ELSEVIER Journal of Pure and Applied Algebra 179 (2003) 65–86
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Proper factorization systems in 2-categories

M. Dupont, E.M. Vitale*

Département de Mathématiques, Université catholique de Louvain. Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

Received 4 April 2002
Communicated by J. Adámek

En mémoire de René Lavendhomme

Abstract

Starting from known examples of factorization systems in 2-categories, we discuss possible definitions of proper factorization system in a 2-category. We focus our attention on the construction of the free proper factorization system on a given 2-category.

MSC: 18A20; 18A32; 18D05

0. Introduction

The notion of factorization system in a category is well established and has a lot of applications to basic category theory [7] as well as to some more specific topic, like categorical topology [11] or categorical Galois theory [8]. When a relevant construction emerges in mathematics the question of existence of such free structure is always important. Korostenski and Tholen [19], study the free category with factorization system on a given category $\mathscr{C}$. They prove that it is given by the embedding $\mathscr{C} \to \mathscr{C}^2$ of $\mathscr{C}$ into its category of morphisms.

In general, given a factorization system $(\mathscr{E}, \mathscr{M})$ in a category and the corresponding factorization $f = (m \in \mathscr{M}) \circ (e \in \mathscr{E})$ of an arrow f, it is a common intuition to think to e as the "surjective" part of f and to m as the "injective" part of f. This is the case for the standard factorization system in Set, as well as for many other natural examples, but it is by no way a consequence of the definition of factorization system.

* Corresponding author.
E-mail addresses: dupont@math.ucl.ac.be (M. Dupont), vitale@math.ucl.ac.be (E.M. Vitale).

PII: S0022-4049(02)00244-X

A factorization system such that the class $\mathscr{E}$ is contained in the class of epimorphisms and the class $\mathscr{M}$ in that of monomorphisms is called proper. The free category $\mathrm{Fr}\,\mathscr{C}$ with proper factorization system on a given category $\mathscr{C}$ has been studied by Grandis [15], where it is proved that $\mathrm{Fr}\mathscr{C}$ is a quotient of $\mathscr{C}^2$, so that we can picture the situation with the diagram

$$\mathscr{C} \to \mathscr{C}^2 \to \mathrm{Fr}\mathscr{C}.$$

The category $\mathrm{Fr}\mathscr{C}$ is of special interest for its applications to the stable homotopy category (in this case it is also called the Freyd completion of $\mathscr{C}$, which explains the notation), to homology theories and to triangulated categories (see [5,10,13,14,22,24]).

For the needs of two-dimensional homological algebra, Kasangian and the second author introduced in [18] the notion of factorization system in a 2-category with invertible 2-arrows, showing the existence of two such factorization systems in the 2-category SCG of symmetric categorical groups. Subsequently, the definition has been extended by Milius to arbitrary 2-categories in [21], where the basic theory is developed. In particular, Milius exhibits the free 2-category with factorization system $\mathbb{C} \to \mathbb{C}^2$ on a given 2-category $\mathbb{C}$, which is the two-dimensional analogue of the Korostenski–Tholen construction. The aim of this note is to complete the picture, giving the two-dimensional analogue of Grandis construction, that is the free 2-category with proper factorization system.

For this, let us look more carefully at the two factorization systems for symmetric categorical groups discussed in [18]. In the first one, an arrow F factors through the kernel of its cokernel; in the second one it factors through the cokernel of its kernel

$$\begin{tikzcd}
 & & \mathrm{Ker}\,(P_F) \arrow[dr, "M_1"] & & \\
\mathrm{Ker}\,F \arrow[r, "e_F"] & \mathscr{A} \arrow[ur, "E_1"] \arrow[rr, "F"] \arrow[dr, "E_2"'] & & \mathscr{B} \arrow[r, "P_F"] & \mathrm{Coker}\,F \\
 & & \mathrm{Coker}\,(e_F) \arrow[ur, "M_2"'] & &
\end{tikzcd}$$

and one has that E_1 is full and essentially surjective, M_1 is faithful, E_2 is essentially surjective and M_2 is full and faithful.

