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External derivations of internal groupoids

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Abstract

If H is a G-crossed module, the set of derivations of G in H is a monoid under the Whitehead product of derivations. We interpret the Whitehead product using the correspondence between crossed modules and internal groupoids in the category of groups. Working in the general context of internal groupoids in a finitely complete category, we relate derivations to holomorphisms, translations, affine transformations, and to the embedding category of a groupoid.

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1. Introduction

Let G be a group and $\varphi\colon G \to \operatorname{Aut} H$ a G-group. A derivation of G in H is a map $d\colon G \to H$ such that $d(xy) = d(x) + x \cdot d(y)$. If H is a G-module, i.e. if H is Abelian, the set $\operatorname{Der}(G, H)$ of derivations is an Abelian group w.r.t. the point-wise sum. If H is not Abelian, in general $\operatorname{Der}(G, H)$ is just a pointed set (the zero-morphism $0\colon G \to H$ is a derivation). Whitehead [26] discovered the following fact.

Theorem 1.1. *Let* $(H \xrightarrow{\partial} G \xrightarrow{\varphi} \operatorname{Aut} H)$ *be a crossed module of groups. The set* $\operatorname{Der}(G, H)$ *is a monoid w.r.t.* $(d_1 + d_2)(x) = d_1(\partial(d_2(x))x) + d_2(x)$.

The aim of this note is to understand in a more conceptual way the Whitehead product of derivations. The idea is to replace crossed modules of groups by the equivalent notion of internal groupoids in the category of groups. Using the language of internal groupoids, the Whitehead product becomes clear: it is nothing but the composition in the internal category. The surprise is that, once expressed in terms of internal groupoids, Whitehead's theorem,

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as well as some other basic properties of derivations (notably, the characterization of regular derivations and the left exactness of $\mathrm{Der}(G, H)$ as a functor of the second variable), have nothing to do with groups, but hold in the very general context of internal groupoids in an arbitrary category $\mathcal{G}$ with finite limits [2,9]. Just to quote some categories where internal crossed modules and internal groupoids are intensively studied, let us mention the category of Lie algebras [6] (in fact, Lie algebras as well as groups constitute semi-Abelian categories [1,16], and for semi-Abelian categories, internal crossed modules and internal groupoids coincide [15]), categories of groups with operations [8], the category of topological spaces and continuous maps [10,13,20], the category of topological spaces and local homeomorphisms (whose internal groupoids are called étale groupoids) [23], the category of smooth manifolds (whose internal groupoids are called Lie groupoids) [19,20,23], and of course the category of sets, which gives ordinary groupoids [3,14,25].

In his paper [11], Gilbert explains the Whitehead product of derivations replacing crossed modules by the equivalent notion of groups in the category of groupoids, whereas we use groupoids in the category of groups. Even if the equivalence between groups in groupoids and groupoids in groups is a trivial fact, the advantage of working with groupoids in groups is that this immediately suggests the more general context of internal groupoids in any finitely complete category. This gain of generality allows us, for example, to include in the same theory the holomorph of a group: this is possible because a group is a particular groupoid in sets. In fact, this easy example is the guiding example to describe derivations using holomorphisms and translations as in Sections 5 and 6. Moreover, it is a fact that several definitions, constructions and proofs become more transparent once having in mind the set-theoretical case instead of the group-theoretical case. Finally, since internal groupoids are the objects of a 2-category, we can exploit some general 2-categorical facts to define derivations, translations, and the category of embeddings of a groupoid.

For the reader's convenience, let us quote here Metatheorem 0.1.3 from [1]. It is used in most of the proofs of Sections 5 and 6.

Theorem 1.2. *The statement $\varphi \Rightarrow \psi$ is valid in every category, provided that it is valid in the category of sets and that φ, ψ can be expressed as conjunctions of properties of the following kind: "some finite diagram is commutative", "some finite diagram is a limit diagram", "some morphism is a monomorphism", "some morphism is an isomorphism", "some arrow $A \to B$ factors through some specified monomorphism $S \to B$".*

Warning: the composition of $f : x \to y$ and $g : y \to z$ is written $f \cdot g$.

2. The monoid of derivations

In this section, we construct the monoid of $\mathbb{C}$-derivations, for $\mathbb{C}$ an internal groupoid.

We fix, once and for all, a category $\mathcal{G}$ with finite limits. The notation for an internal groupoid $\mathbb{C}$ in $\mathcal{G}$ is

$$\mathbb{C} = \left(\begin{tikzcd} C_0 \arrow[r, "u" description] & C_1 \arrow[l, shift right=2, "\mathrm{dom}"'] \arrow[l, shift left=2, "\mathrm{cod}"] & C_1 \times_{C_0} C_1 \arrow[l, "\circ"'] \end{tikzcd} , \; \begin{tikzcd} C_1 \arrow[r, "()^{-1}"] & C_1 \end{tikzcd} \right)$$

where $C_1 \times_{C_0} C_1$ is the object of "composable pairs", that is

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "\mathrm{dom}"] \\ C_1 \arrow[r, "\mathrm{cod}"'] & C_0 \end{tikzcd}$$

is a pullback in $\mathcal{G}$. We also write $\circ^2 : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1$ for the diagonal of the commutative square

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1\times\circ"] \arrow[d, "\circ\times 1"'] & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\circ"'] & C_1 \end{tikzcd}$$

where

$$\begin{tikzcd} C_1 & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_3"] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "\mathrm{dom}"] \\ C_0 \arrow[from=1-1, "\mathrm{cod}"'] & C_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C_0 \end{tikzcd}$$

is a limit in $\mathcal{G}$.

We denote by Grpd($\mathcal{G}$) the 2-category of internal groupoids, internal functors, and internal natural transformations (which always are natural isomorphisms). For $\mathbb{C}, \mathbb{B}$ internal groupoids, we denote by $\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{B})$ the corresponding hom-category (which is a groupoid), and we write

$$\begin{tikzcd} {[\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{B})]_1} \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, shift right, "\mathrm{cod}"'] & {[\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{B})]_0} \end{tikzcd}$$

for its sets of arrows and of objects, together with the domain and the codomain maps. In particular, $\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{C})$ is a strict monoidal groupoid : the tensor product is the composition of internal functors and the horizontal composition of internal natural transformations, the unit object is the identity functor on $\mathbb{C}$. As with any strict monoidal category, the map

$$\mathrm{cod}\colon [\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0$$

is a homomorphism of monoids.

Definition 2.1. The monoid of $\mathbb{C}$-derivations is the kernel of the codomain map

$$\mathrm{Der}\,\mathbb{C} = \mathrm{Ker}(\mathrm{cod}) \to [\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{C})]_1 \to [\mathrm{Grpd}(\mathcal{G})(\mathbb{C}, \mathbb{C})]_0.$$

Explicitly, a $\mathbb{C}$-derivation is a pair (D, d)

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "D"{name=U}] \arrow[r, bend right, "Id"'{name=L}] & \mathbb{C} \arrow[from=U, to=L, phantom, "\Downarrow d"] \end{tikzcd}$$

with D an internal functor and d an internal natural transformation. The product of derivations is the horizontal composition of natural transformations.

When $\mathcal{G}$ is the category of sets, to give a $\mathbb{C}$-derivation just means to choose, for each object x of $\mathbb{C}$, an arrow

$$d(x)\colon \mathrm{dom}(d(x)) \to x$$

with codomain x. This suggests to describe derivations as sections of the codomain arrow.

Proposition 2.2. *To give a $\mathbb{C}$-derivation amounts to giving an arrow $d\colon C_0 \to C_1$ such that the diagram*

$$\begin{tikzcd} & C_1 \arrow[d, "\mathrm{cod}"] \\ C_0 \arrow[ur, "d"] \arrow[r, "1"'] & C_0 \end{tikzcd} \tag{1}$$

commutes.

