Localizations of algebraic categories II

Enrico M. Vitale

Abstract. Using the exact completion of a weakly left exact category, we specialize previous results on monadic categories over $\mathcal{SET}$ to obtain a characterization of localizations of finitary algebraic categories. As a corollary, we have the Popescu-Gabriel representation theorem for Grothendieck categories.

Introduction

Localizations of (finitary) algebraic categories in general, and of module categories in particular, are a widely studied topic in algebra; in this note we look for a characterization of such categories. In fact the following statement is proved:

> a category is equivalent to a localization of an algebraic category if and only if it is exact and has a regular generator which admits all copowers, and directed colimits exist and commute with finite limits.

The idea followed to find these axioms is that algebraic categories are monadic over $\mathcal{SET}$. We can therefore try to specialize to algebraic categories the following fact, proved in [14]: a category is equivalent to a localization of a monadic category over $\mathcal{SET}$ if and only if it is exact and has a regular generator which admits all copowers. For this, we use the exact completion of a weakly left exact category introduced in [4].

Since module categories are exactly abelian algebraic categories, as a corollary we have the Popescu-Gabriel representation theorem for Grothendieck categories.

To recapture algebraic categories and module categories between monadic categories, we use the finitary part of a monad, a construction which exhibits the category of monads with finite rank as a coreflective subcategory of the category of monads over $\mathcal{SET}$.

I would like to thank A. Carboni, Y. Diers and H. Simmons for some useful comments. I have also benefitted from numerous suggestions the anonimous referee has made on an earlier version of this work.

1 Localizations of algebraic categories

Let $\mathbb{A}$ be a locally small category and consider an object A in $\mathbb{A}$ which admits all copowers (if S is a set, we write $S \bullet A$ for the S-indexed copower of A). We have a pair of functors

$$- \bullet A \colon \mathcal{SET} \longrightarrow \mathbb{A} \qquad \mathbb{A}(A, -) \colon \mathbb{A} \longrightarrow \mathcal{SET}$$

with $- \bullet A$ left adjoint to $\mathbb{A}(A, -)$, and we can compare three conditions on A:
i) A is finitely presentable, that is $\mathbb{A}(A, -) \colon \mathbb{A} \longrightarrow \mathcal{SET}$ preserves filtered colimits
ii) the composite $\mathbb{A}(A, - \bullet A) \colon \mathcal{SET} \longrightarrow \mathcal{SET}$ has finite rank, that is preserves filtered colimits
iii) A is abstractly finite, that is each arrow $A \longrightarrow S \bullet A$ factors through the arrow $S' \bullet A \longrightarrow S \bullet A$ induced by the inclusion $S' \hookrightarrow S$ of some finite subset S' of S.
Clearly condition i) implies condition ii). Moreover, condition ii) implies condition iii) because each set is the filtered union of its finite subsets. What about the implication iii) $\Rightarrow$ i)?

Recall that an algebraic category is the category of models of an algebraic theory in the sense of Lawvere (cf. [8]). Algebraic categories can be characterized as follows (cf. [8]):
a category $\mathbb{A}$ is equivalent to an algebraic category if and only if it is exact (in the sense of Barr, cf. [1]), has finite colimits and an abstractly finite regular projective regular generator. Under these conditions (a little bit redundant, cf. [9]), the free-algebra functor is given, up to the equivalence, by

$$- \bullet A \colon \mathcal{SET} \longrightarrow \mathbb{A}$$

where A is the generator.
Since the forgetful functor from an algebraic category to $\mathcal{SET}$ preserves filtered colimits (cf. chapter 3 in [2]), we have that, at least in the presence of the other axioms characterizing algebraic categories, condition iii) implies condition i).

