ON THE CHARACTERIZATION OF MONADIC CATEGORIES OVER SET

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1. Introduction

In this work we look for a new proof of the theorem characterizing monadic categories over Set (see for example [1]); more precisely, we want to stress the role of the exactness condition. Let us recall the theorem (in the following "epi" means regular epimorphism and "projective" means regular projective object):

Let $\mathcal{A}$ be a category; the following conditions are equivalent

1) $\mathcal{A}$ is equivalent to the category of algebras $\mathrm{EM}(\mathbb{T})$ for a monad $\mathbb{T}$ over Set

2) $\mathcal{A}$ is an exact category and there exists an object $G \in \mathcal{A}$ such that

   - G is projective
   - $\forall\, I \in \mathrm{Set}\ \ \exists\, I \bullet G$ (the $I-$indexed copower of G)
   - $\forall\, A \in \mathcal{A}\ \ \exists\, I \bullet G \to A$ epi

To prove that 1) implies 2) one takes as G the free algebra over the singleton; viceversa the hypothesis over G imply that $\mathcal{A}$ has enough projectives. So this theorem leads us to study exact categories with enough projectives and, on the other hand, to find conditions such that $\mathrm{EM}(\mathbb{T})$ is exact and the free algebras are projective.

2. Regularity and exactness of EM($\mathbb{T}$)

In this section we sketch some elementary facts about $\mathrm{EM}(\mathbb{T})$ to obtain a topos theoretic example of a free exact category, i.e. of an exact category with enough projectives (cf. [4]).

Proposition 1 *Let $\mathcal{A}$ be a regular category and $\mathbb{T}$ a monad over $\mathcal{A}$ (with functor part T);*

1) T preserves epi's if and only if the forgetful functor $U\colon EM(\mathbb{T}) \to \mathcal{A}$ preserves epi's

2) if T preserves epi's, then $EM(\mathbb{T})$ is regular and U preserves and reflects the epi-mono factorization.

Key words and phrases: monadic categories, exact categories.

Proposition 2 *Let $\mathcal{A}$ be a regular category and $\mathbb{T}$ a monad over $\mathcal{A}$;*

1) T sends epi's in split epi's (i.e. epi's with a section) if and only if U sends epi's in split epi's

2) if T sends epi's in split epi's, then the free algebras are projectives.

Sketch of the proof: 2) Let $f\colon (D,d) \to (TC,\mu_C)$ be an epi in EM($\mathbb{T}$), where (TC,μ_C) is the free algebra over $C \in \mathcal{A}$ $\quad(\mu\colon T^2 \to T$ is the multiplication of $\mathbb{T}$); f is an epi in EM($\mathbb{T}$) and so in $\mathcal{A}$, then Tf is a split epi in $\mathcal{A}$ and using the section of Tf one can construct the section of f in EM($\mathbb{T}$); the proof of 1) is analogous. ∎

Lemma 3 *Let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; consider an equivalence relation $e_1, e_2\colon (E,e) \rightrightarrows (X,x)$ in EM($\mathbb{T}$) and its coequalizer $q\colon X \to Q$ in $\mathcal{A}$; if*

$$TE \underset{Te_2}{\overset{Te_1}{\rightrightarrows}} TX \xrightarrow{Tq} TQ$$

is a coequalizer diagram in $\mathcal{A}$, then $e_1, e_2\colon (E,e) \rightrightarrows (X,x)$ is effective.

Proposition 4 *Let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; if $\mathbb{T}$ preserves the coequalizers in $\mathcal{A}$ of the equivalence relations in EM($\mathbb{T}$) and the epi's, then EM($\mathbb{T}$) is exact.*

Corollary 5 *Let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$;*

1) if T is left exact and preserves epi's, then EM($\mathbb{T}$) is exact

2) if the coequalizer in $\mathcal{A}$ of an equivalence relation in EM($\mathbb{T}$) is a split epi in $\mathcal{A}$, then EM($\mathbb{T}$) is exact and free algebras are projectives

3) the axiom of choice holds in $\mathcal{A}$ if and only if for every monad $\mathbb{T}$ over $\mathcal{A}$ the category EM($\mathbb{T}$) is exact and the free algebras are projectives.