Proof. With such an arrow d given, we have to construct an internal functor $D\colon \mathbb{C} \to \mathbb{C}$ in such a way that d becomes an internal natural transformation $d\colon D \Rightarrow Id$. The following picture explains the set-theoretical idea behind the construction of D.

$$\begin{tikzcd}
x \arrow[d, "a"'] & \mapsto & D_0(x) = \mathrm{dom}(d(x)) \arrow[r, "d(x)"] \arrow[d, dotted, "D_1(a)"'] & x \arrow[d, "a"] \\
y & & D_0(y) = \mathrm{dom}(d(y)) & y \arrow[l, "d(y)^{-1}"]
\end{tikzcd}$$

It suffices now to internalize this idea:

- On objects, the functor $D\colon \mathbb{C} \to \mathbb{C}$ is defined by

$$D_0\colon\ C_0 \xrightarrow{d} C_1 \xrightarrow{\mathrm{dom}} C_0.$$

- As far as arrows are concerned, we consider the diagram

$$\begin{tikzcd}
C_0 \arrow[d, "d"'] & & C_1 \arrow[ll, "\mathrm{dom}"'] \arrow[r, "\mathrm{cod}"] \arrow[d, "1"] & C_0 \arrow[r, "d"] & C_1 \arrow[d, "()^{-1}"] \\
C_1 \arrow[r, "\mathrm{cod}"'] & C_0 & C_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & C_0 & C_1 \arrow[l, "\mathrm{dom}"]
\end{tikzcd}$$

By Eq. (1), this diagram commutes, and we get a unique factorization of the projective cone through the object of composable triples, say

$$\overline{d} = \langle \mathrm{dom} \cdot d, 1, \mathrm{cod} \cdot d \cdot ()^{-1} \rangle \colon C_1 \to C_1 \times_{C_0} C_1 \times_{C_0} C_1.$$

Finally, the functor $D\colon \mathbb{C} \to \mathbb{C}$ is defined on arrows by

$$D_1\colon\ C_1 \xrightarrow{\overline{d}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\circ^2} C_1. \quad \square$$

We wish now to describe explicitly the operations in $\mathrm{Der}\,\mathbb{C}$ using Proposition 2.2:

- The unit in $\mathrm{Der}\,\mathbb{C}$ is $u\colon C_0 \to C_1$.
- The multiplication in $\mathrm{Der}\,\mathbb{C}$ is the internal version of

$$z \xrightarrow{d_1(y)} y = \mathrm{dom}(d_2(x)) \xrightarrow{d_2(x)} x.$$

(In other words, the Whitehead product of derivations is just the internal composition in $\mathbb{C}$.) This means that, given two derivations $d_1, d_2\colon C_0 \to C_1$, we start constructing the arrow

$$d_1 \star d_2 = \langle d_2 \cdot \mathrm{dom} \cdot d_1, d_2 \rangle \colon C_0 \to C_1 \times_{C_0} C_1$$

and we get the product of d_1 and d_2 by composing internally

$$d_1 \otimes d_2\colon\ C_0 \xrightarrow{d_1 \star d_2} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1.$$

Example 2.3. When $\mathcal{G}$ is the category of groups, we recapture the classical notion of derivation. Indeed, it is well-known that to a crossed module of groups

$$(\ H \xrightarrow{\partial} G \xrightarrow{\varphi} \mathrm{Aut}\,H\ )$$

we can associate an internal groupoid $\mathbb{C}$, with

$$C_0 = G, \qquad C_1 = H \rtimes_{\varphi} G, \qquad m((a, x), (b, y)) = (a + b, y)$$
$$\mathrm{cod}(a, x) = x, \qquad \mathrm{dom}(a, x) = \partial(a)x, \qquad u(x) = (0, x)$$

(see [4,15,17]). Moreover, $\mathbb{C}$-derivations in the sense of Definition 2.1 are in bijection with derivations of G in H : a morphism $d\colon C_0 \to C_1$ is a $\mathbb{C}$-derivation precisely when its second component is the identity on C_0 and its first component is a derivation of G in H. (Let us recall here also the converse construction, which is needed later. Given an internal groupoid $\mathbb{C}$ in groups, we get a crossed module with $G = C_0$ and $H = \mathrm{Ker}(\mathrm{cod})$; the map ∂ is the restriction of dom to H, and the action of G on H is given by $x \cdot a = u(x) + a - u(x)$.)

3. The group of regular derivations

In this section, we characterize the invertible (or regular) elements of the monoid $\mathrm{Der}\,\mathbb{C}$.

From Definition 2.1, we get three morphisms of monoids:

- $\mathcal{U}\colon \mathrm{Der}\,\mathbb{C} \to [\mathrm{Grpd}(\mathcal{G})(\mathbb{C},\mathbb{C})]_0$, $(D,d) \mapsto (D\colon \mathbb{C} \to \mathbb{C})$
- $()_0\colon \mathrm{Der}\,\mathbb{C} \to \mathrm{End}\,C_0$, $(D,d) \mapsto (D_0\colon C_0 \to C_0)$
- $()_1\colon \mathrm{Der}\,\mathbb{C} \to \mathrm{End}\,C_1$, $(D,d) \mapsto (D_1\colon C_1 \to C_1)$.

As with any morphism of monoids, these morphisms restrict to the groups of invertible elements:

$$\begin{tikzcd} \mathrm{Der}\,\mathbb{C} \arrow[r, "\mathcal{U}"] & {[\mathrm{Grpd}(\mathcal{G})(\mathbb{C},\mathbb{C})]_0} \\ \mathrm{Der}^*\,\mathbb{C} \arrow[u] \arrow[r, dotted] & {[\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Der}\,\mathbb{C} \arrow[r, "()_0"] & \mathrm{End}\,C_0 \\ \mathrm{Der}^*\,\mathbb{C} \arrow[u] \arrow[r, dotted] & \mathrm{Aut}\,C_0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Der}\,\mathbb{C} \arrow[r, "()_1"] & \mathrm{End}\,C_1 \\ \mathrm{Der}^*\,\mathbb{C} \arrow[u] \arrow[r, dotted] & \mathrm{Aut}\,C_1 \arrow[u] \end{tikzcd}$$

where $\mathrm{Grpd}(\mathcal{G})^*$ is the sub-2-category of $\mathrm{Grpd}(\mathcal{G})$ of those internal functors which are isomorphisms. In fact, more is true: the previous diagrams are pullbacks. This is a corollary of the following general fact.

Lemma 3.1. *Let*

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \end{tikzcd}$$

be a 2-cell in $\mathrm{Grpd}(\mathcal{G})$*. The following conditions are equivalent:*

1. *α is invertible with respect to the horizontal composition;*
2. *$F, G\colon \mathbb{C} \to \mathbb{B}$ are in* $\mathrm{Grpd}(\mathcal{G})^*$*;*
3. *$F_1, G_1\colon C_1 \to B_1$ are isomorphisms in $\mathcal{G}$;*
4. *$F_1\colon C_1 \to B_1$ and $G_0\colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$;*
5. *$G_1\colon C_1 \to B_1$ and $F_0\colon C_0 \to B_0$ are isomorphisms in $\mathcal{G}$.*

Proof. Implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious.

$3 \Rightarrow 4$: If G_1 is an isomorphism, then G_0 also is an isomorphism, with $G_0^{-1} = u \cdot G_1^{-1} \cdot \mathrm{dom}$. The same holds for $3 \Rightarrow 5$. This argument also gives implication $3 \Rightarrow 2$, because if (F_1, F_0) is an internal functor with F_1 and F_0 invertible, then (F_1^{-1}, F_0^{-1}) also is an internal functor.

$4 \Rightarrow 3$: One has to internalize the following set-theoretical argument : if $g\colon a \to b$ is in B_1, then one defines $G_1^{-1}(g) = F_1^{-1}(\alpha(x) \cdot g \cdot \alpha(y)^{-1})$, where $x = G_0^{-1}(a)$ and $y = G_0^{-1}(b)$.

$2 \Rightarrow 1$: This implication holds in any 2-categories : assume that the 2-cell α is invertible w.r.t. the vertical composition, and let $F \dashv F^*$, $G \dashv G^*$ be adjunctions, with units and counits given by $\eta\colon Id_{\mathbb{C}} \to F \circ F^*$, $\epsilon\colon F^* \circ F \to Id_{\mathbb{B}}$, $\gamma\colon Id_{\mathbb{C}} \to G \circ G^*$, $\beta\colon G^* \circ G \to Id_{\mathbb{B}}$. Using triangular identities, one checks that the following diagrams commute

$$\begin{tikzcd} F \cdot F^* \arrow[r, "\alpha\circ(\alpha^{-1})^*"] & G \cdot G^* \\ Id_{\mathbb{C}} \arrow[u, "\eta"] \arrow[r, "1"'] & Id_{\mathbb{C}} \arrow[u, "\gamma"'] \end{tikzcd} \qquad \begin{tikzcd} F^* \cdot F \arrow[r, "(\alpha^{-1})^*\circ\alpha"] \arrow[d, "\epsilon"'] & G^* \cdot G \arrow[d, "\beta"] \\ Id_{\mathbb{B}} \arrow[r, "1"'] & Id_{\mathbb{B}} \end{tikzcd}$$

where $(\alpha^{-1})^*$ is defined by the following composition

$$F^* = F^* \cdot Id_{\mathbb{C}} \xrightarrow{1\circ\gamma} F^* \cdot G \cdot G^* \xrightarrow{1\circ\alpha^{-1}\circ 1} F^* \cdot F \cdot G^* \xrightarrow{\epsilon\circ 1} Id_{\mathbb{B}} \cdot G^* = G^*.$$

In particular, if F and G are isomorphisms, one can choose η, ϵ, γ and β to be identities, and the proof is complete. $\square$