Now, for a morphism F in SCG (that is, F is a monoidal functor compatible with the symmetry), one has the following situation:

- F is faithful (respectively, full and faithful) iff for any $\mathscr{G} \in \mathrm{SCG}$, the hom-functor $\mathrm{SCG}(\mathscr{G}, F)\colon \mathrm{SCG}(\mathscr{G}, \mathscr{A}) \to \mathrm{SCG}(\mathscr{G}, \mathscr{B})$ is faithful (respectively, full and faithful);
- F is essentially surjective (respectively, full and essentially surjective) iff for any $\mathscr{G} \in \mathrm{SCG}$, the hom-functor $\mathrm{SCG}(F, \mathscr{G})\colon \mathrm{SCG}(\mathscr{B}, \mathscr{G}) \to \mathrm{SCG}(\mathscr{A}, \mathscr{G})$ is faithful (respectively, full and faithful).

This situation suggests to analyze the following variants of the notion of proper factorization system in a 2-category $\mathbb{C}$: a factorization system $(\mathscr{E}, \mathscr{M})$ is

- (1,1)-proper if for any $f \in \mathscr{M}$ the hom-functors $\mathbb{C}(X, f)$ are faithful and for any $f \in \mathscr{E}$ the hom-functors $\mathbb{C}(f, X)$ are faithful (with X varying in $\mathbb{C}$);
- (2,1)-proper if it is (1,1)-proper and moreover for any $f \in \mathscr{E}$ the hom-functors $\mathbb{C}(f, X)$ are full;
- (1,2)-proper if it is (1,1)-proper and moreover for any $f \in \mathscr{M}$ the hom-functors $\mathbb{C}(X, f)$ are full;
- (2,2)-proper if it is (2,1)-proper and (1,2)-proper, i.e. if for any $f \in \mathscr{M}$ the hom-functors $\mathbb{C}(X, f)$ are fully faithful and for any $f \in \mathscr{E}$ the hom-functors $\mathbb{C}(f, X)$ are fully faithful.

For these four kinds of proper factorization systems, we give the construction of the free 2-category with proper factorization system on a given 2-category $\mathbb{C}$. The situation can be summarized in the following diagram (where $\mathrm{Fr}^{i,j}\mathbb{C}$ is the free 2-category with (i, j)-proper factorization system)

$$\begin{tikzcd} & & & \mathrm{Fr}^{1,2}\mathbb{C} \arrow[dr] & \\ \mathbb{C} \arrow[r] & \mathbb{C}^2 \arrow[r] & \mathrm{Fr}^{1,1}\mathbb{C} \arrow[ur] \arrow[dr] & & \mathrm{Fr}^{2,2}\mathbb{C} \\ & & & \mathrm{Fr}^{2,1}\mathbb{C} \arrow[ur] & \end{tikzcd}$$

(conditions on $\mathbb{C}$ are needed to define $\mathrm{Fr}^{2,2}\mathbb{C}$, see Section 6).

The embedding $\mathscr{C} \to \mathrm{Fr}\mathscr{C}$ is a step in the construction of the free regular, exact or abelian category on $\mathscr{C}$ (see [20,24]). From this point of view, the present paper is part of a program devoted to study similar notions for 2-categories, and it is intended to clarify the delicate notions of monomorphism and epimorphism in a 2-categorical setting (see also [1,3,6,9,17,25]).

The paper is organized as follows. In Section 1 we give the definition of factorization system in a 2-category as it appears in [12,21]. It is slightly different from that given in [18], but they are equivalent if the 2-cells are invertible. In Section 2 we recall, from [21], the construction of the free 2-category with factorization system. In Section 3 we fix the terminology for arrows in a 2-category. In Sections 4–6 we describe the various $\mathrm{Fr}^{i,j}\mathbb{C}$ and we prove their universal property. Section 7 is devoted to examples and to an open problem. Finally, in Section 8, we give a glance at the relation between factorization systems in 2-categories and in categories. If $\mathbb{C}$ is a locally discrete 2-category (that is, a category), then our definition coincide with the usual definition of factorization system. But a factorization system in a 2-category $\mathbb{C}$ does not induce a factorization system (in the usual sense) neither in the underlying category of $\mathbb{C}$ nor in the homotopy category $H(\mathbb{C})$ of $\mathbb{C}$. The best we can say is that it induces in $H(\mathbb{C})$ a weak factorization system (a structure of interest especially for Quillen approach to homotopy theory, see [2,4,16,23]), and even this fact is not completely obvious to prove.

