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Localizations of algebraic categories

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Abstract

We characterize localizations of monadic categories over $\mathcal{S}\mathcal{E}\mathcal{T}$ using the fact that the category of algebras for a monad over $\mathcal{S}\mathcal{E}\mathcal{T}$ is the exact completion of the full subcategory of free algebras. This also constitutes an unifying argument to characterize reflections and epi-reflections.

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1. Preliminaries

The aim of this work is to characterize localizations of monadic categories over $\mathcal{S}\mathcal{E}\mathcal{T}$. For this, let us consider a monad $\mathbb{T}$ over $\mathcal{S}\mathcal{E}\mathcal{T}$ and write $\mathbf{KL}(\mathbb{T})$ and $\mathbf{EM}(\mathbb{T})$, respectively, for the Kleisli category of $\mathbb{T}$ and for the Eilenberg–Moore category of $\mathbb{T}$. They can be characterized as follows (all categories are supposed to be locally small):

- (I) consider a category $\mathbb{C}$; the following conditions are equivalent:
 - (1) $\mathbb{C}$ is equivalent to the category $\mathbf{KL}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{S}\mathcal{E}\mathcal{T}$;
 - (2) there exists an object G in $\mathbb{C}$ which admits all copowers and such that each object of $\mathbb{C}$ is isomorphic to a copower of G ;
- (II) consider a category $\mathbb{A}$; the following conditions are equivalent:
 - (1) $\mathbb{A}$ is equivalent to the category $\mathbf{EM}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{S}\mathcal{E}\mathcal{T}$;
 - (2) $\mathbb{A}$ is exact and has a regular projective regular generator which admits all copowers.

An easy proof can be found in [8], where it is shown that the basic fact to obtain the second characterization is to observe that $\mathbf{KL}(\mathbb{T})$ is a *projective cover* of $\mathbf{EM}(\mathbb{T})$, that is each object of $\mathbf{KL}(\mathbb{T})$ is regular projective in $\mathbf{EM}(\mathbb{T})$ and for each object X of $\mathbf{EM}(\mathbb{T})$ there exists an object P in $\mathbf{KL}(\mathbb{T})$ together with a regular epimorphism $P \rightarrow X$.

The fact that $\mathbf{KL}(\mathbb{T})$ is a projective cover of the exact category $\mathbf{EM}(\mathbb{T})$ means exactly that $\mathbf{EM}(\mathbb{T})$ is the *exact completion* of $\mathbf{KL}(\mathbb{T})$. We recall here some basic features of

the exact completion; for more details we refer to [2] and to [7], whereas for an introduction to exact and monadic categories we refer to [1].

Definition 1.1. Let $F: \mathbb{C} \rightarrow \mathbb{A}$ be a functor between a weakly lex category $\mathbb{C}$ and a lex category $\mathbb{A}$; F is left covering if for each finite diagram $\mathcal{L}: \mathcal{D} \rightarrow \mathbb{C}$ and for each (equivalently, for one) weak limit $\text{wlim } \mathcal{L}$, the canonical factorization $p: F(\text{wlim } \mathcal{L}) \rightarrow \lim \mathcal{L} \cdot F$ is a strong epimorphism.

Proposition 1.2. For each weakly lex category $\mathbb{C}$ there exist an exact category $\mathbb{C}_{\text{ex}}$ and a left covering functor $\Gamma: \mathbb{C} \rightarrow \mathbb{C}_{\text{ex}}$ such that, for each exact category $\mathbb{B}$, composing with Γ induces an equivalence

$$\Gamma \cdot -: \text{Ex}[\mathbb{C}_{\text{ex}}, \mathbb{B}] \rightarrow \text{Lco}[\mathbb{C}, \mathbb{B}]$$

between the category of exact functors from $\mathbb{C}_{\text{ex}}$ to $\mathbb{B}$ and the category of left covering functors from $\mathbb{C}$ to $\mathbb{B}$. Moreover, if $\mathbb{A}$ is an exact category and $\mathbb{P}$ is a projective cover of $\mathbb{A}$, then $\mathbb{P}$ is weakly lex and $\mathbb{A}$ is equivalent to $\mathbb{P}_{\text{ex}}$. $\square$

In what follows, we use not only the universal property of $\Gamma: \mathbb{C} \rightarrow \mathbb{C}_{\text{ex}}$, but also an explicit description of $\mathbb{C}_{\text{ex}}$:

- an object of $\mathbb{C}_{\text{ex}}$ is a pseudo equivalence relation in $\mathbb{C}$, that is a pair of arrows $r_1, r_2: R \rightrightarrows X$ (not necessarily monomorphic) such that there exist reflexivity $r_R: X \rightarrow R$, symmetry $s_R: R \rightarrow R$ and transitivity $t_R: R * R \rightarrow R$ (where $R * R$ is a weak pullback of r_1 and r_2) satisfying the usual equations;
- an arrow in $\mathbb{C}_{\text{ex}}$ is an equivalence class of pairs of arrows $(\bar{f}, f)$ (as in the following diagram) such that $\bar{f} \cdot s_1 = r_1 \cdot f$ and $\bar{f} \cdot s_2 = r_2 \cdot f$

$$\begin{array}{ccc} R & \xrightarrow{\bar{f}} & S \\ r_1 \downarrow & & \downarrow s_1 \\ X & \xrightarrow{f} & Y \\ & & \downarrow s_2 \end{array}$$

two pairs $(\bar{f}, f)$ and $(\bar{g}, g)$ from $(r_1, r_2: R \rightrightarrows X)$ to $(s_1, s_2: S \rightrightarrows Y)$ are equivalent if there exists an arrow $\Sigma: X \rightarrow S$ such that $\Sigma \cdot s_1 = f$ and $\Sigma \cdot s_2 = g$.

2. Localizations

We are now ready to prove our characterization.

Proposition 2.1. Consider a category $\mathbb{B}$; the following conditions are equivalent:

- (1) $\mathbb{B}$ is equivalent to a localization of $EM(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SET}$ (that is a reflective subcategory such that the reflector is lex);
- (2) $\mathbb{B}$ is exact and has a regular generator which admits all copowers.

Proof. The implication $1 \Rightarrow 2$ is quite obvious, so let us look at the implication $2 \Rightarrow 1$. Let G be a regular generator as in condition (2) and let us fix some notations: if S is a set, $S \bullet G$ is the S -indexed copower of G and $i_s: G \rightarrow S \bullet G$ is the s th canonical injection ($s \in S$); if $\alpha: S \rightarrow T$ is in $\mathcal{SEF}$, $\alpha': S \bullet G \rightarrow T \bullet G$ is the arrow in $\mathbb{B}$ defined by $i_s \cdot \alpha' = i_{\alpha(s)}$

$$\begin{array}{ccc} S \bullet G & \xrightarrow{\alpha'} & T \bullet G \\ & \swarrow i_t \quad \searrow i_{\alpha(s)} & \\ & G & \end{array}$$

Given an object A in $\mathbb{B}$, the canonical cover of A by G is the unique arrow $a: \mathbb{B}(G, A) \bullet G \rightarrow A$ such that for each $f: G \rightarrow A$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}(G, A) \bullet G & \xrightarrow{a} & A \\ & \swarrow i_f \quad \searrow f & \\ & G & \end{array}$$

The fact that G is a regular generator means exactly that, for each object A of $\mathbb{B}$, such cover is a regular epimorphism. Now consider the full subcategory $\mathbb{C}$ of $\mathbb{B}$ spanned by copowers of G .

Step 1: the full inclusion $F: \mathbb{C} \rightarrow \mathbb{B}$ is a left covering functor.

Consider a finite diagram $\mathcal{L}: \mathcal{D} \rightarrow \mathbb{C}$ and its limit in $\mathbb{B}$

$$\lim \mathcal{L} \cdot F = (\pi_D: L \longrightarrow \mathcal{L}(D))_{D \in \mathcal{D}_0}.$$

We obtain a weak limit in $\mathbb{C}$ precomposing each projection with the canonical cover by G , $l: \mathbb{B}(G, L) \bullet G \rightarrow L$. By assumption, l is a regular epimorphism. This means exactly that F is left covering (recall that, in a regular category, strong epimorphisms coincide with regular epimorphisms). By the universal property of the exact completion $\Gamma: \mathbb{C} \rightarrow \mathbb{C}_{\text{ex}}$, this implies that there exists an exact functor $\hat{F}: \mathbb{C}_{\text{ex}} \rightarrow \mathbb{B}$ such that F and $\Gamma \cdot \hat{F}$ are naturally isomorphic. Let us recall that, if

$$\begin{array}{ccc} R \bullet G & \xrightarrow{\bar{f}} & S \bullet G \\ \downarrow r_1 & & \downarrow s_1 \\ \downarrow r_2 & & \downarrow s_2 \\ X \bullet G & \xrightarrow{f} & Y \bullet G \end{array}$$

is an arrow in $\mathbb{C}_{\text{ex}}$, $\hat{F}[\bar{f}, f]$ is the unique extension to the quotient as in the following diagram:

$$\begin{array}{ccccc}
 R \bullet G & \xrightarrow[r_2]{r_1} & X \bullet G & \xrightarrow{q_1} & A \\
 \bar{f} \downarrow & & \downarrow f & & \downarrow \hat{F}[\bar{f}, f] \\
 S \bullet G & \xrightarrow[s_2]{s_1} & Y \bullet G & \xrightarrow{q_2} & B
 \end{array}$$

Step 2: embedding of $\mathbb{B}$ in $\mathbb{C}_{\text{ex}}$.