Corollary 3.2. *Let (D, d) be a $\mathbb{C}$-derivation. The following conditions are equivalent:*

(1) *(D, d) is a regular derivation;*
(2) *$D: \mathbb{C} \to \mathbb{C}$ is in* $\mathrm{Grpd}(\mathcal{G})^*$;
(3) *$D_0: C_0 \to C_0$ is an isomorphism (i.e. $C_0 \xrightarrow{d} C_0 \xrightarrow{\mathrm{dom}} C_1$ is an isomorphism);*
(4) *$D_1: C_1 \to C_1$ is an isomorphism.*

Example 3.3. When $\mathcal{G}$ is the category of groups and $\mathbb{C}$ is the internal groupoid associated with a crossed module $H \to G \to \mathrm{Aut}\, H$ as in Example 2.3, the previous corollary extends the following characterization of regular derivations, due to Whitehead [26]:
There are morphisms of monoids $\sigma: \mathrm{Der}(G, H) \to \mathrm{End}\, G: \sigma_d(x) = \partial(d(x))x$ and $\theta: \mathrm{Der}(G, H) \to \mathrm{End}\, H: \theta_d(a) = d(\partial(a)) + a$. Moreover, a derivation d is invertible iff $\sigma_d \in \mathrm{Aut}\, G$ iff $\theta_d \in \mathrm{Aut}\, H$.
Our definition of a derivation also explains why the group of regular derivations $\mathrm{Der}^*(G, H)$ enters in the construction of Norrie's actor of a crossed module (cf. [24]; see also Theorem 3.3 in [11]). In fact, for any internal groupoid $\mathbb{C}$ in any finitely complete category $\mathcal{G}$, the data

$$\mathrm{Act}\,\mathbb{C}: \begin{cases} \mathrm{Der}^*\,\mathbb{C} \to [\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 & (D, d) \mapsto (D: \mathbb{C} \to \mathbb{C}) \\ \\ [\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0 \times \mathrm{Der}^*\,\mathbb{C} \to \mathrm{Der}^*\,\mathbb{C} & F, (D, d) \mapsto F_0 \cdot d \cdot {F_1}^{-1} \end{cases}$$

define a crossed module of groups: $\mathrm{Act}\,\mathbb{C}$ precisely is the crossed module associated with $\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$, which is an internal groupoid in groups. Recall now that the actor $\mathrm{Act}(G, H)$ of a crossed module is a new crossed module intended to recapture, in the category of crossed modules, the idea of "group of automorphisms of a group". If we look at the crossed module $H \to G \to \mathrm{Aut}\, H$ as an internal groupoid $\mathbb{C}$ in groups, then the group of automorphisms must be replaced by $\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})$, and $\mathrm{Act}(G, H)$ is nothing but $\mathrm{Act}\,\mathbb{C}$.

4. Left exactness of Der $\mathbb{C}$

If ($H \xrightarrow{\partial} G \xrightarrow{\varphi} \mathrm{Aut}\, H$) is a crossed module of groups, one of the main properties of the group of regular derivations $\mathrm{Der}^*(G, H)$ is that, when it is seen as a functor of the second variable, it preserves kernels. Indeed, this allows one to apply the kernel–cokernel lemma for groups, obtaining in this way the fundamental exact sequence in non-Abelian group cohomology. The aim of this section is to study the main properties of $\mathrm{Der}\,\mathbb{C}$ and $\mathrm{Der}^*\,\mathbb{C}$ as functors.

Consider two internal groupoids $\mathbb{C}$ and $\mathbb{C}'$ in $\mathcal{G}$ having the same object of objects, and an internal functor $F: \mathbb{C} \to \mathbb{C}'$, which is the identity on objects

$$\begin{tikzcd} C_1 \arrow[r, "F_1"] \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] & C_1' \arrow[d, shift right, "\mathrm{dom}'"'] \arrow[d, shift left, "\mathrm{cod}'"] \\ C_0 \arrow[r, "F_0=1"'] & C_0 \end{tikzcd}$$

Composing with F_1 gives a morphism of monoids

$$\mathrm{Der}\, F: \mathrm{Der}\,\mathbb{C} \to \mathrm{Der}\,\mathbb{C}' \qquad C_0 \xrightarrow{d} C_1 \quad \mapsto \quad C_0 \xrightarrow{d} C_1 \xrightarrow{F_1} C_1'$$

and its restrictions to the groups of regular derivations $\mathrm{Der}^*\, F: \mathrm{Der}^*\,\mathbb{C} \to \mathrm{Der}^*\,\mathbb{C}'$.

In fact, this construction is a functor

$$\mathrm{Der}\colon \mathcal{F}_{C_0} \to \mathrm{Mon}$$

where Mon is the category of monoids, and $\mathcal{F}_{C_0}$ is the fibre over C_0 of the functor

$$\mathrm{Grpd}(\mathcal{G}) \to \mathcal{G}$$

which associates to an internal groupoid $\mathbb{C}$ its object of objects C_0. Moreover, the functor Der factorizes through the comma category

$$\mathrm{Mon}/\mathrm{End}\, C_0$$

because $\mathrm{Der}\,\mathbb{C}$ is equipped with a canonical morphism

$$\mathrm{Der}\,\mathbb{C} \to \mathrm{End}\, C_0 \qquad C_0 \xrightarrow{d} C_1 \ \mapsto \ C_0 \xrightarrow{d} C_1 \xrightarrow{\mathrm{dom}} C_0$$

(cf. Proposition 2.2). In the same way, using regular derivations instead of arbitrary derivations, we obtain two functors

$$\mathrm{Der}^*\colon \mathcal{F}_{C_0} \to \mathrm{Grp}/\mathrm{Aut}\, C_0 \qquad \mathrm{Der}^*\colon \mathcal{F}_{C_0} \to \mathrm{Grp}$$

where Grp is the category of groups.

Proposition 4.1. (1) *The functor* $\mathrm{Der}\colon \mathcal{F}_{C_0} \to \mathrm{Mon}/\mathrm{End}\, C_0$ *preserves finite limits;*
(2) *The functor* $\mathrm{Der}\colon \mathcal{F}_{C_0} \to \mathrm{Mon}$ *preserves equalizers;*
(3) *The functor* $\mathrm{Der}^*\colon \mathcal{F}_{C_0} \to \mathrm{Grp}/\mathrm{Aut}\, C_0$ *preserves finite limits;*
(4) *The functor* $\mathrm{Der}^*\colon \mathcal{F}_{C_0} \to \mathrm{Grp}$ *preserves equalizers.*

Proof. The functor $\mathrm{Mon} \to \mathrm{Grp}$ associating to a monoid the group of its invertible elements preserves limits, so that points (3) and (4) follow from points (1) and (2). Moreover, the canonical forgetful functor from a comma category to the base category preserves equalizers, so that point (2) follows from point (1). As far as point (1) is concerned, it is enough to give a glance to finite limits in the fibre $\mathcal{F}_{C_0}$.

- The object of arrows of the terminal object in $\mathcal{F}_{C_0}$ is the product $C_0 \times C_0$. Domain and codomain are the projections. Composition $C_0 \times C_0 \times C_0 \to C_0 \times C_0$ is the projection on the first and third components. The inverse $C_0 \times C_0 \to C_0 \times C_0$ is the twist.
- The object of arrows of the equalizer in $\mathcal{F}_{C_0}$ of $F, G\colon \mathbb{C} \to \mathbb{C}'$ is the equalizer in $\mathcal{G}$

$$\begin{tikzcd} E \arrow[r, "e"] & C_1 \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] & C_1' \end{tikzcd}$$

  with domain and codomain given by $\mathrm{dom} \cdot e$, $\mathrm{cod} \cdot e$. The rest of the structure is inherited from that of $\mathbb{C}$ using the universal property of E.
- The object of arrows of the product in $\mathcal{F}_{C_0}$ of $\mathbb{C}$ and $\mathbb{C}'$ is the limit L as in the following diagram

$$\begin{tikzcd} & L \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C_1 \arrow[d, "\mathrm{dom}"'] \arrow[drr, "\mathrm{cod}"] & & C_1' \arrow[d, "\mathrm{cod}'"] \arrow[dll, "\mathrm{dom}'"'] \\ C_0 & & C_0 \end{tikzcd}$$

  The domain is $\mathrm{dom} \cdot p_1 = \mathrm{dom}' \cdot p_2$, and the codomain is $\mathrm{cod} \cdot p_1 = \mathrm{cod}' \cdot p_2$. The rest of the structure is inherited from those of $\mathbb{C}$ and $\mathbb{C}'$ via the universal property of L.