1. Factorization systems in 2-categories

To define the notion of factorization system in a 2-category, we need the orthogonality condition. A first 2-categorical version of this condition was introduced in [18] for a 2-category with invertible 2-cells. Since we work in an arbitrary 2-category, we need a stronger version, as in [12,21].

Definition 1.1. Let $\mathbb{C}$ be a 2-category and consider two arrows $f : C \to C'$ and $g : D \to D'$ in $\mathbb{C}$. We say that f *is orthogonal to* g, denoted by $f \downarrow g$, if the following diagram is a bi-pullback in Cat:

$$\begin{tikzcd}
\mathbb{C}(C',D) \arrow[r, "-\circ f"] \arrow[d, "g\circ -"'] & \mathbb{C}(C,D) \arrow[d, "g\circ -"] \\
\mathbb{C}(C',D') \arrow[r, "-\circ f"'] & \mathbb{C}(C,D')
\end{tikzcd}$$

If $\mathscr{H}$ is a class of arrows of $\mathbb{C}$, we write $\mathscr{H}^{\uparrow} = \{e \mid e \downarrow h \text{ for all } h \in \mathscr{H}\}$ and $\mathscr{H}^{\downarrow} = \{m \mid h \downarrow m \text{ for all } h \in \mathscr{H}\}$.

To make the previous definition more explicit, we need some point of terminology.

Definition 1.2. The 2-*category of arrows* of $\mathbb{C}$, denoted by $\mathbb{C}^2$, is the 2-category whose objects are arrows of $\mathbb{C}$, whose 1-cells are triples (u, φ, v), as in the following diagram, where φ is invertible,

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi \nearrow"] & D \arrow[d, "g"] \\
C' \arrow[r, "v"'] & D'
\end{tikzcd}$$

and whose 2-cells $(u, \varphi, v) \Rightarrow (w, \psi, x) : f \to g$ are pairs (α, β) of 2-cells of $\mathbb{C}$, with $\alpha : u \Rightarrow w$ and $\beta : v \Rightarrow x$ such that

$$(g * \alpha) \circ \varphi = \psi \circ (\beta * f).$$

Definition 1.3. Let (u, φ, v) be an arrow from f to g in $\mathbb{C}^2$. A *fill-in* for (u, φ, v) is a triple (α, s, β), as in the following diagram, with $\alpha : sf \Rightarrow u$ and $\beta : gs \Rightarrow v$ invertible and such that $g * \alpha = \varphi(\beta * f)$.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "u"'] & C' \arrow[d, "v"] \arrow[dl, "s"'] \\
D \arrow[r, "g"'] & D'
\end{tikzcd}$$

The fill-in (α, s, β) is *universal* if for any other fill-in (γ, t, δ) for (u, φ, v), there is a unique invertible $\omega : t \Rightarrow s$ such that $\gamma = \alpha(\omega * f)$ and $\delta = \beta(g * \omega)$.

Proposition 1.4. *Let $f : C \to C'$ and $g : D \to D'$ be two arrows in a 2-category $\mathbb{C}$. Then $f \downarrow g$ if and only if the following conditions hold*:

1. *each morphism $(u, \varphi, v) : f \to g$ has a universal fill-in*;
2. *for each $(u, \varphi, v), (u', \varphi', v') : f \to g$, for each $(\mu, \nu) : (u, \varphi, v) \Rightarrow (u', \varphi', v')$ in $\mathbb{C}^2$, for each universal fill-in (α, s, β) and (α', s', β') respectively for (u, φ, v) and for (u', φ', v'), there is a unique $\sigma : s \Rightarrow s'$ such that*

$$\mu \circ \alpha = \alpha' \circ (\sigma * f) \quad \textit{and} \quad \nu \circ \beta = \beta' \circ (g * \sigma). \tag{1}$$

The former version of the orthogonality condition, in [18], consists only of condition 1 of the previous proposition. When all 2-cells are invertible, condition 2 follows from condition 1.

The following lemma is sometimes useful to check the orthogonality condition.

