

Regularity of the Category of Kelley Spaces

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Abstract. We show that the cartesian closed category of compactly generated Hausdorff spaces is regular, but is neither exact, nor locally cartesian closed. In fact we find a coequalizer of an equivalence relation which is not stable under pullback.

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Introduction

It is well known that the category **Top** of topological spaces has not a good behaviour with respect to two fundamental constructions, namely function spaces and coequalizers of equivalence relations, since **Top** is neither cartesian closed, nor exact, nor regular (by exact category we mean a regular category with effective equivalence relations, cf. [1]). (Day and Kelly in [5] characterized both cartesian objects and universal quotients of **Top**.)

In order to avoid these difficulties which arise in **Top**, one can try to find appropriate categories of topological spaces which do have such properties.

From Manes theorem, we know that the category **Hcompt** of compact Hausdorff spaces is exact, but it is not cartesian closed (see the characterization of exponentiable objects and maps in **Hcompt** in [4]). We can enlarge this category in order to obtain cartesian closedness: it is sufficient to consider the category **Kelley** of Kelley spaces (compactly generated Hausdorff spaces) (cf. [6, 10]). The problem now is to see if the property of exactness still remains or at least to see whether some weaker property holds.

The first result of this paper is to show that **Kelley** is *regular*.

Since a necessary condition for exactness is that coequalizers of equivalence relations are pullback stable, in order to know if **Kelley** is exact, we could investigate the behaviour of pullback functors with respect to the preservation of coequalizers. Via the special adjoint functor theorem, in this case the preservation of coequalizers and coproducts is equivalent to the existence of a right adjoint to

pullback functors, that is to the local cartesian closedness. Furthermore in **Kelley** coproducts coincide with topological sums and they are stable under pullback, so that preservation of coequalizers would tell us that **Kelley** is also locally cartesian closed. But this is not the case. In fact what we find is that in this category pullback functors preserve regular epimorphisms, but not coequalizers, not even coequalizers of equivalence relations, which means that **Kelley** is also not exact. (Notice that, since in the case of **Hcomp** pullback functors preserve coequalizers, but fail to preserve coproducts, both the categories we are interested in are not locally cartesian closed, but for opposite reasons.)

As a final remark, we can look at the exact completion, say $\mathcal{E}_{\text{ex/reg}}$, of **Kelley**, which exists because **Kelley** is regular (cf. [3]). Since in **Kelley** coproducts are disjoint and universal, the same happens in $\mathcal{E}_{\text{ex/reg}}$. This means that what $\mathcal{E}_{\text{ex/reg}}$ needs for being a Grothendieck topos is, according to the Giraud's conditions (cf., for example [2, p. 238]), a set of regular generators. But the existence of such a set $\mathcal{E}_{\text{ex/reg}}$ would imply the existence of a set of regular generators for **Kelley** and this is impossible, as Johnstone shows in [7, 10.8 and 10.9].

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Let us start with the first problem faced in the introduction, namely the regularity of **Kelley**.

DEFINITION (cf. [1, 8]). We recall that a category $\mathcal{C}$ is said to be *regular* if the following holds:

- (1) $\mathcal{C}$ has finite limits.
- (2) Every morphism has the (regular epimorphism, monomorphism)-factorization.
- (3) Regular epimorphisms are stable under pullback, that is a pullback of a regular epimorphism along any morphism is a regular epimorphism.

$\mathcal{C}$ is said to be *exact* when it is regular and has *effective* equivalence relations (which means that they coincide with the kernel pair of their coequalizer).

The first two conditions are satisfied in any coreflective subcategory (the inclusion functor has a right adjoint) of the category **Haus** of Hausdorff spaces. For the last condition, we need some additional hypothesis, as the following shows:

LEMMA. *Let $\mathcal{C}$ be a cartesian closed category with regular epimorphisms stable under pullback along regular monomorphisms.*

Then regular epimorphisms are stable under pullback along any morphism.

Proof. Let $q: X \rightarrow Y$ be a regular epimorphism in $\mathcal{C}$ and consider its pullback q' along any morphism $f: T \rightarrow Y$

$$\begin{array}{ccc}
 T \times_Y X & \xrightarrow{f'} & X \\
 \downarrow q' & & \downarrow q \\
 T & \xrightarrow{f} & Y
 \end{array}
 .$$

As a consequence the following diagram is again a pullback diagram:

$$\begin{array}{ccc}
 T \times_Y X & \xrightarrow{\langle q', f' \rangle} & T \times X \\
 \downarrow q' & & \downarrow \text{id}_T \times q \\
 T & \xrightarrow{\langle \text{id}_T, f \rangle} & T \times Y
 \end{array}
 .$$

$(\text{id}_T \times q): T \times X \rightarrow T \times Y$ is a regular epimorphism, since $T \times (-)$ preserves colimits ($\mathcal{C}$ is cartesian closed). Furthermore, if we denote by p_T, p_Y the projections of $T \times Y$ onto its factors, $\langle \text{id}_T, f \rangle: T \rightarrow T \times Y$ is the equalizer of (fp_T, p_Y) . Then q' is obtained as the pullback of a regular epimorphism along a regular monomorphism, so it is itself a regular epimorphism. $\square$

Applying the previous lemma to the category of Kelley spaces we obtain (cf. [11]):

THEOREM. *The category of Kelley spaces is regular.*

Proof. Since **Kelley** is coreflective (and then bicoreflective) in the category of Hausdorff spaces, it is closed under quotients in **Haus** and the regular epimorphisms in **Kelley** are exactly the topological quotients. Thus **Kelley** has the (regular epimorphism = quotient, monomorphism)-factorization. Furthermore, **Kelley** is closed under the formation of equalizers in **Haus** and then regular monomorphisms coincide with closed embeddings. Since topological quotients are stable under pullback along closed embeddings, we can apply the previous lemma, showing that **Kelley** is regular. $\square$

Remark. The previous theorem holds in any cartesian closed category of topological spaces having the (quotient, injection)-factorization and equalizers which are closed embeddings. In particular we have the regularity of the category of weakly Hausdorff k -spaces (cf., e.g., [9]) and of the category of sequential Hausdorff spaces (cf. [7]).