It is now easy to verify that the functor $\mathrm{Der}\colon \mathcal{F}_{C_0} \to \mathrm{Mon}/\mathrm{End}\, C_0$ preserves finite limits. Let us look, for example, at the case of products. Consider a pair

$$(d_1, d_2) \in \mathrm{Der}\,\mathbb{C} \times_{\mathrm{End}\, C_0} \mathrm{Der}\,\mathbb{C}'$$

(which is the product in the comma category Mon/End C_0). Since $\mathrm{cod} \cdot d_1 = 1 = \mathrm{cod}' \cdot d_2$ and $\mathrm{dom} \cdot d_1 = \mathrm{dom}' \cdot d_2$, there is a unique arrow $d \colon C_0 \to L$ such that $p_1 \cdot d = d_1$ and $p_2 \cdot d = d_2$. Moreover, $\mathrm{cod} \cdot p_1 \cdot d = \mathrm{cod} \cdot d_1 = 1$, so that d is a derivation. Conversely, if $d \colon C_0 \to L$ is a derivation, then $cod \cdot p_1 \cdot d = 1 = \mathrm{cod}' \cdot p_2 \cdot d$ and $\mathrm{dom} \cdot p_1 \cdot d = \mathrm{dom}' \cdot p_2 \cdot d$, so that the pair $(p_1 \cdot d, p_2 \cdot d)$ is an element of the pullback $\mathrm{Der}\,\mathbb{C} \times_{\mathrm{End}\, C_0} \mathrm{Der}\,\mathbb{C}'$. □

5. The 2-category of holomorphisms

In this section, we give a different description of 2-cells in $\mathrm{Grpd}(\mathcal{G})$. For this, we introduce the notion of holomorphism between two groupoids. Our terminology is justified by the following example.

Example 5.1. We can consider a group G as a groupoid (in sets) with just one object, having the elements of G as arrows. Group homomorphisms correspond, then, to internal functors. If $f, g \colon G \to H$ are group homomorphisms, a natural transformation $h \colon f \Rightarrow g$ is just an element $h_* \in H$ such that, for all $a \in G$, one has $f(a) + h_* = h_* + g(a)$. We can therefore define a map $h \colon G \to H$ by $h(a) = f(a) + h_*$, so that $h(0) = h_*$. Such a map satisfies the equation $h(a + c) = h(a) - h(0) + h(c)$, which is also equivalent to the equation $h(a - b + c) = h(a) - h(b) + h(c)$ (compare with Lemma 5.2(1)). A map satisfying these equivalent conditions is called a group holomorphism (see, for example, Section IV.1 in [21]). Conversely, a holomorphism $h \colon G \to H$ is a homomorphism precisely when it is pointed, that is when $h(0) = 0$. We can therefore construct two homomorphisms from a holomorphism h :

$$\delta_h \colon G \to H, \qquad \delta_h(a) = h(a) - h(0); \qquad \gamma_h \colon G \to H, \qquad \gamma_h(a) = -h(0) + h(a).$$

The element $h(0)$ gives then a natural transformation $h(0) \colon \delta_h \Rightarrow \gamma_h$ (compare with Lemma 5.4).

The set-theoretical idea behind the general notion of holomorphism is quite easy: given a 2-cell

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \end{tikzcd}$$

in $\mathrm{Grpd}(\mathcal{G})$, F, G and α itself are completely determined by the map associating to an internal arrow $(a \colon x \to y) \in C_1$ the diagonal $(Fx \to Gy) \in B_1$ of the commutative square

$$\begin{tikzcd} Fx \arrow[r, "\alpha(x)"] \arrow[d, "Fa"'] & Gx \arrow[d, "Ga"] \\ Fy \arrow[r, "\alpha(y)"'] & Gy \end{tikzcd}$$

To make this more precise, we need some preliminary work. Consider two internal groupoids $\mathbb{C}$, $\mathbb{B}$, and let $h \colon C_1 \to B_1$ be an arrow making commutative the following diagrams

$$\begin{tikzcd} C_1 \arrow[r, "h"] \arrow[d, "\mathrm{dom}"'] & B_1 \arrow[r, "\mathrm{dom}"] & B_0 \\ C_0 \arrow[r, "u"'] & C_1 \arrow[r, "h"'] & B_1 \arrow[u, "\mathrm{dom}"'] \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} C_1 \arrow[r, "h"] \arrow[d, "\mathrm{cod}"'] & B_1 \arrow[r, "\mathrm{cod}"] & B_0 \\ C_0 \arrow[r, "u"'] & C_1 \arrow[r, "h"'] & B_1 \arrow[u, "\mathrm{cod}"'] \end{tikzcd} \tag{3}$$

Thanks to conditions (2) and (3), we get two arrows

$$\widehat{h} = \langle \pi_1 \cdot h, \pi_1 \cdot \mathrm{cod} \cdot u \cdot h \cdot ()^{-1}, \pi_2 \cdot h \rangle \colon C_1 \times_{C_0} C_1 \to B_1 \times_{B_0} B_1 \times_{B_0} B_1$$
$$\widetilde{h} = \langle \pi_1 \cdot h, \pi_2 \cdot h, \pi_3 \cdot h \rangle \colon P \to Q$$

where P and Q are defined by the following limits in $\mathcal{G}$

$$\begin{tikzcd} C_1 \arrow[d, "\mathrm{cod}"'] & P \arrow[l, "\pi_1"'] \arrow[r, "\pi_3"] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "\mathrm{dom}"] \\ C_0 & C_1 \arrow[l, "\mathrm{cod}"] \arrow[r, "\mathrm{dom}"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[d, "\mathrm{cod}"'] & Q \arrow[l, "\pi_1"'] \arrow[r, "\pi_3"] \arrow[d, "\pi_2"'] & B_1 \arrow[d, "\mathrm{dom}"] \\ B_0 & B_1 \arrow[l, "\mathrm{cod}"] \arrow[r, "\mathrm{dom}"'] & B_0 \end{tikzcd}$$

Lemma 5.2. (1) *Diagram (4) commutes iff diagram (5) commutes*

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\circ"] \arrow[d, "\widehat{h}"'] & C_1 \arrow[d, "h"] \\ B_1 \times_{B_0} B_1 \times_{B_0} B_1 \arrow[r, "\circ^2"'] & B_1 \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} P \arrow[rr, "\widetilde{h}"] \arrow[d, "\langle \pi_1, \pi_2 \cdot (\,)^{-1}, \pi_3 \rangle"'] & & Q \arrow[d, "\langle \pi_1, \pi_2 \cdot (\,)^{-1}, \pi_3 \rangle"] \\ C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[d, "\circ^2"'] & & B_1 \times_{B_0} B_1 \times_{B_0} B_1 \arrow[d, "\circ^2"] \\ C_1 \arrow[rr, "h"'] & & B_1 \end{tikzcd} \tag{5}$$

(2) *Diagram (6) commutes iff diagram (7) commutes*

$$\begin{tikzcd} C_0 \arrow[r, "u"] \arrow[d, "u"'] & C_1 \arrow[r, "h"] & B_1 \\ C_1 \arrow[r, "h"'] & B_1 \arrow[r, "\mathrm{dom}"'] & B_0 \arrow[u, "u"'] \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} C_0 \arrow[r, "u"] \arrow[d, "u"'] & C_1 \arrow[r, "h"] & B_1 \\ C_1 \arrow[r, "h"'] & B_1 \arrow[r, "\mathrm{cod}"'] & B_0 \arrow[u, "u"'] \end{tikzcd} \tag{7}$$

Proof. Part (2) being quite obvious, let us concentrate on part (1). If the base category, $\mathcal{G}$ is the category of sets and $a: x \to y, b: z \to y, c: z \to w$ are elements of C_1, condition (5) means that $h(a \cdot b^{-1} \cdot c) = h(a) \cdot h(b)^{-1} \cdot h(c)$. Condition (4) expresses the special case of condition (5) where $z = y$ and $b = 1_y$. It is therefore clear that (5) $\Rightarrow$ (4). Conversely, assume that h satisfies (4) and put

$$\delta_h : C_1 \to B_1 \quad \delta_h(\, x \xrightarrow{a} y \,) = h(a) \cdot h(1_y)^{-1}.$$

Clearly, δ_h preserves units. Moreover, (4) immediately implies that δ_h preserves composition too. Therefore,

$$\begin{aligned} h(a \cdot b^{-1} \cdot c) &= \delta_h(a \cdot b^{-1} \cdot c) \cdot h(1_w) = \delta_h(a) \cdot \delta_h(b)^{-1} \cdot \delta_h(c) \cdot h(1_w) \\ &= h(a) \cdot h(1_y)^{-1} \cdot (h(b) \cdot h(1_y)^{-1})^{-1} \cdot h(c) \cdot h(1_w)^{-1} \cdot h(1_w) = h(a) \cdot h(b)^{-1} \cdot h(c). \end{aligned}$$

This concludes the proof when $\mathcal{G}$ is the category of sets. Following Theorem 1.2, the result holds for any finitely complete category $\mathcal{G}$. $\square$

Definition 5.3. Consider two groupoids $\mathbb{C}$, $\mathbb{B}$ in $\mathcal{G}$.