Lemma 1.5. (1) *If there exists a universal fill-in for $(u, \varphi, v) : f \to g$, then every fill-in for (u, φ, v) is universal.*

(2) *The following conditions are equivalent*:

(a) *$f \downarrow g$*;

(b) *the functor $\mathbb{C}(C', D) \to \mathbb{C}^2(f, g)$ which maps $d : C' \to D$ to (df, gd) is an equivalence*;

(c) (i) *each morphism $(u, \varphi, v) : f \to g$ has a fill-in*;

(ii) *for each $(u, \varphi, v), (u', \varphi', v') : f \to g$, for each $(\mu, \nu) : (u, \varphi, v) \Rightarrow (u', \varphi', v')$, for each fill-in (α, s, β) and (α', s', β'), respectively, for (u, φ, v) and for (u', φ', v'), there is a unique $\sigma : s \Rightarrow s'$ such that Eq. (1) hold.*

Proof. Just observe that $\mathbb{C}^2(f, g)$ is the bi-pullback in Cat of $- \circ f : \mathbb{C}(C', D') \to \mathbb{C}(C, D')$ and $g \circ - : \mathbb{C}(C, D) \to \mathbb{C}(C, D')$. □

Here is the definition of a factorization system in $\mathbb{C}$.

Definition 1.6. A *factorization system* in a 2-category $\mathbb{C}$ is a pair $(\mathscr{E}, \mathscr{M})$ of classes of arrows in $\mathbb{C}$ such that:

1. if $m \in \mathscr{M}$ and i is an equivalence then $mi \in \mathscr{M}$, and if $e \in \mathscr{E}$ and i is an equivalence then $ie \in \mathscr{E}$;
2. $\mathscr{E}$ and $\mathscr{M}$ are stable under invertible 2-cells (i.e. if $e \in \mathscr{E}$ and $\alpha : f \Rightarrow e$ is invertible, then $f \in \mathscr{E}$, and the same property holds for $\mathscr{M}$);
3. for each arrow f in $\mathbb{C}$, there exists $e \in \mathscr{E}$, $m \in \mathscr{M}$ and an invertible 2-cell $\varphi : me \Rightarrow f$ (such a factorization φ of f is called an $(\mathscr{E}, \mathscr{M})$-*factorization* of f);
4. for each $e \in \mathscr{E}$ and for each $m \in \mathscr{M}$, $e \downarrow m$.

The proof of the basic properties of factorization systems in 2-categories can be found in [18,21].

Proposition 1.7. *Let $(\mathscr{E},\mathscr{M})$ be a factorization system in $\mathbb{C}$. The following properties hold*:
1. $\mathscr{E}\cap\mathscr{M}=\{equivalences\}$;
2. *$\mathscr{E}$ and $\mathscr{M}$ are closed under composition*;
3. $\mathscr{E}=\mathscr{M}^{\uparrow}$ *and* $\mathscr{M}=\mathscr{E}^{\downarrow}$;
4. *The $(\mathscr{E},\mathscr{M})$-factorization of an arrow of $\mathbb{C}$ is essentially unique (i.e. if $\varphi: me\Rightarrow f$ and $\varphi': m'e'\Rightarrow f$ are two such factorizations, there exist an equivalence i and invertible 2-cells $\alpha: ie\Rightarrow e'$ and $\beta: m'i\Rightarrow m$ such that $\varphi\circ(\beta * e)=\varphi'\circ(m' * \alpha)$)*;
5. *(Cancellation property) If $m',m\in\mathscr{M}$ and if $\theta: m'g\Rightarrow m$ is an invertible 2-cell, then $g\in\mathscr{M}$; dually, if $e',e\in\mathscr{E}$ and if $\theta: fe'\Rightarrow e$ is an invertible 2-cell, then $f\in\mathscr{E}$*;
6. *$\mathscr{M}$ is stable under bi-limits and $\mathscr{E}$ is stable under bi-colimits.*

Remark. In Definition 1.6, conditions 1, 2 and 4 can be equivalently replaced by point 3 of Proposition 1.7.