Given an object A in $\mathbb{B}$, consider its canonical cover by G , $a: \mathbb{B}(G, A) \bullet G \rightarrow A$, the kernel pair of a

$$a_1, a_2: N(a) \rightrightarrows \mathbb{B}(G, A) \bullet G$$

and again the canonical cover by G , $n: \mathbb{B}(G, N(a)) \bullet G \rightarrow N(a)$. It is only a straightforward calculation to prove that the pair of arrows

$$n \cdot a_1, n \cdot a_2: \mathbb{B}(G, N(a)) \bullet G \rightrightarrows \mathbb{B}(G, A) \bullet G$$

is an object of $\mathbb{C}_{\text{ex}}$. Consider now an arrow $\varphi: A \rightarrow B$ in $\mathbb{B}$; we can build up the following diagram, commutative in each part

$$\begin{array}{ccccccc}
 \mathbb{B}(G, N(a)) \bullet G & \xrightarrow{n} & N(a) & \xrightarrow[a_2]{a_1} & \mathbb{B}(G, A) \bullet G & \xrightarrow{a} & A \\
 \downarrow \bar{\alpha}' & & \downarrow t & & \downarrow \alpha' & & \downarrow \varphi \\
 \mathbb{B}(G, N(b)) \bullet G & \xrightarrow{m} & N(b) & \xrightarrow[b_2]{b_1} & \mathbb{B}(G, B) \bullet G & \xrightarrow{b} & B
 \end{array}$$

The construction of the horizontal lines has just been explained; as far as the columns are concerned, α' is induced by $\alpha: \mathbb{B}(G, A) \rightarrow \mathbb{B}(G, B)$ which sends $h: G \rightarrow A$ into $h \cdot \varphi$; the existence of a unique t such that $t \cdot b_1 = a_1 \cdot \alpha'$ and $t \cdot b_2 = a_2 \cdot \alpha'$ follows from $a \cdot \varphi = \alpha' \cdot b$ and the universal property of $N(b)$; $\bar{\alpha}'$ is induced by $\bar{\alpha}: \mathbb{B}(G, N(a)) \rightarrow \mathbb{B}(G, N(b))$ which sends $h: G \rightarrow N(a)$ into $h \cdot t$. In particular, $[\bar{\alpha}', \alpha']$ gives us an arrow in $\mathbb{C}_{\text{ex}}$ which we take as value of a functor $r: \mathbb{B} \rightarrow \mathbb{C}_{\text{ex}}$. Once again, it is a straightforward calculation to prove that r is full and faithful.

Step 3: adjunction $\hat{F} \dashv r$.

Let $(r_1, r_2: R \bullet G \rightrightarrows X \bullet G)$ be an object in $\mathbb{C}_{\text{ex}}$, consider its coequalizer $q: X \bullet G \rightarrow A$ in $\mathbb{B}$ (that is $A = \hat{F}(r_1, r_2)$) and build up

$$r(A) = (n \cdot a_1, n \cdot a_2: \mathbb{B}(G, N(a)) \bullet G \rightrightarrows \mathbb{B}(G, A) \bullet G).$$

The unit of the adjunction $\hat{F} \dashv r$ must be an arrow in $\mathbb{C}_{\text{ex}}$ of the following kind:

$$\begin{array}{ccc}
 R \bullet G & \xrightarrow{\bar{\eta}'} & \mathbb{B}(G, N(a)) \bullet G \\
 \downarrow r_1 \quad \downarrow r_2 & & \downarrow n \cdot a_1 \quad \downarrow n \cdot a_2 \\
 X \bullet G & \xrightarrow{\eta'} & \mathbb{B}(G, A) \bullet G
 \end{array}$$

As η' we take the arrow induced by $\eta: X \rightarrow \mathbb{B}(G, A)$ which sends $x \in X$ into $i_x \cdot q: G \rightarrow X \bullet G \rightarrow A$. Now observe that with this definition $\eta' \cdot a = q$ and then $r_1 \cdot \eta' \cdot a = r_2 \cdot \eta' \cdot a$. This implies that there exists a unique arrow $\tau: R \bullet G \rightarrow N(a)$ such that $\tau \cdot a_1 = r_1 \cdot \eta'$ and $\tau \cdot a_2 = r_2 \cdot \eta'$. Now as $\bar{\eta}'$ we can take the arrow induced by $\bar{\eta}: R \rightarrow \mathbb{B}(G, N(a))$ which sends $r \in R$ into $i_r \cdot \tau: G \rightarrow R \bullet G \rightarrow N(a)$.

