Groupoids and the Brauer group

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Résumé. On utilise les bigroupoïdes pour analyser la suite exacte reliant le groupe de Picard et le groupe de Brauer, aussi bien que la description K-théorétique des groupes de Picard et de Brauer.

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1 Preliminaries

The existence of an exacte sequence between the Picard group and the Brauer group of a commutative unital ring is a well known fact in algebraic K-theory. Similar exact sequences have been obtained starting for example from a ringed space, a Krull domain, etc. [1, 9, 16, 20, 23, 24, 25]. All these constructions are particular cases of the exact sequence between the Picard group and the Brauer group of a symmetric monoidal category [9, 10, 26].

Nevertheless, there are generalizations/enrichements of the Brauer group and of the related exact sequences which do not fit into the general theory developed in [10, 26]. Two interesting (and, for us, motivating) examples are the Brauer-Taylor group and the categorical Brauer group [5, 6, 7, 17, 18, 19, 21, 22]. We look for a more general approach which contains these new examples.

What makes this possible is that it remains a deep analogy between the Brauer group and these generalizations: they can be described as a kind of Picard group associated to a convenient monoidal bicategory [5, 12, 18, 19]. The aim of our note is to show that this fact suffices to obtain some

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relevant results: the above mentioned exact sequence and the K-theoretical description of Picard and Brauer groups.

Let us describe now the general situation using the formalism of bicategories and tricategories. For the basic definitions, the reader can consult [3, 8, 11]. The situation to be kept in mind is given in the following commutative diagram.

$$\begin{tikzcd}
\text{Tricategories} \arrow[r, "cl_2"] \arrow[d, "P_3"'] & \text{Bicategories} \arrow[r, "cl_1"] \arrow[d, "P_2"'] & \text{Categories} \arrow[d, "P_1"'] \\
\text{Trigroupoids} \arrow[r, "cl_2"] & \text{Bigroupoids} \arrow[r, "cl_1"] & \text{Groupoids}
\end{tikzcd}$$

where P_i takes invertibles at each level (for an n-cell invertible means invertible up to invertible $n+1$-cell); cl_1 is the classifying category of a bicategory as in [3] (take 2-isomorphism classes of 1-cells as arrows) and cl_2 is the analogous construction performed taking 3-isomorphism classes of 2-cells. Given a monoidal category (i.e., a bicategory with a single 0-cell) $\mathbb{C}$, its Picard group is the group $P_1(cl_1(\mathbb{C}))$. If $\mathbb{C}$ is symmetric and has stable coequalizers (cf. [8, 26]), we can build up the monoidal bicategory (i.e., tricategory with a single 0-cell) $\mathrm{Mon}\mathbb{C}$ of unital monoids and unital bimodules. Then the Brauer group of $\mathbb{C}$ is the group $P_1(cl_1(cl_2(\mathrm{Mon}\mathbb{C})))$.

In the final remark to section 2 in [26], it is suggested to take $\mathrm{Mon}\mathbb{C}$ as primitive notion. The point is that, despite of the previous definitions of Picard and Brauer groups, the majority part of the computations are performed passing through

$$\text{Bicategories} \xrightarrow{P_2} \text{Bigroupoids} \xrightarrow{cl_1} \text{Groupoids}$$

instead of

$$\text{Bicategories} \xrightarrow{cl_1} \text{Categories} \xrightarrow{P_1} \text{Groupoids}.$$

Once this is clearly recognized, it becomes reasonable to take as primitive the bigroupoid $\mathbb{B} = P_2(cl_2(\mathrm{Mon}\mathbb{C}))$, called in [27] the Brauer cat-group of $\mathbb{C}$. We test this point of view in the next two sections.

2 The K-theoretical description

For the definition of the Grothendieck group K_0 and of the Whitehead group K_1 the reader can see [2]. Let $\mathbb{B} = (\mathbb{B}, \otimes, I, ()^*, \ldots)$ be a compact closed groupoid (or symmetric cat-group), i.e. a bigroupoid with a single 0-cell and symmetric (braided is enough) as monoidal category [13, 14, 15, 27].

Proposition 1 *(i)* $K_0(\mathbb{B})$ *is isomorphic to* $cl_1(\mathbb{B})$

(ii) $K_1(\mathbb{B})$ *is isomorphic to* $\mathbb{B}(I, I)$ *(the group of automorphisms of* $\mathbb{B}$ *at* I*).*

Proof (i) is obvious from the universal property of K_0.
(ii) Consider the category $\Omega\mathbb{B}$ whose objects are arrows $f : A \to A$ in $\mathbb{B}$ and whose arrows are commutative diagrams of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & A \arrow[d, "x"] \\ B \arrow[r, "g"'] & B. \end{tikzcd}$$