2. Free 2-categories with factorization system

In this section, we describe the free 2-category with factorization system on a given 2-category $\mathbb{C}$

$$E_{\mathbb{C}}:\mathbb{C}\to\mathbb{C}^2.$$

In fact, $\mathbb{C}^2$ is provided with the following factorization system $(\mathscr{E}_{\mathbb{C}},\mathscr{M}_{\mathbb{C}})$

$$\mathscr{E}_{\mathbb{C}}=\{(u,\varphi,v)\mid u \text{ is an equivalence}\},$$

$$\mathscr{M}_{\mathbb{C}}=\{(u,\varphi,v)\mid v \text{ is an equivalence}\}.$$

An arrow $(u,\varphi,v): f\to g$ in $\mathbb{C}^2$ factors as in the following diagram.

$$\begin{tikzcd} C \arrow[r, "1c"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow\varphi"] & C \arrow[r, "u"] \arrow[d, "gu"] & D \arrow[d, "g"] \\ C' \arrow[r, "v"'] & D' \arrow[r, "1_{D'}"'] & D' \end{tikzcd} \tag{2}$$

We write $e_{(u,\varphi,v)}=(1_C,\varphi,v)$ and $m_{(u,\varphi,v)}=(u,1_{gu},1_{D'})$. The 2-functor $E_{\mathbb{C}}:\mathbb{C}\to\mathbb{C}^2$ maps an object $C\in\mathbb{C}$ to 1_C, an arrow $f\in\mathbb{C}(C,C')$ to $(f,1_f,f)$, and a 2-cell $\alpha: f\Rightarrow g: C\to C'$ to (α,α).

If $\mathbb{C}$ and $\mathbb{D}$ are 2-categories, we write $\mathrm{PS}(\mathbb{C},\mathbb{D})$ for the 2-category of pseudo-functors from $\mathbb{C}$ to $\mathbb{D}$, pseudo-natural transformations and modifications. If $\mathbb{C}$ and $\mathbb{D}$ are 2-categories with factorization system, $\mathrm{PS}_{\mathrm{fs}}(\mathbb{C},\mathbb{D})$ is the 2-category of pseudo-functors preserving the factorization system (i.e. $F(\mathscr{E})\subseteq\mathscr{E}$ and $F(\mathscr{M})\subseteq\mathscr{M}$), pseudo-natural transformations and modifications. Here is the universal property of $E_{\mathbb{C}}:\mathbb{C}\to\mathbb{C}^2$.

Proposition 2.1. *For each* 2*-category* $\mathbb{C}$ *and for each* 2*-category* $(\mathbb{D},(\mathscr{E},\mathscr{M}))$ *with factorization system, the* 2*-functor*

$$-\circ E_{\mathbb{C}}: \ \mathrm{PS}_{\mathrm{fs}}(\mathbb{C}^2,\mathbb{D}) \to \mathrm{PS}(\mathbb{C},\mathbb{D})$$

is a biequivalence.

Proof. A proof can be found in [21]. For reader's convenience, we recall how to construct, from an arbitrary pseudo-functor $G:\mathbb{C}\to\mathbb{D}$, a pseudo-functor $F:\mathbb{C}^2\to\mathbb{D}$ preserving the factorization system and such that $FE_{\mathbb{C}}\cong G$.

Observe that, given an object $f:C\to C'$ in $\mathbb{C}^2$, we get a commutative diagram

$$\begin{tikzcd} C \arrow[r, "1_C"] \arrow[d, "1_C"'] & C \arrow[r, "f"] \arrow[d, "f"] & C' \arrow[d, "1_{C'}"] \\ C \arrow[r, "f"'] & C' \arrow[r, "1_{C'}"'] & C' \end{tikzcd}$$

where the square on the left is an arrow in $\mathscr{E}_{\mathbb{C}}$ and the square on the right is an arrow in $\mathscr{M}_{\mathbb{C}}$. This means that f is the image of $E_{\mathbb{C}}(f)$ in $\mathbb{C}^2$. Now, if we want F to preserve the factorization system and if we want an equivalence $FE_{\mathbb{C}}\cong G$, we have to define $F(f)$ as the image of $G(f)$ in $\mathbb{D}$. The definition of F on 1-cells and on 2-cells follows now from the orthogonality condition. □

3. Arrows in a 2-category

We introduce now a terminology to name various kinds of arrows in a 2-category. Our terminology will be justified by the examples $\mathbb{C}=\mathrm{Cat}$ and SCG discussed in Section 7.