Moreover, it is a well-known fact that algebraic categories are exactly categories of algebras for monads with finite rank.
Now, the category of monads with finite rank is coreflective in the category of monads over $\mathcal{SET}$. This is more or less implicit in Linton's contribution to La Jolla proceedings (cf. [10]). It can also be deduced from the fact that the inclusion of the category of finitary functors into the category of endofunctors of $\mathcal{SET}$ has a right adjoint which is a lax monoidal functor, so that it carries monoids (= monads over $\mathcal{SET}$) into monoids (= monads with finite rank) (cf. [5]).
For the proof of proposition 1.1, we need an explicit description of the finitary coreflection of a monad: let $\mathbb{T} = (T, \mu, \eta)$ be a monad over $\mathcal{SET}$; consider the

corresponding Kleisli category together with the free algebra functor and the forgetful functor

$$F, U \colon KL(\mathbb{T}) \rightleftarrows \mathcal{SET} \ .$$

If G is the free algebra over the singleton, F and U are given, up to natural isomorphisms, by

$$- \bullet G \colon \mathcal{SET} \longrightarrow KL(\mathbb{T}) \ \text{ and } \ KL(\mathbb{T})(G, -) \colon KL(\mathbb{T}) \longrightarrow \mathcal{SET} \ .$$

Now consider the subcategory $\mathbb{C}$ of $KL(\mathbb{T})$ with the same objects of $KL(\mathbb{T})$ and take as arrows $f \colon R \bullet G \longrightarrow S \bullet G$ the arrows of $KL(\mathbb{T})$ such that, for each $r \in R$, $\sigma_r \cdot f \colon G \longrightarrow R \bullet G \longrightarrow S \bullet G$ factors through $S' \bullet G \longrightarrow S \bullet G$ for some finite subset S' of S, where $\sigma_r \colon G \longrightarrow R \bullet G$ is the $r-$th injection in the coproduct. We obtain a new adjunction

$$- \bullet G, \mathbb{C}(G, -) \colon \mathbb{C} \rightleftarrows \mathcal{SET} \ .$$

The monad $\mathbb{T}' = (T', \mu', \eta')$ over $\mathcal{SET}$ induced by this new adjunction is the finitary part of $\mathbb{T}$. In fact, $KL(\mathbb{T}')$ is equivalent to $\mathbb{C}$ and G is abstractly finite in $\mathbb{C}$, so that $\mathbb{T}'$ has finite rank. The counit

$$\epsilon \colon \mathbb{T}' \longrightarrow \mathbb{T}$$

is given, for each set S, by the inclusion $\mathbb{C}(G, S \bullet G) \hookrightarrow KL(\mathbb{T})(G, S \bullet G)$.

Let us fix some more notations: the counit $\epsilon \colon \mathbb{T}' \longrightarrow \mathbb{T}$ induces a functor between Eilenberg-Moore categories

$$\epsilon^* \colon EM(\mathbb{T}) \longrightarrow EM(\mathbb{T}')$$

which has a left adjoint

$$\hat{\epsilon} \colon EM(\mathbb{T}') \longrightarrow EM(\mathbb{T}) \ .$$

The functor ϵ^* is faithful but, in general, it is not full (think about compact Hausdorff spaces or about Sup-lattices). Now consider a localization $\mathbb{B}$ of $EM(\mathbb{T})$

$$L, I \colon \mathbb{B} \rightleftarrows EM(\mathbb{T}) \qquad L \dashv I \ .$$

Up to replacing $\mathbb{T}$ with the monad induced by the adjunction

$$- \bullet G, \mathbb{B}(G, -) \colon \mathbb{B} \rightleftarrows \mathcal{SET} \ ,$$

where G is the reflection in $\mathbb{B}$ of the free $\mathbb{T}-$algebra on the singleton, we can suppose that the full subcategory $KL(\mathbb{T})$ of free $\mathbb{T}-$algebras is contained in $\mathbb{B}$ (cf. [14]).

Proposition 1.1 *With the above notations and conventions, if in $\mathbb{B}$ directed colimits commute with finite limits, then the composite*

$$I \cdot \epsilon^* \colon \mathbb{B} \longrightarrow EM(\mathbb{T}) \longrightarrow EM(\mathbb{T}')$$

is full and faithful, and the composite

$$\hat{\epsilon} \cdot L \colon EM(\mathbb{T}') \longrightarrow EM(\mathbb{T}) \longrightarrow \mathbb{B}$$

is left exact. In particular, $\mathbb{B}$ is equivalent to a localization of the algebraic category $EM(\mathbb{T}')$.