$\Omega\mathbb{B}$ is a monoidal category with a composition between objects. To have an isomorphism between $K_1(\mathbb{B})$ and $\mathbb{B}(I, I)$, we need a surjective map γ from the objects of $\Omega\mathbb{B}$ to $\mathbb{B}(I, I)$, constant on connected components, which sends tensor product and composition on the composition of $\mathbb{B}(I, I)$, and such that the following condition holds: if $\gamma(f : A \to A) = (1_I : I \to I)$, then $f : A \to A$ is isomorphic to $1_A : A \to A$ in $\Omega\mathbb{B}$. We define γ in the following way:

$$\gamma(f : A \to A) = I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{f \otimes 1_{A^*}} A \otimes A^* \xrightarrow{\eta_A^{-1}} I$$

where η_A is the unit of the duality $A^* \dashv A$. The proof that γ verifies the various conditions is a quite straightforward step-by-step transcription of the proof of proposition 2.4 in [26]. For this reason, we will only show that γ sends tensor product on composition. This probably is the simplest one between the five conditions, but it seems to us enough to give the flavour of the proof and to illustrate the role of the compact closed structure of $\mathbb{B}$.

Given two objects $f : A \to A$ and $g : B \to B$ in $\mathbb{B}$, we have to show that the following diagram commutes

$$\begin{tikzcd} A \otimes B \otimes (A \otimes B)^* \arrow[rr, "f \otimes g \otimes 1"] & & A \otimes B \otimes (A \otimes B)^* \arrow[d, "\eta_{A \otimes B}^{-1}"] \\ I \arrow[u, "\eta_{A \otimes B}"] \arrow[r, "\eta_A \cdot (f \otimes 1) \cdot \eta_A^{-1}"'] & I \arrow[r, "\eta_B \cdot (g \otimes 1) \cdot \eta_B^{-1}"'] & I \end{tikzcd}$$

But, as in every compact closed category, there is a natural isomorphism $u_{AB} : (A \otimes B)^* \to B^* \otimes A^*$ such that

$$\begin{tikzcd}
I \arrow[r, "\eta_{A\otimes B}"] \arrow[d, "\eta_A"'] & A \otimes B \otimes (A \otimes B)^* \arrow[d, "1\otimes 1\otimes u_{AB}"] \\
A \otimes A^* \cong A \otimes I \otimes A^* \arrow[r, "1\otimes\eta_B\otimes 1"'] & A \otimes B \otimes B^* \otimes A^*
\end{tikzcd}$$

commutes. Then, the following diagram also commutes

$$\begin{tikzcd}
A \otimes B \otimes (A \otimes B)^* \arrow[r, "f\otimes g\otimes 1"] & A \otimes B \otimes (A \otimes B)^* \arrow[d, "\eta^{-1}_{A\otimes B}"] \\
I \arrow[u, "\eta_{A\otimes B}"] \arrow[d, "\eta_A"'] & I \\
A \otimes A^* \cong A \otimes I \otimes A^* \arrow[d, "1\otimes\eta_B\otimes 1"'] & A \otimes I \otimes A^* \cong A \otimes A^* \arrow[u, "\eta_A^{-1}"'] \\
A \otimes B \otimes B^* \otimes A^* \arrow[r, "f\otimes g\otimes 1\otimes 1"'] & A \otimes B \otimes B^* \otimes A^* \arrow[u, "1\otimes\eta_B^{-1}\otimes 1"']
\end{tikzcd}$$

Now the fact that

$$\begin{tikzcd}
I \arrow[r, "\eta_A\cdot(f\otimes 1)\cdot\eta_A^{-1}"] \arrow[d, "\eta_A"'] & I \arrow[r, "\eta_B\cdot(g\otimes 1)\cdot\eta_B^{-1}"] & I \\
A \otimes A^* \cong A \otimes I \otimes A^* \arrow[d, "1\otimes\eta_B\otimes 1"'] & & A \otimes I \otimes A^* \cong A \otimes A^* \arrow[u, "\eta_A^{-1}"'] \\
A \otimes B \otimes B^* \otimes A^* \arrow[rr, "f\otimes g\otimes 1\otimes 1"'] & & A \otimes B \otimes B^* \otimes A^* \arrow[u, "1\otimes\eta_B^{-1}\otimes 1"']
\end{tikzcd}$$

commutes is a routine diagram argument using the functoriality of $\otimes$ and the various natural and coherent isomorphisms of a symmetric monoidal category. $\square$

3 The exact sequence

The K-theoretical description given in the previous proposition is the key result used in [2, 10, 9] to obtain a Picard-Brauer exact sequence. Here we sketch a more direct method, which follows the lines of section 2 in [26].