As each algebra is a quotient of a free algebra, if free algebras are projective then EM($\mathbb{T}$) has enough projectives; if, moreover, EM($\mathbb{T}$) is exact, one has that EM($\mathbb{T}$) is the free exact category over its full subcategory KL($\mathbb{T}$) of free algebras (cf. [4]). An obvious example of such a situation is when $\mathcal{A}$ is Set, or a power of Set, and we can apply the third point of corollary 5. Another example is the following:

Example 6 *Let $\mathcal{E}$ be an elementary topos; the category of sup-lattices in $\mathcal{E}$ is the free exact category over the category of relations in $\mathcal{E}$.*

Proof: Let us consider the covariant monad "power-set" $\mathcal{P}\colon \mathcal{E} \to \mathcal{E}$, for which EM($\mathcal{P}$) = SL($\mathcal{E}$) and KL($\mathcal{P}$) = Rel($\mathcal{E}$); the corresponding forgetful functor SL($\mathcal{E}$) $\to \mathcal{E}$ sends epi's in split epi's (cf. [5]) and so SL($\mathcal{E}$) is a regular category and the objects of Rel($\mathcal{E}$) are projectives in SL($\mathcal{E}$). It remains to prove that the second point of corollary 5 is satisfied; we sketch the proof using the internal language of $\mathcal{E}$: let $e_1, e_2\colon E \rightrightarrows X$ be an equivalence relation in SL($\mathcal{E}$) and $q\colon X \to Q$ its coequalizer in $\mathcal{E}$; we obtain a section $s\colon Q \to X$ defining $\forall\, y \in Y \quad s(y) = \mathrm{Sup}\{x \in X \mid q(x) = y\}$. ∎

For “estetic reasons”, let us observe that the condition stated in 5.2 is also necessary; in fact we have the folowing lemma:

Lemma 7 *Let $\mathbb{T}$ be a monad over a category $\mathcal{A}$;*

1) *if $EM(\mathbb{T})$ is regular and free algebras are projectives, then U sends epi's in split epi's*
2) *if U sends epi's in (split) epi's, then the coequalizer in $\mathcal{A}$ of an exact sequences in $EM(\mathbb{T})$ is a (split) epi in $\mathcal{A}$*

Now we can summarize the previous discussion as follows:

Proposition 8 *let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; the following conditions are equivalent:*

1) *$EM(\mathbb{T})$ is exact and free algebras are projectives*
2) *the coequalizer in $\mathcal{A}$ of an equivalence relation in $EM(\mathbb{T})$ is a split epi in $\mathcal{A}$*

3. Exact categories with enough projectives

In this section we obtain a property of exact categories which, in the case of monadic categories over Set, will allow us to give a short proof of the characterizing theorem.

Definition 9 *A full subcategory $P_{\mathcal{A}}$ of a category $\mathcal{A}$ is said to be a projective cover of $\mathcal{A}$ if*

- *every object of $P_{\mathcal{A}}$ is projective in $\mathcal{A}$*
- *every object of $\mathcal{A}$ is a quotient of an object of $P_{\mathcal{A}}$*

Lemma 10 *Let $\mathcal{A}$ be a category with kernel pairs and $P_{\mathcal{A}}$ a projective cover of $\mathcal{A}$; $P_{\mathcal{A}}$ “generates” $\mathcal{A}$ via coequalizers.*

(The assertion means that, given a morphism $f\colon A \to B$ in $\mathcal{A}$, we are able to build up a commutative diagram

$$\begin{tikzcd} P' \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "f'"'] & P \arrow[r, "p"] \arrow[d, "\bar{f}"'] & A \arrow[d, "f"] \\ Q' \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & Q \arrow[r, "q"] & B \end{tikzcd}$$

such that the left square is in $P_{\mathcal{A}}$ and the two horizontal lines are coequalizers, so that f is the unique extension to the quotient.)