Proof: To follow as closely as possible the proof of proposition 2.1 in [14], we use the theory of the exact completion (cf. [4]). Recall that, if $\mathbb{C}$ is a weakly left exact category, its exact completion $\mathbb{C}_{\mathrm{ex}}$ has, for objects, pseudo equivalence relation $r_0, r_1 \colon R \rightrightarrows X$ in $\mathbb{C}$; an arrow in $\mathbb{C}_{\mathrm{ex}}$ is a class of equivalence of pairs of arrows $(\overline{f}, f)$ such that $\overline{f} \cdot s_0 = r_0 \cdot f$ and $\overline{f} \cdot s_1 = r_1 \cdot f$

$$\begin{tikzcd} R \arrow[r, "\overline{f}"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & S \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

two pairs $(\overline{f}, f)$ and $(\overline{g}, g)$ of this kind are equivalent if there exists an arrow $\Sigma \colon X \longrightarrow S$ such that $\Sigma \cdot s_0 = f$ and $\Sigma \cdot s_1 = g$.

Let G be the free $\mathbb{T}-$algebra on the singleton; $KL(\mathbb{T}')$ is equivalent to the (not full) subcategory $\mathbb{C}$ of $\mathbb{B}$ having for objects the copowers of G and for arrows the arrows $f \colon R \bullet G \longrightarrow S \bullet G$ such that, for each $r \in R$, $\sigma_r \cdot f \colon G \longrightarrow R \bullet G \longrightarrow S \bullet G$ factors through $S' \bullet G \longrightarrow S \bullet G$ for some finite subset S' of S. Via the equivalences

$$(KL(\mathbb{T}))_{ex} \simeq EM(\mathbb{T}) \quad \text{and} \quad \mathbb{C}_{ex} \simeq (KL(\mathbb{T}'))_{ex} \simeq EM(\mathbb{T}')$$

(cf. [4]), the composite

$$I \cdot \epsilon^* \colon \mathbb{B} \longrightarrow EM(\mathbb{T}) \longrightarrow EM(\mathbb{T}')$$

can be described as follows: given an object A in $\mathbb{B}$, consider its canonical cover

$$a \colon \mathbb{B}(G, A) \bullet G \longrightarrow A$$

together with its kernel pair

$$a_0, a_1 \colon N(a) \rightrightarrows \mathbb{B}(G, A) \bullet G \,.$$

Now let $\mathbb{C}(G,a)$ be the set of arrows $y: G \longrightarrow N(a)$ such that the two composites $y \cdot a_0: G \longrightarrow \mathbb{B}(G,A) \bullet G$ and $y \cdot a_1: G \longrightarrow \mathbb{B}(G,A) \bullet G$ are in $\mathbb{C}$ and consider the canonical factorization

$$n: \mathbb{C}(G,a) \bullet G \longrightarrow N(a) \ .$$

Then

$$i(A) = (n \cdot a_0, n \cdot a_1: \mathbb{C}(G,a) \bullet G \rightrightarrows \mathbb{B}(G,A) \bullet G)$$

is the value on A of the functor

$$i: \mathbb{B} \longrightarrow \mathbb{C}_{ex}$$

corresponding to $I \cdot \epsilon^*$.
Consider now an arrow $\varphi: A \longrightarrow B$ in $\mathbb{B}$; we can build up the following diagram, commutative in each part

$$\begin{tikzcd}
\mathbb{C}(G,a) \bullet G \arrow[r, "n"] \arrow[d, "\overline{\alpha}'"'] & N(a) \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] \arrow[d, "t"'] & \mathbb{B}(G,A) \bullet G \arrow[r, "a"] \arrow[d, "\alpha'"] & A \arrow[d, "\varphi"] \\
\mathbb{C}(G,b) \bullet G \arrow[r, "m"] & N(b) \arrow[r, shift left, "b_0"] \arrow[r, shift right, "b_1"'] & \mathbb{B}(G,B) \bullet G \arrow[r, "b"] & B
\end{tikzcd}$$