(1) A holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is an arrow $h\colon C_1 \to B_1$ making commutative diagram (2), diagram (3), and diagram (4).
(2) A holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is pointed if it makes commutative diagram (6).

Lemma 5.4. (1) *Holomorphisms and pointed holomorphisms are stable under composition in $\mathcal{G}$.*
(2) *If $h\colon \mathbb{C} \to \mathbb{B}$ is a holomorphism, then the arrows*

$$\delta_h\colon\ C_1 \xrightarrow{\langle h,\, \mathrm{cod}\cdot u\cdot h\cdot(\ )^{-1}\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

$$\gamma_h\colon\ C_1 \xrightarrow{\langle \mathrm{dom}\cdot u\cdot h\cdot(\ )^{-1},\, h\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

are pointed holomorphisms from $\mathbb{C}$ to $\mathbb{B}$. We call δ_h the domain of h and γ_h the codomain of h.

Proof. The proof is straightforward if $\mathcal{G}$ is the category of sets. One obtains the result using once again Theorem 1.2. □

We are ready to describe the 2-category $\mathrm{Hol}(\mathcal{G})$ of holomorphisms:

- Objects are internal groupoids in $\mathcal{G}$.
- If $\mathbb{C}$ and $\mathbb{B}$ are internal groupoids, 1-cells $\mathbb{C} \to \mathbb{B}$ are pointed holomorphisms.
- Composition of 1-cells $f\colon \mathbb{C} \to \mathbb{B}$, $g\colon \mathbb{B} \to \mathbb{D}$ is the composition of $f\colon C_1 \to B_1$ and $g\colon B_1 \to D_1$ in $\mathcal{G}$.
- If $f, g\colon \mathbb{C} \to \mathbb{B}$ are pointed holomorphisms, 2-cells $f \Rightarrow g$ are holomorphisms $h\colon \mathbb{C} \to \mathbb{B}$ such that $\delta_h = f$ and $\gamma_h = g$.
- Horizontal composition of 2-cells $h\colon f \Rightarrow g\colon \mathbb{C} \to \mathbb{B}$, $k\colon f' \Rightarrow g'\colon \mathbb{B} \to \mathbb{D}$ is the composition of $h\colon C_1 \to B_1$ and $k\colon B_1 \to D_1$ in $\mathcal{G}$.
- If $h, k\colon \mathbb{C} \to \mathbb{B}$ are holomorphisms with $\gamma_h = \delta_k$, their vertical composition is given by

$$C_1 \xrightarrow{\langle \mathrm{dom}\cdot u\cdot h,\, k\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle h,\, \mathrm{cod}\cdot u\cdot k\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1.$$

- The identity 2-cell on a 1-cell $f\colon \mathbb{C} \to \mathbb{B}$ is f itself.

Because of the way holomorphisms compose, we have the following fact.

Corollary 5.5. *A holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ is invertible with respect to horizontal composition iff $h\colon C_1 \to B_1$ is an isomorphism in $\mathcal{G}$.*

As announced at the beginning of this section, $\mathrm{Hol}(\mathcal{G})$ provides an equivalent description of $\mathrm{Grpd}(\mathcal{G})$. In fact, we have the following result.

Proposition 5.6. *There is a 2-functor $\epsilon\colon \mathrm{Hol}(\mathcal{G}) \to \mathrm{Grpd}(\mathcal{G})$ which is the identity on objects and an isomorphism on hom-categories. The 2-functor ϵ restricts to the sub-2-categories of isomorphisms $\mathrm{Hol}(\mathcal{G})^* \to \mathrm{Grpd}(\mathcal{G})^*$.*

Proof. If $f\colon \mathbb{C} \to \mathbb{B}$ is a pointed holomorphism, we get an internal functor $\epsilon(f) = (F_1, F_0)\colon \mathbb{C} \to \mathbb{B}$ by $F_1 = f\colon C_1 \to B_1$ and $F_0 = u \cdot f \cdot \mathrm{dom}\colon C_0 \to B_0$.
If $h\colon \mathbb{C} \to \mathbb{B}$ is a holomorphism, we get an internal natural transformation $\epsilon(h)\colon \epsilon(\delta_h) \Rightarrow \epsilon(\gamma_h)$ by $\epsilon(h) = u \cdot h\colon C_0 \to C_1 \to B_1$.
Conversely, if $\alpha\colon F = (F_1, F_0) \Rightarrow G = (G_1, G_0)\colon \mathbb{C} \to \mathbb{B}$ is an internal natural transformation (with $\alpha\colon C_0 \to B_1$), we get a holomorphism $h\colon \mathbb{C} \to \mathbb{B}$ by

$$C_1 \xrightarrow{\langle F_1,\, \mathrm{cod}\cdot\alpha\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1$$

or, equivalently, by

$$C_1 \xrightarrow{\langle \mathrm{dom}\cdot\alpha,\, G_1\rangle} B_1 \times_{B_0} B_1 \xrightarrow{\ \circ\ } B_1.$$

Details are routine and are left to the reader. □

6. Translations

In this section, we specialize the notion of holomorphism to get a different description of derivations in terms of what we call translations. Once again, the name is justified by the case of a group G seen as a groupoid with just one object: in that case, a translation in the sense of Definition 6.1 is precisely a translation in the usual sense, see Example 6.4.

Definition 6.1. The monoid of $\mathbb{C}$-translations is the kernel of the codomain map

$$\mathrm{Tr}\,\mathbb{C} = \mathrm{Ker}(\mathrm{cod}) \to [\mathrm{Hol}(\mathcal{G})(\mathbb{C},\mathbb{C})]_1 \to [\mathrm{Hol}(\mathcal{G})(\mathbb{C},\mathbb{C})]_0.$$

As we did in Proposition 2.2 with derivations, we give now a more geometrical description of translations. Fix an arrow $t\colon C_1 \to C_1$ such that the diagram

$$\begin{tikzcd} C_1 \arrow[rr, "t"] \arrow[dr, "\mathrm{cod}"'] & & C_1 \arrow[dl, "\mathrm{cod}"] \\ & C_1 & \end{tikzcd} \tag{8}$$

commutes, and consider the factorizations

$$\widehat{t} = \langle \mathrm{dom}\cdot u\cdot t, 1\rangle \colon C_1 \to C_1 \times_{C_0} C_1, \qquad \widetilde{t} = \langle \pi_1\cdot t, \pi_2\rangle \colon C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1.$$

Lemma 6.2. *Diagram (9) commutes iff diagram (10) commutes*

$$\begin{tikzcd} C_1 \arrow[rr, "t"] \arrow[dr, "\widehat{t}"'] & & C_1 \\ & C_1 \times_{C_0} C_1 \arrow[ur, "\circ"'] & \end{tikzcd} \tag{9}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\circ"] \arrow[d, "\widetilde{t}"'] & C_1 \arrow[d, "t"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\circ"'] & C_1 \end{tikzcd} \tag{10}$$

Proof. Let us sketch the proof in the case where $\mathcal{G}$ is the category of sets. The first condition means that for any arrow $a\colon x \to y$, the diagram

$$\begin{tikzcd} \cdot \arrow[rr, "t(a)"] \arrow[dr, "t(1_x)"'] & & y \\ & x \arrow[ur, "a"'] & \end{tikzcd}$$

commutes; the second condition means that for any composable pair of arrows

$$z \xrightarrow{b} x \xrightarrow{a} y$$

the diagram

$$\begin{tikzcd} \cdot \arrow[rr, "t(ba)"] \arrow[dr, "t(b)"'] & & y \\ & x \arrow[ur, "a"'] & \end{tikzcd}$$

commutes. Clearly, the first condition is a special case of the second one (take $b = 1_x$). Conversely, using the first condition, both sides of the diagram expressing the second condition are equal to

$$\xrightarrow{t(1_x)} z \xrightarrow{b} y \xrightarrow{a} x.$$

The general case follows using Theorem 1.2. □

Proposition 6.3. *To give a $\mathbb{C}$-translation amounts to giving an arrow $t\colon C_1 \to C_1$ such that diagrams (8) and (9) commute.*

Proof. Routine. □

Example 6.4. Consider once again a group G as a groupoid $\mathbb{C}$ with a single object. Thanks to Proposition 6.3, a $\mathbb{C}$-translation in the sense of Definition 6.1 is nothing but a map $t\colon G \to G$ such that $t(a) = t(0) + a$ for all $a \in G$. That is, t is the right translation by $t(0)$. Therefore, in this case $\mathrm{Tr}\,\mathbb{C}$ is a group isomorphic to G.