Proof: Given A in $\mathcal{A}$, there exists P in $P_{\mathcal{A}}$ and an epi $p\colon P \to A$; now consider the kernel pair

$$\begin{tikzcd} N(p) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & P \arrow[r, "p"] & A \end{tikzcd}$$

and again there exists an epi $p'\colon P' \to N(p)$ with P' in $P_{\mathcal{A}}$, so that p is the coequalizer of p_1 and p_2 and then of $p'p_1 = a_1$ and $p'p_2 = a_2$; analogously one can work over B and now the three dotted arrows making the following diagram commutative arise respectively from the fact that P is projective and q is an epi, from the universality of $q_1, q_2\colon N(q) \rightrightarrows Q$ and from the fact that P' is projective and q' is an epi

$$\begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, dotted, "f"'] & N(p) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, dotted, "\tilde{f}"'] & P \arrow[r, "p"] \arrow[d, dotted, "\bar{f}"] & A \arrow[d, "f"] \\ Q' \arrow[r, "q'"] & N(q) \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & Q \arrow[r, "q"] & B \end{tikzcd}$$

■

Proposition 11 *Let $\mathcal{A}$ and $\mathcal{B}$ be two exact categories with enough projectives, $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ two projective covers and $P(\mathcal{A})$ and $P(\mathcal{B})$ the full subcategories of projective objects;*

1) $\mathcal{A}$ is equivalent to $\mathcal{B}$ if and only if $P(\mathcal{A})$ is equivalent to $P(\mathcal{B})$

2) if $P_{\mathcal{A}}$ is equivalent to $P_{\mathcal{B}}$, then $P(\mathcal{A})$ is equivalent to $P(\mathcal{B})$

Proof: 1) the non-trivial implication is the "if": let $F\colon P(\mathcal{A}) \to P(\mathcal{B})$ be an equivalence; define $F'\colon \mathcal{A} \to \mathcal{B}$ as follows: if $f\colon A \to B$ is in $\mathcal{A}$, consider its presentation as in the previous lemma

$$\begin{tikzcd} P' \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] \arrow[d, "f'"'] & P \arrow[r, "p"] \arrow[d, "\bar{f}"'] & A \arrow[d, "f"] \\ Q' \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & Q \arrow[r, "q"] & B \end{tikzcd}$$

and put $F'f$ as the unique extension to the quotient of

$$\begin{tikzcd} FP' \arrow[r, shift left, "Fa_1"] \arrow[r, shift right, "Fa_2"'] \arrow[d, "Ff'"'] & FP \arrow[r, "F'p"] \arrow[d, "F\bar{f}"'] & F'A \arrow[d, "F'f"] \\ FQ' \arrow[r, shift left, "Fb_1"] \arrow[r, shift right, "Fb_2"'] & Q \arrow[r, "F'q"] & F'B \end{tikzcd}$$

The existence of F' depends on the fact that the (jointly) monic part (i_1, i_2) of the epi-(jointly) mono factorization

$$\begin{tikzcd} FP' \arrow[r, shift left, "Fa_1"] \arrow[r, shift right, "Fa_2"'] \arrow[dr, "n"'] & FP \\ & N \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] \end{tikzcd}$$

is an equivalence relation in $\mathcal{B}$; this follows from the fact that the pair (a_1, a_2) is a pseudo-equivalence relation in $P(\mathcal{A})$ (i.e. as an equivalence relation but we do not require that a_1 and a_2 are jointly monic) and so the same holds for (Fa_1, Fa_2) in $P(\mathcal{B})$. See for instance the transitivity condition: consider the following diagram

$$\begin{tikzcd}
M' \arrow[rr, "l_1"] \arrow[dd, "l_2"'] \arrow[dr, "m"'] & & FP' \arrow[d, "n"] \\
 & M \arrow[r, "n_1"] \arrow[d, "n_2"'] & N \arrow[d, "i_2"] \\
FP' \arrow[r, "n"'] & N \arrow[r, "i_1"'] & FP
\end{tikzcd}$$