The construction of the horizontal lines has just been explained; as far as the columns are concerned, α' is induced by $\alpha: \mathbb{B}(G,A) \longrightarrow \mathbb{B}(G,B)$ which sends $h: G \longrightarrow A$ into $h \cdot \varphi$; the existence of a unique t such that $t \cdot b_0 = a_0 \cdot \alpha'$ and $t \cdot b_1 = a_1 \cdot \alpha'$ follows from $a \cdot \varphi = \alpha' \cdot b$ and the universal property of $N(b)$; $\overline{\alpha}'$ is induced by $\overline{\alpha}: \mathbb{C}(G,a) \longrightarrow \mathbb{C}(G,b)$ which sends $h: G \longrightarrow N(a)$ into $h \cdot t$. In particular $[\overline{\alpha}', \alpha']$ gives us an arrow in $\mathbb{C}_{ex}$ which we take as value of a functor $r: \mathbb{B} \longrightarrow \mathbb{C}_{ex}$.

The faithfulness of $i: \mathbb{B} \longrightarrow \mathbb{C}_{ex}$ is obvious. As far as its fulness is concerned, consider an arrow $[\overline{f}, f]: i(A) \longrightarrow i(B)$ in $\mathbb{C}_{ex}$. If we can show that $n: \mathbb{C}(G,a) \bullet G \longrightarrow N(a)$ is an epimorphism, then $a: \mathbb{B}(G,A) \bullet G \longrightarrow A$ is the coequalizer of the pair $(n \cdot a_0, n \cdot a_1)$. This implies that the pair $(\overline{f}, f)$ gives rise to a unique extension $\varphi: A \longrightarrow B$ to the quotient, and it is straightforward to prove that $i(\varphi) = [\overline{f}, f]$.
So, the crucial point is to show that $n: \mathbb{C}(G,a) \bullet G \longrightarrow N(a)$ is a (regular) epimorphism. This is certainly true if $\mathbb{B}(G,A)$ is finite, because in this case $\mathbb{C}(G,a) = \mathbb{B}(G, N(a))$ and G is a regular generator (in $EM(\mathbb{T})$ and then in $\mathbb{B}$). In general, recall that $\mathbb{B}(G,A)$ is the directed union of its finite subsets and then $\mathbb{B}(G,A) \bullet G$ is the directed colimit

$$\mathrm{colim}_{\mathcal{S}}(S' \bullet G)$$

in $\mathbb{B}$, where $\mathcal{S}$ is the partially ordered set of finite subsets S' of $\mathbb{B}(G, A)$. Now, consider the following diagram, where the horizontal lines are kernel pairs, $a_{S'} = i_{S'} \cdot a$ and $j_{S'}$ is induced by the canonical injection $i_{S'}$ via the universal property of $N(a)$

$$\begin{tikzcd}
N(a_{S'}) \arrow[r, shift left, "a_{S',0}"] \arrow[r, shift right, "a_{S',1}"'] \arrow[d, "j_{S'}"'] & S' \bullet G \arrow[r, "a_{S'}"] \arrow[d, "i_{S'}"'] & A \arrow[d, "\mathrm{id}"] \\
N(a) \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \mathbb{B}(G, A) \bullet G \arrow[r, "a"] & A
\end{tikzcd}$$

Since S' is finite, $n_{S'} \colon \mathbb{C}(G, a_{S'}) \bullet G \longrightarrow N(a_{S'})$ is a regular epimorphism and, since $\mathbb{B}$ has kernel pairs, we can use an interchange argument for colimits to obtain a regular epimorphism $\vec{n}$ at the level of colimits

$$\begin{tikzcd}
\mathrm{colim}_{\mathcal{S}}(\mathbb{C}(G, a_{S'}) \bullet G) \arrow[d, "\vec{n}"'] & \mathbb{C}(G, a_{S'}) \bullet G \arrow[l] \arrow[d, "n_{S'}"] \\
\mathrm{colim}_{\mathcal{S}} N(a_{S'}) & N(a_{S'}) \arrow[l]
\end{tikzcd}$$

