpha'$ follows from $a \cdot \varphi = \alpha' \cdot b$ and the universal property of $N(b)$; $\overline{\alpha}'$ is induced by $\overline{\alpha}\colon \mathbb{B}(G,N(a)) \longrightarrow \mathbb{B}(G,N(b))$ which sends $h\colon G \longrightarrow N(a)$ into $h \cdot t$. In particular $[\overline{\alpha}', \alpha']$ gives us an arrow in $\mathbb{C}_{\text{ex}}$ which we take as value of a functor $r\colon \mathbb{B} \longrightarrow \mathbb{C}_{\text{ex}}$. Once again, it is a straightforward calculation to prove that r is full and faithful.

Third step: adjunction $\hat{F} \dashv r$.
Let $(r_1, r_2\colon R \bullet G \rightrightarrows X \bullet G)$ be an object in $\mathbb{C}_{\text{ex}}$, consider its coequalizer $q\colon X \bullet G \longrightarrow A$ in $\mathbb{B}$ (that is $A = \hat{F}(r_1, r_2)$) and build up

$$r(A) = (n \cdot a_1, n \cdot a_2\colon \mathbb{B}(G,N(a)) \bullet G \rightrightarrows \mathbb{B}(G,A) \bullet G$$

The unit of the adjunction $\hat{F} \dashv r$ must be an arrow in $\mathbb{C}_{\text{ex}}$ of the following kind

$$\begin{tikzcd}
R \bullet G \arrow[r, "\overline{\eta}'"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \mathbb{B}(G,N(a)) \bullet G \arrow[d, shift right, "n \cdot a_1"'] \arrow[d, shift left, "n \cdot a_2"] \\
X \bullet G \arrow[r, "\eta'"'] & \mathbb{B}(G,A) \bullet G
\end{tikzcd}$$

As η' we take the arrow induced by $\eta: X \longrightarrow \mathbb{B}(G, A)$ which sends $x \in X$ into $i_x \cdot q: G \longrightarrow X \bullet G \longrightarrow A$. Now observe that with this definition $\eta' \cdot a = q$ and then $r_1 \cdot \eta' \cdot a = r_2 \cdot \eta' \cdot a$. This implies that there exists a unique arrow $\tau: R \bullet G \longrightarrow N(a)$ such that $\tau \cdot a_1 = r_1 \cdot \eta'$ and $\tau \cdot a_2 = r_2 \cdot \eta'$. Now as $\overline{\eta}'$ we can take the arrow induced by $\overline{\eta}: R \longrightarrow \mathbb{B}(G, N(a))$ which sends $r \in R$ into $i_r \cdot \tau: G \longrightarrow R \bullet G \longrightarrow N(a)$.

Conclusion: we have just shown that $\mathbb{B}$ is (equivalent to) a localization of $\mathbb{C}_{\text{ex}}$. But, from the preliminaries, we know that the full subcategory $\mathbb{C}$ of $\mathbb{B}$ is equivalent to KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ and then its exact completion $\mathbb{C}_{\text{ex}}$ is equivalent to EM($\mathbb{T}$). The proof of proposition 2.1 is now complete. ■

3 Related results

Let us look more carefully at the proof of proposition 2.1. If, instead of exact, $\mathbb{B}$ is assumed to be only left exact but with coequalizers, we can again define $\hat{F}: \mathbb{C}_{\text{ex}} \longrightarrow \mathbb{B}$ as at the end of the first step. Since in the second and the third steps we do not use the exactness of $\mathbb{B}$, we have the following

Proposition 3.1 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) $\mathbb{B}$ is equivalent to a reflective subcategory of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$

2) $\mathbb{B}$ is left exact with coequalizers and has a regular generator which admits all copowers. ■

Working essentially in the same way (that is working with the formal description of $\mathbb{C}_{\text{ex}}$ and forgetting that it is equivalent to EM($\mathbb{T}$)) one can also prove the following proposition (more details can be found in [7]):