where M is the pullback of i_1 and i_2 and M' the pullback of Fa_1 and Fa_2, so that the unique factorization $m\colon M' \to M$ is an epi; consider again a projective cover $m'\colon R \to M'$; the transitivity of (Fa_1, Fa_2) in $P(\mathcal{B})$ means exactly that there exists a morphism $t\colon R \to FP'$ making commutative the following diagram

$$\begin{tikzcd}
R \arrow[r, "m'"] \arrow[d, "t"'] & M' \arrow[r, "m"] & M \arrow[r, "(n_1 i_1, n_2 i_2)"] & FP \times FP \arrow[d, "id"] \\
FP' \arrow[rr, "n"'] & & N \arrow[r, "(i_1, i_2)"'] & FP \times FP
\end{tikzcd}$$

The fact that $m'm$ is an epi and (i_1, i_2) is a mono implies the existence of a morphism $\tau\colon M \to N$ which exhibits the transitivity of $i_1, i_2\colon N \rightrightarrows FP$. To show that F' is a full and essentially surjective functor is quite obvious (for this recall that F is an equivalence); the faithfulness of F' essentially depends on the fact that the image of (Fb_1, Fb_2), being an equivalence relation in $\mathcal{B}$, is the kernel pair of its coequalizer $F'q$.
2) is trivial under the only condition that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. ■

The previous proposition explains the name "free" given to an exact category with enough projectives: it is completely determined by the full subcategory of projective objects. In [4] we have discussed the universal property satisfied by this kind of categories.

4. Characterization theorem

Proposition 12 *Let $\mathcal{C}$ be a category; the following conditions are equivalent:*

1) $\mathcal{C}$ is equivalent to the category $KL(\mathbb{T})$ for a monad $\mathbb{T}$ over Set

2) there exists an object $G \in \mathcal{C}$ such that
- $\forall\, I \in Set\ \exists\, I \bullet G$
- $\forall\, X \in \mathcal{C}\ \exists\, I \in Set$ such that $X \cong I \bullet G$

Proof: 2) $\Rightarrow$ 1) consider the pair of functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift right, "-\bullet G"'] & \mathcal{C} \arrow[l, shift right, "\mathcal{C}(G,-)"'] \end{tikzcd}$$

The first condition says that $-\bullet G$ is left adjoint to $\mathcal{C}(G,-)$; the second condition says that the comparison functor $\mathrm{KL}(\mathbb{T}) \to \mathcal{C}$ is essentially surjective and so it is an equivalence (here $\mathbb{T}$ is the monad induced by $-\bullet G \dashv \mathcal{C}(G,-)$). ■

Proposition 13 *Let $\mathcal{A}$ be a category; the following conditions are equivalent:*

1) $\mathcal{A}$ is equivalent to the category EM$(\mathbb{T})$ for a monad $\mathbb{T}$ over Set

2) $\mathcal{A}$ is an exact category and there exists an object $G \in \mathcal{A}$ such that
- G is projective
- $\forall\, I \in$ Set $\exists\, I \bullet G$
- $\forall\, A \in \mathcal{A}\ \exists\, I \bullet G \to A$ epi

Proof: 2) $\Rightarrow$ 1) let $\mathcal{C}$ be the full subcategory of $\mathcal{A}$ spanned by $I \bullet G$ for $I \in$ Set; by proposition 12, $\mathcal{C} \simeq \mathrm{KL}(\mathbb{T})$ for a monad $\mathbb{T}$ over Set; so, by proposition 11, $\mathcal{A} \simeq \mathrm{EM}(\mathbb{T})$ because $\mathcal{C}$ is a projective cover of $\mathcal{A}$ and $\mathrm{KL}(\mathbb{T})$ is a projective cover of $\mathrm{EM}(\mathbb{T})$. ■

5. Presheaf categories

The two previous propositions can be generalized to characterize $\mathrm{KL}(\mathbb{T})$ and $\mathrm{EM}(\mathbb{T})$ when $\mathbb{T}$ is a monad over Set^X for $X \in$ Set (to get examples as presheaf categories); the short proof suggested for proposition 13 remains, of course, unchanged. It is not surprising (cf. [6]) that proposition 11 allows us also to give a short proof for the characterization of presheaf categories (cf. [2, 3]). In the next lemma, Fam$\mathbb{C}$ is the sum completion of a small category $\mathbb{C}$.