Moreover, the various $j_{S'} \colon N(a_{S'}) \longrightarrow N(a)$ give rise to a unique factorization

$$j \colon \mathrm{colim}_{\mathcal{S}} N(a_{S'}) \longrightarrow N(a) \ .$$

For each finite subset S' of S, we can define

$$\alpha_{S'} \colon \mathbb{C}(G, a_{S'}) \longrightarrow \mathbb{C}(G, a) \quad y \mapsto y \cdot j_{S'} \colon G \longrightarrow N(a_{S'}) \longrightarrow N(a)$$

and we can consider the induced arrow

$$\alpha'_{S'} \colon \mathbb{C}(G, a_{S'}) \bullet G \longrightarrow \mathbb{C}(G, a) \bullet G \ .$$

The various $\alpha'_{S'}$ give rise to a unique factorization

$$\alpha \colon \mathrm{colim}_{\mathcal{S}}(\mathbb{C}(G, a_{S'}) \bullet G) \longrightarrow \mathbb{C}(G, a) \bullet G \ .$$

Finally, we have built up a commutative square

$$\begin{array}{ccc}\mathrm{colim}_{\mathcal{S}}(\mathbb{C}(G,a_{S'})\bullet G) & \xrightarrow{\vec{n}} & \mathrm{colim}_{\mathcal{S}}N(a_{S'}) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle j} \\ \mathbb{C}(G,a)\bullet G & \xrightarrow[n]{} & N(a)\end{array}$$

Since in $\mathbb{B}$ directed colimits commute with finite limits, j is an isomorphism and then n is a regular epimorphism.

Now we will show that the (not full) inclusion $F: \mathbb{C} \longrightarrow \mathbb{B}$ is a left covering functor (so that it extends in an essentially unique way to an exact functor $\hat{F}: \mathbb{C}_{ex} \longrightarrow \mathbb{B}$, cf. [4]). This means that, for each finite category $\mathcal{D}$ and for each functor $\mathcal{L}: \mathcal{D} \longrightarrow \mathbb{C}$, if L is a weak limit of $\mathcal{L}$ and L' is the limit of $\mathcal{L} \cdot F: \mathcal{D} \longrightarrow \mathbb{C} \longrightarrow \mathbb{B}$, then the comparison $F(L) \longrightarrow L'$ must be a regular epimorphism. Since $\mathbb{B}$ is regular, it suffices to show the left covering character of F with respect to the terminal object, binary products and equalizers of pairs of parallel arrows (cf. [4]). Let us start with the case of equalizers: consider two parallel arrows in $\mathbb{C}$ and their equalizer in $\mathbb{B}$

$$E \xrightarrow{\quad e \quad} S \bullet G \underset{g}{\overset{f}{\rightrightarrows}} R \bullet G\ .$$

Let $\mathbb{C}(G,e)$ be the set of arrows $x: G \longrightarrow E$ such that $x \cdot e: G \longrightarrow E \longrightarrow S \bullet G$ is in $\mathbb{C}$ and consider the canonical factorization

$$\Sigma: \mathbb{C}(G,e) \bullet G \longrightarrow E\ .$$

Then $\Sigma \cdot e: \mathbb{C}(G,e) \bullet G \longrightarrow E \longrightarrow S \bullet G$ is a weak equalizer of f and g in $\mathbb{C}$ and we have to show that Σ is a regular epimorphism. This is true if S is finite, because in this case $\mathbb{C}(G,e) = \mathbb{B}(G,E)$ and G is a regular generator. The general case follows once again by induction, using that S is the directed colimit of its finite subsets and using the commutativity of directed colimits and finite limits. The case of binary products is similar, but it needs two inductions, one for each factor of the product. The case of the terminal object is trivial because G is a weak terminal object in $\mathbb{C}$.

The proof that $\hat{F}: \mathbb{C}_{ex} \longrightarrow \mathbb{B}$ is left adjoint to $i: \mathbb{B} \longrightarrow \mathbb{C}_{ex}$ runs, up to straightforward modifications, as in the third step of the proof of proposition 2.1 in [14] and we omit details. ∎

We can restate the previous proposition as a characterization theorem.