In contrast with the situation described in Example 6.4, the monoid $\mathrm{Tr}\,\mathbb{C}$ in general is not a group.

Corollary 6.5. *The group of regular translation $\mathrm{Tr}^*\,\mathbb{C}$ is given by $\mathrm{Tr}\,\mathbb{C} \cap \mathrm{Aut}\ C_1$.*

By Proposition 4.7, we get the following corollary.

Corollary 6.6. *The 2-functor $\epsilon\colon \mathrm{Hol}(\mathcal{G}) \to \mathrm{Grpd}(\mathcal{G})$ induces two isomorphisms of monoids $\mathrm{Tr}\,\mathbb{C} \simeq \mathrm{Der}\,\mathbb{C}$ and $\mathrm{Tr}^*\,\mathbb{C} \simeq \mathrm{Der}^*\,\mathbb{C}$.*

We can describe the isomorphism $\mathrm{Tr}\,\mathbb{C} \simeq \mathrm{Der}\,\mathbb{C}$ using Propositions 2.2 and 6.3:

- Given $(t\colon C_1 \to C_1) \in \mathrm{Tr}\,\mathbb{C}$, we get $(u \cdot t\colon C_0 \to C_1 \to C_1) \in \mathrm{Der}\,\mathbb{C}$.
- Given $(d\colon C_0 \to C_1) \in \mathrm{Der}\,\mathbb{C}$, we get $C_1 \xrightarrow{\langle \mathrm{dom}\cdot d,\, 1\rangle} C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \ \in \mathrm{Tr}\,\mathbb{C}$.

Let us summarize the situation we have so far with the following picture, where the unlabelled vertical arrows are the inclusion of the kernel.

$$\begin{tikzcd}
\mathrm{Tr}^*\,\mathbb{C} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Der}^*\,\mathbb{C} \arrow[d] \\
{[\mathrm{Hol}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1} \arrow[r, "\simeq"] \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] & {[\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1} \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] \\
{[\mathrm{Hol}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0} \arrow[r, "\simeq"] & {[\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0}
\end{tikzcd}$$

Example 6.7. Since $\mathrm{Hol}(\mathcal{G})^*(\mathbb{C},\mathbb{C})$ is a groupoid in groups, using the constructions described in Example 2.3 we can pass to a crossed module of groups, and then come back to a groupoid isomorphic to $\mathrm{Hol}(\mathcal{G})^*(\mathbb{C},\mathbb{C})$. Using also the isomorphisms of the previous picture, we get a group isomorphism

$$[\mathrm{Hol}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_1 \simeq \mathrm{Tr}^*\,\mathbb{C} \rtimes [\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C},\mathbb{C})]_0.$$

If we specialize this isomorphism to the case where $\mathcal{G}$ is the category of sets and $\mathbb{C}$ is the one-object groupoid associated to a group G (Examples 5.1 and 6.4), we get the classical isomorphism

$$\mathrm{Hol}\,G \simeq G \rtimes \mathrm{Aut}\,G$$

where $\mathrm{Hol}G$ is the group of bijective holomorphisms from G to G (see [21], Section IV.1).

Example 6.8. Let us consider a K-vector space $\mathcal{V}$ as a 1-object (commutative) groupoid with an additional multiplicative structure. It is easy to verify that

$$[\mathrm{Hol}^*(\mathcal{V})]_0 \simeq GL(\mathcal{V})$$

i.e. the general linear group of $\mathcal{V}$. It is less obvious that $[\mathrm{Hol}^*(\mathcal{V})]_1$ corresponds to the group of affine transformations of $\mathbb{A}$, the affine space of the "points" of $\mathcal{V}$. Moreover, this group is canonically isomorphic to the semidirect product

$$Tr^*\mathcal{V} \rtimes GL(\mathcal{V})$$

where $Tr^*\mathcal{V}(\simeq \mathcal{V})$ is indeed the group of translations of $\mathbb{A}$.
More precisely, if a is the affine structure of $\mathbb{A}$ (*i.e.* $a(P, Q) = Q - P$), an affine transformation is a pair of bijections

$$(F : \mathbb{A} \to \mathbb{A}, \varphi : \mathcal{V} \to \mathcal{V})$$

resp. in **Set** and in $\mathbf{Vect_K}$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A} \times \mathbb{A} \arrow[r, "a"] \arrow[d, "F \times F"'] & \mathcal{V} \arrow[d, "\varphi"] \\
\mathbb{A} \times \mathbb{A} \arrow[r, "a"'] & \mathcal{V}
\end{tikzcd}$$

The linear map φ is determined by F in the following way: fix a point P and set

$$\varphi_P : \mathbf{v} \mapsto F(P + \mathbf{v}) - F(P).$$

Now, given a bijection $F : \mathbb{A} \to \mathbb{A}$, F induces an affine transformation iff φ_P is linear. This expresses exactly the condition of diagram (4). In fact,

$$\varphi_P(\mathbf{v} + \mathbf{w}) = \varphi_P(\mathbf{v}) + \varphi_P(\mathbf{w})$$
$$F(P + \mathbf{v} + \mathbf{w}) - F(P) = F(P + \mathbf{v}) - F(P) + F(P + \mathbf{w}) - F(P)$$
$$F(P + \mathbf{v} + \mathbf{w}) = F(P + \mathbf{v}) - F(P) + F(P + \mathbf{w})$$
$$F(\mathbf{v} + \mathbf{w}) = F(\mathbf{v}) - F(0) + F(\mathbf{w})$$

where in the last line we identify $\mathbb{A}$ with $\mathcal{V}$.
This example may be further generalized to modules, and to not-necessarily bijective maps. Under this perspective, group holomorphisms are group transformations that preserve the affine structure of a group, namely equivalent bi-points.

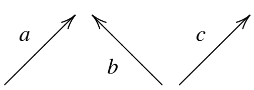
Example 6.9. As in Example 6.7, we have a group isomorphism

$$[\mathrm{Hol}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1 \simeq \mathrm{Der}^*\,\mathbb{C} \rtimes [\mathrm{Grpd}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_0.$$

This generalizes the isomorphism established by Lue in the case where $\mathcal{G}$ is the category of groups, and groupoids are replaced by crossed modules (see [18], Theorem 9). It is interesting to observe that, in that case, only the analogue of our conditions (2) and (3) are used to define the analogue of $[\mathrm{Hol}(\mathcal{G})^*(\mathbb{C}, \mathbb{C})]_1$. This is because the category of groups is a Mal'cev category (see [1] for the notion of Mal'cev category). In fact, we have the following result.

Lemma 6.10. *Let $\mathcal{G}$ be a Mal'cev category, and consider two groupoids $\mathbb{C}, \mathbb{B}$ in $\mathcal{G}$. If an arrow $h : C_1 \to B_1$ satisfies conditions (2) and (3), then it is a holomorphism.*

Proof. We shall use set theoretic notations, using once again Theorem 1.2. By results in [7], we know that, if $\mathbb{C}$ is a groupoid in a Mal'cev category, composition is uniquely determined and it corresponds to a (unique) partial associative Mal'cev operation p on C_1. This operation p is defined on those (a, b, c) in C_1 such that $\mathrm{cod}(a) = \mathrm{cod}(b)$ and $\mathrm{dom}(b) = \mathrm{dom}(c)$:



and, in this case,

$$p(a, b, c) = a \circ b^{-1} \circ c.$$

Now, given two groupoids $\mathbb{C}, \mathbb{B}$ and (a, b, c) in C_1 such that $\mathrm{cod}(a) = \mathrm{cod}(b)$ and $\mathrm{dom}(b) = \mathrm{dom}(c)$, if an arrow $h\colon C_1 \to B_1$ satisfies conditions (2) and (3), $h(a), h(b)^{-1}, h(c)$ are composable; that is $\mathrm{cod}(h(a)) = \mathrm{cod}(h(b))$ and $\mathrm{dom}(h(b)) = \mathrm{dom}(h(c))$. This means that the (unique) partial associative Mal'cev operation p' on B_1 associated to composition in $\mathbb{B}$ is defined on $(h(a), h(b), h(c))$ and h satisfies condition (5) if and only if

$$h(p(a, b, c)) = p'(h(a), h(b), h(c)).$$

But this is true because in any Mal'cev category, every relation $\mathcal{R}$ is difunctional; that is $\mathcal{R} \circ \mathcal{R}^{op} \circ \mathcal{R} = \mathcal{R}$. In fact, let $\mathcal{R}$ be the relation on $C_1^2 \times C_1$ given by

$$(a, b)\mathcal{R}c \Leftrightarrow \mathrm{cod}(a) = \mathrm{cod}(b), \qquad \mathrm{dom}(b) = \mathrm{dom}(c) \quad \text{and} \quad h(p(a, b, c)) = p'(h(a), h(b), h(c)).$$

Now, if $\mathrm{cod}(a) = \mathrm{cod}(b)$, $\mathrm{dom}(b) = \mathrm{dom}(c)$, the following three conditions hold:

- $(a, b)\mathcal{R}b$, since $h(p(a, b, b)) = h(a) = p'(h(a), h(b), h(b))$
- $(b, b)\mathcal{R}b$, as above, with $a = b$
- $(b, b)\mathcal{R}c$, since $h(p(b, b, c)) = h(c) = p'(h(b), h(b), h(c))$.