Now we want to investigate the behaviour of pullback functors related to coproducts and coequalizers. In the case of **Hcomp**, where coequalizers are preserved by pullbacks along any morphism, coproducts are not preserved. In fact, if you take the Stone-Čech compactification of the natural numbers $\beta N = \coprod_1^{+\infty} \{n\}$, the pullbacks of the inclusions $i_n: \{n\} \rightarrow \beta N$, along the inclusion of $\beta N \setminus N$

in βN are the inclusions of the empty set into $\beta N \setminus N$, which clearly does not coincide with the coproducts of countable copies of the empty set.

In the case of **Kelley**, coproducts coincide with topological sums and are preserved by pullbacks along any morphism. In fact, consider a coproduct $\sum_I X_i$ in **Kelley**: since it is a topological sum, its pullback P in **Haus** (which is the same as in **Top**) is the topological sum of the pullbacks X'_i in **Haus** of any cofactor. But the pullback cP in **Kelley** of $\sum_I X_i$ is the coreflection of $P = \sum_I X'_i$ in **Kelley**, and, since this coreflection preserves topological sums, cP coincides with the coproduct $\sum_I cX'_i$ of the coreflection cX'_i of any cofactor X'_i . But any of these cX'_i is exactly the pullback of X_i in **Kelley** and then we have that coproducts are universal also in **Kelley** (cf. 2.2.III in [6]).

The reason which does not allow **Kelley** to be locally cartesian closed lays in the subtle, but substantial difference between preservation by pullbacks of regular epimorphisms and preservation of coequalizers. In particular we will find an equivalence relation whose coequalizer is not preserved. This example shows at the same time that **Kelley** can be neither locally cartesian closed, nor exact, since in exact categories equivalence relations are effective and therefore their coequalizers must be stable under pullback.

PROPOSITION. *The category of Kelley spaces is neither locally cartesian closed nor exact*

Proof. Consider the unit interval $I = [0, 1]$ and the coproduct $I + I$ of two copies of I with the two coprojections $i_1, i_2: I \rightarrow I + I$. Denote by I_1 and by I_2 the copies of I in $I + I$ given respectively by $i_1(I)$ and $i_2(I)$. Then the product $(I + I) \times (I + I)$ can be identified with $(I_1)^2 + (I_2)^2 + I_1 \times I_2 + I_2 \times I_1$. Let $\Delta_1, \Delta_2, \Delta_{1,2}$, and $\Delta_{2,1}$ be the diagonals of any of the squares, in the same order. Consider now the subspace R of $(I + I) \times (I + I)$ given (with suitable identifications) by $\Delta_1 + \Delta_2 + \Delta_{1,2} \setminus \{(0, 0)\} + \Delta_{2,1} \setminus \{(0, 0)\}$. R is not compact, but it is still a Kelley space, since it is first countable. Furthermore R is an equivalence relation on $I + I$, namely the relation whose elements are the pairs $(i_j(0), i_j(0))$ for $j = 1, 2$ and $(i_k(x), i_h(x))$, for $k, h = 1, 2$ and $x \in I, x \neq 0$. Roughly speaking, any element of I_1 but 0 is in relation with the same element in the other copy I_2 and any of the copies of 0 is in relation only with itself. The coequalizer of this equivalence relation in **Kelley** is the same as the coequalizer q in **Haus** (**Kelley** is coreflective in **Haus**). On the other hand, q is obtained taking the reflection in **Haus** of the coequalizer $\bar{q}$ in **Top**. It easy to see that $\bar{q}: I + I \rightarrow J$ identifies $I_1 \setminus \{0\}$ with $I_2 \setminus \{0\}$, while $\bar{q}$ distinguishes the two copies of 0 in $I + I$. Furthermore these two points have not disjoint neighbourhoods in J , which means that J is not Hausdorff. So the Hausdorff reflection of J is homeomorphic to I .

Consider now the inclusion $m: \{0\} \rightarrow I$ of the point 0 in I . We want to show that the pullback along m does not preserve the coequalizer q .

The pullback q' of q along m is the quotient of the two-point discrete space D_2 over the singleton:

$$\begin{array}{ccc} D_2 & \xrightarrow{q'} & \{0\} \\ \downarrow m' & & \downarrow m \\ I + I & \xrightarrow{q} & I \end{array} .$$

If coequalizers were preserved by pullbacks, q' should be the coequalizer of the pair given by the pullback of qr_0 and qr_1 along m , where r_0, r_1 are the two projections of R on $I + I$. This pair is given by (id, id) , where id is the identity of D_2 . But the coequalizer of (id, id) must be an isomorphism and so cannot coincide with q' . $\square$

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