Corollary 1.2 *Let $\mathbb{B}$ be a category; the following conditions are equivalent:*

1) $\mathbb{B}$ is equivalent to a localization of an algebraic category

2) *$\mathbb{B}$ is exact and has a regular generator G which admits all copowers, and in $\mathbb{B}$ directed colimits exist and commute with finite limits.*

Proof: The implication 1⇒2 is known, since in an algebraic category filtered colimits commute with finite limits and since this condition is stable under localizations (cf. chapter 3 in [2]). As far as the implication 2⇒1 is concerned, by proposition 2.1 (and its proof) in [14], $\mathbb{B}$ is equivalent to a localization of $EM(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SET}$ and $KL(\mathbb{T})$ is a full subcategory of $\mathbb{B}$. Now proposition 1.1 shows that $\mathbb{B}$ is equivalent to a localization of $EM(\mathbb{T}')$, where $\mathbb{T}'$ is the finitary part of $\mathbb{T}$. ■

The following example, provided to me by R. Boerger, shows that a category may be monadic over $\mathcal{SET}$ and have exact filtered colimits without being algebraic.

Example 1.3 Let $\mathcal{SET}_*$ denote the category of pointed sets; write 0 for the zero object (that is, the singleton) and P for the two-elements set. Let $\mathbb{A} = (\mathcal{SET}_*)^I$ be the $I-$power of $\mathcal{SET}_*$ for any infinite set I. The category $\mathbb{A}$ is monadic over $\mathcal{SET}$ since it is exact and has a regular projective regular generator (take P in each component). An object X of $\mathbb{A}$ is a regular generator iff it is 0 in no component, and X is finitely presentable iff it is 0 in all but finitely many components and finite in all components. So $\mathbb{A}$ does not have a finitely presentable regular generator and then it is not algebraic. Nevertheless, $\mathbb{A}$ is locally presentable (the set of generators being given by the objects $P(i)$ having P in the $i-$th component and 0 in all the other components) and then in $\mathbb{A}$ filtered colimits commute with finite limits.

2 Localizations of module categories

Cocomplete abelian categories with exact directed colimits (Ab5 categories in [6]) and with a generator are known as Grothendieck categories. The next characterization theorem has been originally proved by Gabriel and Popescu using Gabriel topologies (cf. [11] and [12]). We obtain it specializing corollary 1.2 to module categories, which is an easy matter because the condition to be abelian is stable under localizations. Recall that an abelian category is algebraic if and only if it is the category of modules over an associative ring with unit (cf. [8]).

Proposition 2.1 *Let $\mathbb{B}$ be a category; the following conditions are equivalent:*

1) *$\mathbb{B}$ is equivalent to a localization of a module category*

2) *$\mathbb{B}$ is abelian and has a generator G which admits all copowers, and in $\mathbb{B}$ directed colimits exist and commute with finite limits.*

Proof: We use the same notations as in the proof of proposition 1.1. We only have to prove that $\mathbb{C}_{ex}$ is abelian and, for this, preadditivity is enough (cf. chapter 2 in [2]). But $\mathbb{C}_{ex}$ is preadditive if and only if $\mathbb{C}$ is preadditive (same argument as for the exact completion of a left exact category, cf. [3]). It remains only to show that, for each pair of sets R and S, $\mathbb{C}(R \bullet G, S \bullet G)$ is a subgroup of $\mathbb{B}(R \bullet G, S \bullet G)$. We limit to the case $\mathbb{C}(G, S \bullet G)$; the general case follows easily.
- the zero morphism $0\colon G \longrightarrow S \bullet G$ is in $\mathbb{C}$ since it factors through the initial object $\emptyset \bullet G$;
- if $f\colon G \longrightarrow S \bullet G$ factors as $f' \cdot i_{S'}\colon G \longrightarrow S' \bullet G \longrightarrow S \bullet G$, then $-f$ factors as $(-f') \cdot i_{S'}$;
- if $f, g\colon G \longrightarrow S \bullet G$ are in $\mathbb{C}$, we can build up their sum using the additivity of $\mathbb{B}$ in the following way

$$G \xrightarrow{\quad \triangle \quad} G \oplus G \xrightarrow{f \oplus g} S \bullet G \oplus S \bullet G \xrightarrow{\quad \triangledown \quad} S \bullet G$$