Hence, by difunctionality, $(a, b)\mathcal{R}c$, that is $h(p(a, b, c)) = p'(h(a), h(b), h(c))$. □

Example 6.11. Observe that, by Proposition 2.2 and Corollary 3.2, if the domain and codomain maps of an internal groupoid $\mathbb{C}$ are equal, then $\mathbb{C}$-derivations are invertible. This is the case when $\mathbb{C}$ is the groupoid in groups associated with a crossed module of the form $H \xrightarrow{0} G \xrightarrow{\varphi} \mathrm{Aut}\,H$, where $\varphi\colon G \to \mathrm{Aut}\,H$ is a G-module and $0\colon H \to G$ is the zero-morphism. Indeed, in this case, both domain and codomain coincide with the second projection $\pi_2\colon H \rtimes G \to G$. Moreover, in this case a classical result (see [21], Proposition IV.2.1) asserts that the group $\mathrm{Der}^*\,\mathbb{C}$ is isomorphic to the group of isomorphisms $t\colon H \rtimes G \to H \rtimes G$ making commutative the following diagrams

$$\begin{tikzcd} H \rtimes G \arrow[r, "t"] \arrow[dr, "\pi_2"'] & H \rtimes G \arrow[d, "\pi_2"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "i"] \arrow[dr, "i"'] & H \rtimes G \arrow[d, "t"] \\ & H \rtimes G \end{tikzcd}$$

where $i(a) = (a, 1)$. Since the first diagram is precisely diagram (8), and the commutativity of the second diagram is equivalent to the commutativity of diagram (9); this description of $\mathrm{Der}^*\,\mathbb{C}$ is a specialization of the isomorphism $\mathrm{Der}^*\,\mathbb{C} \simeq \mathrm{Tr}^*\,\mathbb{C}$ established in 6.6. Even for an arbitrary crossed module $H \to G \to \mathrm{Aut}\,H$, the group $\mathrm{Der}^*(G, H)$ can be described as a suitable subgroup of $\mathrm{Aut}\,(H \rtimes G)$; see Proposition 3.5 in [11]. Once again, this description is a particular case of the isomorphism $\mathrm{Der}^*\,\mathbb{C} \simeq \mathrm{Tr}^*\,\mathbb{C}$.

7. The embedding category of an internal groupoid

If $(H \xrightarrow{\partial} G \xrightarrow{\varphi} \mathrm{Aut}\,H)$ is a crossed module of groups and $F_0\colon A_0 \to G$ is a group homomorphism, a derivation relative to F_0, or F_0-derivation, is a map $d\colon A_0 \to H$ such that $d(xy) = d(x) + F_0(x) \cdot d(y)$. Equivalently, an F_0-derivation is a group homomorphism $d\colon A_0 \to H \rtimes G$ such that

$$\begin{tikzcd} & H \rtimes G \arrow[d, "\pi"] \\ A_0 \arrow[ur, "d"] \arrow[r, "F_0"'] & G \end{tikzcd}$$

commutes. Relative derivations have been used in non-Abelian cohomology of groups, see for example [12,17,18], and contain derivations as a special case (take as F_0 the identity morphism).

Relative derivations can be defined in the general context of internal groupoids in a finitely complete category $\mathcal{G}$ in much the same way as we did for derivations in Section 2. In this section, we develop a different approach: for a fixed groupoid $\mathbb{C}$ in $\mathcal{G}$, we construct the category of embeddings $\mathrm{Emb}\,\mathbb{C}$ as the underlying category of a comma 2-category, and we show that derivations and relative derivations can be obtained as particular hom-sets from $\mathrm{Emb}\,\mathbb{C}$. The guiding

Example 7.5, which justifies our terminology, is the category of embeddings of an étale groupoid, introduced in [22] to study the homotopy type of the groupoid.

Definition 7.1. (1) Let $\mathbb{A}, \mathbb{C}$ be internal groupoids in $\mathcal{G}$. An internal functor $F : \mathbb{A} \to \mathbb{C}$ is full and faithful if, for every internal groupoid $\mathbb{X}$, the functor $\mathrm{hom}(\mathbb{X}, F) : hom(\mathbb{X}, \mathbb{A}) \to hom(\mathbb{X}, \mathbb{C})$ is full and faithful in the usual sense.

(2) The category of embeddings $\mathrm{Emb}\,\mathbb{C}$ has full and faithful functors $F : \mathbb{A} \to \mathbb{C}$ as objects. An arrow from $F : \mathbb{A} \to \mathbb{C}$ to $G : \mathbb{B} \to \mathbb{C}$ is a pair

$$(D : \mathbb{A} \to \mathbb{B}, d : D \cdot G \Rightarrow F)$$

with D an internal functor and d an internal natural transformation. Composition and identities are the obvious ones.

Clearly, $\mathrm{Emb}\,\mathbb{C}(Id_{\mathbb{C}}, Id_{\mathbb{C}}) = \mathrm{Der}\,\mathbb{C}$. More is true:

Proposition 7.2. *If $F : \mathbb{A} \to \mathbb{C}$ is full and faithful, then*

$$- \cdot F : \mathrm{Der}\,\mathbb{A} \to \mathrm{Emb}\,\mathbb{C}(F, F) \qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "D"] \arrow[r, bend right, "Id"'] \arrow[r, phantom, "\Downarrow d"] & \mathbb{C} \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{A} \arrow[rr, "D"] \arrow[dr, "F"'] & \Leftarrow d \circ F & \mathbb{A} \arrow[dl, "F"] \\ & \mathbb{C} & \end{tikzcd}$$

is an isomorphism of monoids.

Proof. This immediately follows from the definition of full and faithful internal functor. $\square$

To recover relative derivations from the category of embeddings is less trivial. We need the following result from [5].

Lemma 7.3. *An internal functor $F : \mathbb{A} \to \mathbb{C}$ is full and faithful iff the diagram*

$$\begin{tikzcd} & & A_1 \arrow[dll, "\mathrm{dom}"'] \arrow[d, "F_1"] \arrow[drr, "\mathrm{cod}"] & & \\ A_0 \arrow[dr, "F_0"'] & & C_1 \arrow[dl, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"'] & & A_0 \arrow[dl, "F_0"] \\ & C_0 & & C_0 & \end{tikzcd}$$

is a limit diagram in $\mathcal{G}$.

Proof. Assume that the diagram in the statement is a limit diagram. Consider internal functors $L, H : \mathbb{X} \to \mathbb{A}$ and an internal natural transformation $\alpha : L \cdot F \Rightarrow H \cdot F$. Explicitly

$$\begin{tikzcd} X_1 \arrow[r, shift left, "L_1"] \arrow[r, shift right, "H_1"'] \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] & A_1 \arrow[r, "F_1"] & C_1 \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] \\ X_0 \arrow[r, shift left, "L_0"] \arrow[r, shift right, "H_0"'] \arrow[urr, "\alpha"] & A_0 \arrow[r, "F_0"'] & C_0 \end{tikzcd}$$

Therefore, $\alpha \cdot \mathrm{dom} = L_0 \cdot F_0$ and $\alpha \cdot \mathrm{cod} = H_0 \cdot F_0$. From the universal property of the limit A_1, we get a unique arrow $\beta : X_0 \to A_1$ such that $\beta \cdot \mathrm{dom} = L_0$, $\beta \cdot \mathrm{cod} = H_0$ and $\beta \cdot F_1 = \alpha$. This means that β is an internal natural transformation from L to H such that $\beta \circ F = \alpha$. (The naturality of β is expressed by the commutativity of

$$\begin{tikzcd}
X_1 \arrow[r, "\langle \mathrm{dom}\cdot\beta, H_1\rangle"] \arrow[d, "\langle L_1, \mathrm{cod}\cdot\beta\rangle"'] & A_1 \times_{A_0} A_1 \arrow[d, "\circ"] \\
A_1 \times_{A_0} A_1 \arrow[r, "\circ"] & A_1
\end{tikzcd}$$

which is easy to check by composing with the projections of the limit A_1 and using the naturality of α.)
The converse implication follows from the fact that to give a factorization of a commutative diagram

$$\begin{tikzcd}
 & & X_0 \arrow[dll, "L_0"'] \arrow[d, "\alpha"] \arrow[drr, "H_0"] & & \\
A_0 \arrow[dr, "F_0"'] & & C_1 \arrow[dl, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"'] & & A_0 \arrow[dl, "F_0"] \\
 & C_0 & & C_0 &
\end{tikzcd}$$

through the diagram in the statement corresponds to give a (necessarily natural) transformation $\beta \colon L \Rightarrow H$ such that $\beta \circ F = \alpha \colon L \cdot F \Rightarrow H \cdot F$, where L and H are internal functors with discrete domain, as in the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, shift left, "L_0 \cdot u"] \arrow[r, shift right, "H_0 \cdot u"'] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & A_1 \arrow[d, shift right, "\mathrm{dom}"'] \arrow[d, shift left, "\mathrm{cod}"] \\
X_0 \arrow[r, shift left, "L_0"] \arrow[r, shift right, "H_0"'] & A_0
\end{tikzcd} \qquad \square$$