For this, consider two compact closed groupoids $\mathbb{A}$ and $\mathbb{B}$ and a monoidal functor $F : \mathbb{A} \to \mathbb{B}$. We can construct the abelian group $\mathcal{F}$ having as elements classes of triples (A_1, b, A_2) with A_1, A_2 in $\mathbb{A}$ and $b : FA_1 \to FA_2$ in $\mathbb{B}$. Two triples (A_1, b, A_2) and (A'_1, b', A'_2) are equivalent if there exist $a_1 : A_1 \to A'_1$, $a_2 : A_2 \to A'_2$ in $\mathbb{A}$ such that

$$\begin{tikzcd} FA_1 \arrow[r, "b"] \arrow[d, "Fa_1"'] & FA_2 \arrow[d, "Fa_2"] \\ FA'_1 \arrow[r, "b'"'] & FA'_2 \end{tikzcd}$$

commutes. The operation in $\mathcal{F}$ is induced by the tensor product in $\mathbb{B}$. Now consider the subgroup $\mathcal{N}$ of $\mathcal{F}$ spanned by the elements of the form $[A, 1_{FA}, A]$ for A in $\mathbb{A}$, and take the quotient group $\pi : \mathcal{F} \to \mathcal{F}/\mathcal{N} = \overline{\mathcal{F}}$. There are also two morphisms:

- $F_1 : \mathbb{B}(I, I) \to \mathcal{F}, \quad (b : I \to I) \mapsto [I,\ FI \cong I \xrightarrow{b} I \cong FI\ , I]$
- $F_2 : \mathcal{F} \to cl_1(\mathbb{A}), \quad [A_1, b, A_2] \mapsto [A_1 \otimes A_2^*].$

Moreover, since $\mathcal{N}$ is contained in the kernel of F_2, the morphism F_2 factors through π; let $F'_2 : \overline{\mathcal{F}} \to cl_1(\mathbb{A})$ be the factorisation.

Proposition 2 *With the previous notations, the sequence of abelian groups and morphisms*

$$\mathbb{A}(I, I) \longrightarrow \mathbb{B}(I, I) \xrightarrow{F_1 \cdot \pi} \overline{\mathcal{F}} \xrightarrow{F'_2} cl_1(\mathbb{A}) \longrightarrow cl_1(\mathbb{B})$$

is exact.

Proof A direct proof can be done following the proof of Proposition 2.2 in [26]. Alternatively, one can work in the following way: first observe that, even if $\mathbb{B}$ in general does not have coequalizers, one can construct Mon$\mathbb{B}$ because, up to isomorphism, the unique monoid in $\mathbb{B}$ is I and then the needed coequalizers are trivial. Now, following [27], we obtain a 2-exact sequence of symmetric cat-groups

$$\mathbb{A} \to \mathbb{B} \to \mathbb{F} \to cl_2(\text{Mon}\mathbb{A}) \to cl_2(\text{Mon}\mathbb{B}).$$

Taking, for each cat-group, the abelian group of automorphisms of the unit object, we have the requested exact sequence of abelian groups. (As far as $cl_2(\mathrm{Mon}\mathbb{B})$ is concerned, observe that it is the Brauer cat-group of $\mathbb{B}$ [27], so that its group of automorphisms is the Picard group of $\mathbb{B}$. Since $\mathbb{B}$ is already a compact closed groupoid, its Picard group is nothing but $cl_1(\mathbb{B})$.) $\square$

4 Conclusion

The interest of our technique is that now we can apply Proposition 1 and 2 choosing as compact closed groupoids those defined by convenient monoidal bicategories.

1. Consider a unital commutative ring R and let $\mathbb{C} = R$-mod be the category of R-modules. If we put $\mathbb{B} = P_2(cl_2(\mathrm{Mon}\mathbb{C}))$, Proposition 1 gives the well-known K-theoretical interpretation of the Picard and Brauer groups of R, and Proposition 2 gives the classical Picard-Brauer exact sequence.

2. With $\mathbb{C} = R$-mod as in the previous example, instead of $\mathrm{Mon}\mathbb{C}$ consider the monoidal bicategory $\mathrm{Mon}^{reg}\mathbb{C}$ of regular (but not necessarily unital) R-algebras and regular bimodules [12]. Then, taking $\mathbb{B} = P_2(cl_2(\mathrm{Mon}^{reg}\mathbb{C}))$, Proposition 2 gives an exact sequence connecting the Picard and the Brauer-Taylor groups of R, and Proposition 1 provides a K-theoretical interpretation of the Brauer-Taylor group.

3. Again with $\mathbb{C} = R$-mod, consider the monoidal bicategory $\mathrm{Dist}\mathbb{C}$ of small $\mathbb{C}$-categories and distributors [5, 8]. Taking $\mathbb{B} = P_2(cl_2(\mathrm{Dist}\mathbb{C}))$, Proposition 2 gives an exact sequence between the Picard and the categorical Brauer groups of R.

4. The three previous examples can be generalized taking as $\mathbb{C}$ any symmetric monoidal category with stable coequalizers.

5. A curiosity: if we take $\mathbb{B} = P_2(R\text{-mod})$, then Proposition 2, which in the first example gives the Picard-Brauer sequence, gives now the Unit-Picard sequence, because $\mathbb{B}(I, I)$ is isomorphic to the group of units of R.

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