where $\triangle$ and $\triangledown$ are the diagonal and the codiagonal. Clearly $\triangle$ is in $\mathbb{C}$ because $G \oplus G$ is a finite copower. Now we can choose $(S \amalg S) \bullet G$ as $S \bullet G \oplus S \bullet G$ (where $\amalg$ is the disjoint union in $\mathcal{SET}$) and $f \oplus g$ is in $\mathbb{C}$ because if we precompose with the first injection we obtain $f \cdot i_1\colon G \longrightarrow S \bullet G \longrightarrow (S \amalg S) \bullet G$ (where i_1 is induced by the first injection $S \longrightarrow S \amalg S$) and f is in $\mathbb{C}$ and analogously if we precompose with the second injection. Finally the codiagonal $\triangledown\colon (S \amalg S) \bullet G \longrightarrow S \bullet G$ is in $\mathbb{C}$ because it is induced by the codiagonal $S \amalg S \longrightarrow S$ in $\mathcal{SET}$. ■

Remark 2.2 All the previous results can be generalized replacing finite limits and filtered colimits by $\kappa-$limits and $\kappa-$filtered colimits for a regular infinite cardinal κ. Algebraic theories must then be replaced by theories with operations of ariety at most κ, monads with finite rank by monads with rank κ (that is, preserving $\kappa-$filtered colimits) and abstractly finite objects by objects satisfying the obvious $\kappa-$analogous condition. In particular, in proposition 2.1, categories of modules over an associative ring with unit must be replaced by categories of models of a $\kappa-$algebraic theory $\mathcal{T}$ such that $\mathcal{T} \otimes \mathcal{Z} \simeq \mathcal{T}$, where $\otimes$ is the Kronecker product of theories and $\mathcal{Z}$ is the theory of abelian groups (cf. [15]). To adapt the various proofs, one need the "$\kappa-$fication" of the theory of the exact completion. This has been done in [7].

3 *

References

[1] M. Barr: Exact categories, Lecture Notes in Math. 236, Springer Verlag (1971) 1-120.

[2] F. Borceux: Handbook of Categorical Algebra 2, Encyclopedia of Math. 51, Cambridge University Press (1994).

[3] A. Carboni and R. Celia Magno: The free exact category on a left exact one, Jour. Austral. Math. Soc. (Series A) 33 (1982) 295-301.

[4] A. Carboni and E.M. Vitale: Regular and exact completions, Jour. Pure Appl. Algebra (to appear).

[5] B. Day: Note on monoidal localizations, Bull. Austral. Math. Soc. 8 (1973) 1-16.

[6] A. Grothendieck: Sur quelques points d'algèbre homologique, Tohoku Math. Jour. 9 (1957) 119-221.

[7] H. Hu and W. Tholen: A note on free regular and exact completions, and their infinitary generalizations, Theory Appl. Categories 2 (1996) 113-132.

[8] F.W. Lawvere: Functorial semantics of algebraic theories, Proc. Nat. Acad. Sci. 50 (1963) 869-872.

[9] F.W. Lawvere: Category theory over a base topos, University of Perugia (1973).

[10] F. Linton: Some aspects of equational categories, Proc. Conf. Categor. Algebra, La Jolla 1965, Springer Verlag (1966).

[11] N. Popescu and P. Gabriel: Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, Comp. Rend. Acad. Sc. Paris 258 (1964) 4188-4190.

[12] B. Stenström: Rings of quotients, Die Grundlehren der Math. 217, Springer Verlag (1975).

[13] E.M. Vitale: On the characterization of monadic categories over Set, Cahiers de Topologie et G.D. XXXV-4 (1994) 351-358.

[14] E.M. Vitale: Localizations of algebraic categories, Jour. Pure Appl. Algebra 108 (1996) 315-320.

[15] G.C. Wraith: Algebraic theories, Lecture Notes Series 22, Matematisk Institut, Aarhus Universitet (1970).

Enrico M. Vitale
Laboratoire LANGAL, Faculté de Sciences, Université du Littoral
quai Freycinet 1, B.P. 5526, 59379 Dunkerque, France
vitale@lma.univ-littoral.fr