Proposition 7.4. *Let $\mathbb{C}$ be an internal groupoid. The category* $\mathrm{Emb}\,\mathbb{C}$ *can be described in the following way:*

- *Objects are arrows $F_0 \colon A_0 \to C_0$ in $\mathcal{G}$.*
- *A morphism from $F_0 \colon A_0 \to C_0$ to $G_0 \colon B_0 \to C_0$ is a pair $(D_0 \colon A_0 \to B_0, d \colon A_0 \to C_1)$ of arrows in $\mathcal{G}$ such that*

$$\begin{tikzcd}
 & C_1 \arrow[d, "\mathrm{cod}"] \\
A_0 \arrow[ur, "d"] \arrow[r, "F_0"'] & C_0
\end{tikzcd} \qquad
\begin{tikzcd}
 & & C_1 \arrow[d, "\mathrm{dom}"] \\
A_0 \arrow[urr, "d"] \arrow[r, "D_0"'] & B_0 \arrow[r, "G_0"'] & C_0
\end{tikzcd}$$

commute.

Proof. Let us concentrate on objects (the argument for arrows is similar). Given an arrow $F_0 \colon A_0 \to C_0$, the limit

$$\begin{tikzcd}
 & & A_1 \arrow[dll, "\mathrm{dom}"'] \arrow[d, "F_1"] \arrow[drr, "\mathrm{cod}"] & & \\
A_0 \arrow[dr, "F_0"'] & & C_1 \arrow[dl, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"'] & & A_0 \arrow[dl, "F_0"] \\
 & C_0 & & C_0 &
\end{tikzcd}$$

produces an internal groupoid $\mathbb{A}$ and a full and faithful internal functor $F = (F_1, F_0) \colon \mathbb{A} \to \mathbb{C}$. Conversely, if $F \colon \mathbb{A} \to \mathbb{C}$ is full and faithful, then the unit $u \colon A_0 \to A_1$ and the composition $\circ \colon A_1 \times_{A_0} A_1 \to A_1$ of $\mathbb{A}$ are the unique factorizations through the previous limit of the following commutative diagrams

$$\begin{tikzcd}
 & & A_0 \arrow[dll, "1"'] \arrow[d, "F_0\cdot u"] \arrow[drr, "1"] & & \\
A_0 \arrow[dr, "F_0"'] & & C_1 \arrow[dl, "dom"] \arrow[dr, "cod"'] & & A_0 \arrow[dl, "F_0"] \\
 & C_0 & & C_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A_1 \times_{A_0} A_1 \arrow[dll, "\pi_1"'] \arrow[d, "F_1\times_{F_0}F_1"] \arrow[drr, "\pi_2"] & & \\
A_1 \arrow[d, "dom"] & & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] & & A_1 \arrow[d, "cod"'] \\
A_0 \arrow[dr, "F_0"'] & & C_1 \arrow[dl, "dom"] \arrow[dr, "cod"'] & & A_0 \arrow[dl, "F_0"] \\
 & C_0 & & C_0 &
\end{tikzcd}$$ □

Example 7.5. Let $\mathcal{G}$ be the category of topological spaces and local homeomorphisms, and let $\mathbb{C}$ be an internal groupoid in $\mathcal{G}$, that is $\mathbb{C}$ is an étale groupoid. Fix a basis $\mathcal{B}$ of contractible open sets for the space C_0. The category of embeddings $Emb_{\mathcal{B}}\mathbb{C}$ (see [22]) is the full subcategory of Emb $\mathbb{C}$ with objects the inclusions $U \to C_0$, $U \in \mathcal{B}$. Indeed, following Proposition 7.4, if G_0 is a monomorphism, then an arrow (D_0, d) is determined by d, in the sense that if D_0 exists, it is unique. (Note that in [22], the role of dom and cod is inverted: the equations defining an arrow (D_0, d) in 7.4 should be written as $d \cdot dom = F_0$ and $d \cdot cod = D_0 \cdot G_0$. Since $\mathbb{C}$ is a groupoid, this makes no difference.)

Example 7.6. Consider a crossed module $H \to G \to \mathrm{Aut}\, H$, the corresponding internal groupoid $\mathbb{C}$, and a group homomorphism $F_0 \colon A_0 \to G$. The set of F_0-derivations is in bijection with the hom-set $\mathrm{Emb}\,\mathbb{C}(F, Id_{\mathbb{C}})$, where $F \colon \mathbb{A} \to \mathbb{C}$ is the full and faithful internal functor corresponding to $F_0 \colon A_0 \to C_0$ as in the proof of Proposition 7.4. Indeed, following the description of Emb $\mathbb{C}$ given in 7.4, if $G = Id_{\mathbb{C}}$, then D_0 necessarily is $d \cdot dom$.

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References

[1] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Kluwer Academic Publishers, 2004.
[2] F. Borceux, G. Janelidze, Galois Theories, Cambridge University Press, 2001.
[3] R. Brown, Topology and Groupoids, 2006. www.groupoids.org.
[4] R. Brown, C.B. Spencer, G-groupoids, crossed modules and the fundamental groupoid of a topological group, Proceedings of the Koninklijke Nederlandse Akademie Van Wetenschappen Series A—Mathematical Sciences 79 (1976) 296–302.
[5] M. Bunge, R. Paré, Stacks and equivalence of indexed categories, Cahiers de Topologie et Géométrie Différentielle 20 (1979) 373–399.
[6] M. Casas, M. José, Invariantes de módulo cruzados en álgebras de Lie, Alxebra 57 (1991).
[7] A. Carboni, M.C. Pedicchio, N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, in: Category Theory 1991, in: CMS Conference Proceedings, vol. 13, 1992, pp. 97–109.
[8] T. Datuashvili, Whitehead homotopy equivalence and internal category equivalence of crossed modules in categories of groups with operations, Proceedings of A. Razmadze Mathematical Institute 113 (1995) 3–30.
[9] C. Ehresmann, Catégories et Structures, Dunod, 1965.
[10] C. Ehresmann, Catégories topologiques, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen 69 (1966) 133–175.
[11] N.D. Gilbert, Derivations, automorphisms and crossed modules, Communications in Algebra 18 (1990) 2703–2734.
[12] D. Guin, Cohomologie et homologie non abélienne des groupes, Journal of Pure and Applied Algebra 50 (1988) 109–137.
[13] P.J. Higgins, Introduction to Topological Groups, Cambridge University Press, 1974.
[14] P.J. Higgins, Categories and groupoids, Reprints in Theory and Applications of Categories 7 (2005) 1–195.
[15] G. Janelidze, Internal crossed modules, Georgian Mathematical Journal 10 (2003) 99–114.
[16] G. Janelidze, L. Márki, W. Tholen, Semi-Abelian categories, Journal of Pure and Applied Algebra 168 (2002) 367–386.
[17] R. Lavendhomme, J.R. Roisin, Cohomologie non abélienne de structures algébriques, Journal of Algebra 67 (1980) 385–414.
[18] A.S.-T. Lue, Semi-complete crossed modules and holomorphs of groups, Bulletin of the London Mathematical Society 11 (1979) 8–16.
[19] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, Cambridge University Press, 1987.
[20] K. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, Cambridge University Press, 2005.
[21] S. MacLane, Homology, Springer Verlag, 1975.
[22] I. Moerdijk, On the weak homotopy type of étale groupoids, in: Integrable systems and foliations, Birkhäuser, 1997, pp. 147–156.

[23] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids, Cambridge University Press, 2003.
[24] K. Norrie, Actions and automorphisms of crossed modules, Bulletin de la Société Mathématique Française 118 (1990) 129–146.
[25] A. Ramsay, J. Renault (Eds.), Groupoids in Analysis, Geometry, and Physics, American Mathematical Society, 2001.
[26] J.H.C. Whitehead, On operators in relative homotopy groups, Annals of Mathematics 49 (1948) 610–640.