Lemma 14 *Let $\mathbb{C}$ be a small category and $\mathcal{B}$ the full subcategory of $\mathrm{Set}^{\mathbb{C}^{op}}$ spanned by sums of representable functors; $\mathcal{B}$ is equivalent to Fam$\mathbb{C}$.*

Proof: Consider the unique extension $Y' \colon \mathrm{Fam}\mathbb{C} \to \mathcal{B}$ of the Yoneda embedding $Y \colon \mathbb{C} \to \mathcal{B}$; obviously Y' is essentially surjective; its fullness and faithfulness easily follow from Yoneda's lemma. ■

Lemma 15 *Let $\mathcal{B}$ be a category with disjoint sums and strict initial object; the following conditions are equivalent*

(1) $\mathcal{B}$ is equivalent to the category Fam$\mathbb{C}$ for a small category $\mathbb{C}$

(2) there exists a small subcategory $\mathbb{C}$ of $\mathcal{B}$ such that

- $\forall\, B \in \mathcal{B}\ \exists\ \{C_i\}_I$ *with* $C_i \in \mathbb{C}$ *such that* $B \cong \coprod_I C_i$
- $\forall\, f\colon C \to \coprod_I C_i$ *with* $C, C_i \in \mathbb{C}\ \ \exists\, i_0 \in I$ *such that* f *can be factorized through the injection* $C_{i_0} \to \coprod_I C_i$
- *the initial object* $0 \notin \mathbb{C}$

Proof: 2) $\Rightarrow$ 1) consider the unique extension $F\colon \mathrm{Fam}\mathbb{C} \to \mathcal{B}$ of the full inclusion of $\mathbb{C}$ in $\mathcal{B}$; the first condition implies that F is essentially surjective; the second conditions implies that F is full; the third condition (together with the disjointness and the fact that the initial object is strict) implies that F is faithful. ∎

Proposition 16 *Let $\mathcal{A}$ be an exact category with disjoint sums and strict initial objects; the following conditions are equivalent*

(1) $\mathcal{A}$ is equivalent to the category of presheaves on a small category

(2) $\mathcal{A}$ has a set $\{G_j\}_J$ of regular generators such that

- $\forall\, j \in J\ \ G_j$ *is projective*
- $\forall\, f\colon G \to \coprod_I G_i$ *with* $G, G_i \in \{G_j\}_J\ \ \ \exists\, i_0 \in I$ *such that* f *can be factorized through the injection* $G_{i_0} \to \coprod_I G_i$

(3) $\mathcal{A}$ has a family of absolutely presentable generators

Proof: 1) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2) are obvious (recall that an object $G \in \mathcal{A}$ is absolutely presentable if $\mathcal{A}(G, -)\colon \mathcal{A} \to \mathrm{Set}$ preserves colimits).
2) $\Rightarrow$ 1): two cases: first, if the initial object $0 \in \{G_j\}_J$ but $\{G_j\}_J \setminus 0$ is not a family of generators, then $\{G_j\}_J = \{0\}$ and so $\mathcal{A} \simeq 1 \simeq \mathrm{Set}^{\emptyset}$; second, if $0 \notin \{G_j\}_J$ let $\mathbb{C}$ be the full subcategory of generators and $\mathcal{B}$ the full subcategory spanned by sums of generators; by lemma 15, $\mathcal{B} \simeq \mathrm{Fam}\mathbb{C}$ and, by lemma 14, Fam$\mathbb{C}$ is a projective cover of $\mathrm{Set}^{\mathbb{C}^{op}}$; but $\mathcal{B}$ is a projective cover of $\mathcal{A}$, so, by proposition 11, $\mathcal{A} \simeq \mathrm{Set}^{\mathbb{C}^{op}}$. ∎

Acknowledgements: I would like to thank F. Borceux, M. C. Pedicchio, F. Van der Plancke and, in particular, A. Carboni for some useful discussions on this topic.

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