Proposition 3.2 *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

1) $\mathbb{B}$ is equivalent to an epireflective subcategory of EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$ (epireflective = units are regular epimorphisms)

2) $\mathbb{B}$ is regular with coequalizers of equivalence relations and has a regular projective regular generator which admits all copowers. ■

Remarks: i) In proposition 3.1 it does not suffice to assume the existence of coequalizers of pseudo equivalence relations. This is because (with the notations of 2.1) a pseudo equivalence relation in $\mathbb{C}$ is not necessarily a pseudo equivalence relation in the whole category $\mathbb{B}$. On the contrary, in proposition 3.2 we only need coequalizers of equivalence relations because the jointly monic part of the (regular epi, mono) factorization of a pseudo equivalence relation in $\mathbb{C}$ is an equivalence relation in $\mathbb{B}$.

ii) In the characterization of EM($\mathbb{T}$) as well as in proposition 3.2, the regularity of the category is a little bit redundant; in fact the stability of regular epimorphisms under pullbacks follows from the other assumptions (cf. lesson 2 in [4]). This is not true in proposition 2.1, because there the regular generator in general is not regular projective.

iii) Propositions 3.1 and 3.2 (or, at least, the second one) are well-known (see [5], where this kind of characterizations are used to study Malcev conditions in varietal and quasi-varietal categories, [3] and [6]). I have quoted them here because I think it is remarkable that the theory of the exact completion gives us a general framework to prove (in a quite straightforward way) all characterization theorems contained in this work.

iv) All results can be obviously generalized to monads over a power $\mathcal{SET}^X$ of $\mathcal{SET}$. To characterize KL($\mathbb{T}$), we need an X-indexed family of objects G_x in $\mathbb{C}$ such that: i) for each $f\colon S \longrightarrow X$ in $\mathcal{SET}$ there exists $\coprod_{s \in S} G_{f(s)}$; ii) for each object C in $\mathbb{C}$ there exists $f\colon S \longrightarrow X$ in $\mathcal{SET}$ such that $C \simeq \coprod_{s \in S} G_{f(s)}$. Now the other results hold replacing the single generator with an X-indexed family of generators which admit all sums.

v) With a little bit more of effort, one can adapt the proof of proposition 2.1 to obtain an elementary proof of the well-known fact that Giraud axioms characterize localizations of presheaf categories. The major difference is that $\mathbb{C}$ must be the sum-completion of the full subcategory $\mathcal{D}$ of generators, so that $\mathbb{C}_{\mathrm{ex}}$ is equivalent to $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$. For this, we take as arrows from a generator to a sum of generators only the arrows which factor through a canonical injection (generators are indecomposable in $\mathcal{SET}^{\mathcal{D}^{\mathrm{op}}}$). Now the inclusion of $\mathbb{C}$ in $\mathbb{B}$ is not full, but it remains left covering in virtue of the extensivity assumption on $\mathbb{B}$ (sums are disjoint and universal).

4 *

References

[1] **Borceux F.:** Handbook of Categorical Algebra, Cambridge University Press, Cambridge (1994).

[2] **Carboni A. and Vitale E.M.:** *Regular and exact completions,* to appear.

[3] **Keane O.:** *Abstract Horn theories,* SLNM 445, Springer-Verlag, Berlin (1975).

[4] **Lawvere F.W.:** *Category theory over a base topos,* mimeographed notes, Universit di Perugia (1972-1973).

[5] **Pedicchio M.C.:** *On k-permutability for categories of T-algebras,* to appear.

[6] **Tholen W.:** *Relative Bildzerlegungen und algebraische Kategorien,* Ph.D. thesis, Munster (1974).

[7] **Vitale E.M.:** *Left covering functors,* Ph.D. thesis, Louvain-la-Neuve (1994).

[8] **Vitale E.M.:** *On the characterization of monadic categories over Set,* to appear in Cahiers de Topologie et G.D., vol. XXXVI (1995).