Conclusion. We have just shown that $\mathbb{B}$ is (equivalent to) a localization of $\mathbb{C}_{\text{ex}}$. But, from the preliminaries, we know that the full subcategory $\mathbb{C}$ of $\mathbb{B}$ is equivalent to $\text{KL}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SEF}$ and then its exact completion $\mathbb{C}_{\text{ex}}$ is equivalent to $\text{EM}(\mathbb{T})$. The proof of Proposition 2.1 is now complete. $\square$

3. Related results

Let us look more carefully at the proof of Proposition 2.1. If, instead of exact, $\mathbb{B}$ is assumed to be only left exact but with coequalizers, we can again define $\hat{F}: \mathbb{C}_{\text{ex}} \rightarrow \mathbb{B}$ as at the end of the first step. Since in the second and the third steps we do not use the exactness of $\mathbb{B}$, we have the following:

Proposition 3.1. *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

- (1) $\mathbb{B}$ is equivalent to a reflective subcategory of $\text{EM}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SEF}$;
- (2) $\mathbb{B}$ is left exact with coequalizers and has a regular generator which admits all copowers.

Working essentially in the same way (that is working with the formal description of $\mathbb{C}_{\text{ex}}$ and forgetting that it is equivalent to $\text{EM}(\mathbb{T})$) one can also prove the following proposition (more details can be found in [7]):

Proposition 3.2. *Consider a category $\mathbb{B}$; the following conditions are equivalent:*

- (1) $\mathbb{B}$ is equivalent to an eireflective subcategory of $\text{EM}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SEF}$ (eireflective = units are regular epimorphisms);
- (2) $\mathbb{B}$ is regular with coequalizers of equivalence relations and has a regular projective regular generator which admits all copowers.

Remarks: (i) In Proposition 3.1 it does not suffice to assume the existence of coequalizers of pseudo equivalence relations. This is because (with the notations of

Proposition 2.1) a pseudo equivalence relation in $\mathbb{C}$ is not necessarily a pseudo equivalence relation in the whole category $\mathbb{B}$. On the contrary, in Proposition 3.2 we only need coequalizers of equivalence relations because the jointly monic part of the (regular epi, mono) factorization of a pseudo equivalence relation in $\mathbb{C}$ is an equivalence relation in $\mathbb{B}$.

(ii) In the characterization of $\text{EM}(\mathbb{T})$ as well as in Proposition 3.2, the regularity of the category is a little bit redundant; in fact, the stability of regular epimorphisms under pullbacks follows from the other assumptions (cf. Lesson 2 in [4]). This is not true in Proposition 2.1, because there the regular generator in general is not regular projective.

(iii) Propositions 3.1 and 3.2 (or, at least, the second one) are well-known (see [5], where this kind of characterization is used to study Malcev conditions in varietal and quasi-varietal categories, [3, 6]). I have quoted them here because I think it is remarkable that the theory of the exact completion gives us a general framework to prove (in a quite straightforward way) all characterization theorems contained in this work.

(iv) All results can be obviously generalized to monads over a power $\mathcal{SET}^X$ of $\mathcal{SET}$. To characterize $\text{KL}(\mathbb{T})$, we need an X -indexed family of objects G_x in $\mathbb{C}$ such that: (i) for each $f: S \rightarrow X$ in $\mathcal{SET}$ there exists $\coprod_{s \in S} G_{f(s)}$; (ii) for each object C in $\mathbb{C}$ there exists $f: S \rightarrow X$ in $\mathcal{SET}$ such that $C \simeq \coprod_{s \in S} G_{f(s)}$. Now the other results hold replacing the single generator with an X -indexed family of generators which admit all sums.

(v) With a little bit more effort, one can adapt the proof of Proposition 2.1 to obtain an elementary proof of the well-known fact that the Giraud axioms characterize localizations of presheaf categories. The major difference is that $\mathbb{C}$ must be the sum-completion of the full subcategory $\mathcal{G}$ of generators, so that $\mathbb{C}_{\text{ex}}$ is equivalent to $\mathcal{SET}^{\mathcal{G}^{\text{op}}}$. For this, we take as arrows from a generator to a sum of generators only the arrows which factor through a canonical injection (generators are indecomposable in $\mathcal{SET}^{\mathcal{G}^{\text{op}}}$). Now the inclusion of $\mathbb{C}$ in $\mathbb{B}$ is not full, but it remains left covering in virtue of the extensivity assumption on $\mathbb{B}$ (sums are disjoint and universal).

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