Definition 3.1. Let $\mathbb{C}$ be a 2-category and $f:C\to C'$, an arrow in $\mathbb{C}$.
1. We say that f is *faithful* if for each $X\in\mathbb{C}$, the functor $f\circ -:\mathbb{C}(X,C)\to\mathbb{C}(X,C')$ is faithful.
2. We say that f is *fully faithful* if for each $X\in\mathbb{C}$, the functor $f\circ -:\mathbb{C}(X,C)\to\mathbb{C}(X,C')$ is fully faithful.
3. We say that f is *cofaithful* if for each $Y\in\mathbb{C}$, the functor $-\circ f:\mathbb{C}(C',Y)\to\mathbb{C}(C,Y)$ is faithful.
4. We say that f is *fully cofaithful* if for each $Y\in\mathbb{C}$, the functor $-\circ f:\mathbb{C}(C',Y)\to\mathbb{C}(C,Y)$ is fully faithful.

This terminology for arrows generates a terminology for factorization systems, which generalizes the term "proper factorization system" used for usual categories.

Definition 3.2. Let $(\mathscr{E},\mathscr{M})$ be a factorization system on a 2-category $\mathbb{C}$.
1. We say that $(\mathscr{E},\mathscr{M})$ is (1,1)-*proper* if each $e\in\mathscr{E}$ is cofaithful and each $m\in\mathscr{M}$ is faithful.

2. We say that $(\mathscr{E},\mathscr{M})$ is (2,1)-*proper* if each $e\in\mathscr{E}$ is fully cofaithful and each $m\in\mathscr{M}$ is faithful.
3. We say that $(\mathscr{E},\mathscr{M})$ is (1,2)-*proper* if each $e\in\mathscr{E}$ is cofaithful and each $m\in\mathscr{M}$ is fully faithful.
4. We say that $(\mathscr{E},\mathscr{M})$ is (2,2)-*proper* if each $e\in\mathscr{E}$ is fully cofaithful and each $m\in\mathscr{M}$ is fully faithful.

Remark. If $\mathbb{C}$ is locally discrete, then any factorization system $(\mathscr{E},\mathscr{M})$ on $\mathbb{C}$ is (1,1)-proper. It is (2,1)-proper exactly when $\mathscr{E}$ is contained in the class of epimorphisms, and (1,2)-proper when $\mathscr{M}$ is contained in the class of monomorphisms. Finally, $(\mathscr{E},\mathscr{M})$ is (2,2)-proper exactly when it is proper in the usual sense.

In the sequel, we will construct the free 2-category with a (i,j)-proper (for $i=1,2$ and $j=1,2$) factorization system on a given 2-category.

4. (1,1)-proper factorization systems

In this section, we describe the free 2-category with (1,1)-proper factorization system on a given 2-category $\mathbb{C}$

$$E^{1,1}_{\mathbb{C}}:\mathbb{C}\to \mathrm{Fr}^{1,1}\mathbb{C}.$$

Definition 4.1. Let $\mathbb{C}$ be a 2-category. The 2-category $\mathrm{Fr}^{1,1}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell between two arrows (u,φ,v) and $(w,\psi,x):f\to g$ is an equivalence class of 2-cells of $\mathbb{C}^2$ between the same arrows, for the equivalence relation

$$(\alpha,\beta)\sim(\alpha',\beta') \quad \text{iff } g*\alpha=g*\alpha'$$
$$\text{iff } \beta*f=\beta'*f.$$

We write $[\alpha,\beta]$ for the equivalence class of (α,β). The composition of 2-cells is the same as in $\mathbb{C}^2$, modulo $\sim$: $[\alpha',\beta']\circ[\alpha,\beta]=[\alpha'\circ\alpha,\beta'\circ\beta]$ and $[\gamma,\delta]*[\alpha,\beta]=[\gamma*\alpha,\delta*\beta]$.

The 2-category $\mathrm{Fr}^{1,1}\mathbb{C}$ is equipped with a factorization system $(\mathscr{E}^{1,1}_{\mathbb{C}},\mathscr{M}^{1,1}_{\mathbb{C}})$ which factorizes an arrow (u,φ,v) as in $\mathbb{C}^2$, diagram (2). Following the notations of (2):

$$\mathscr{E}^{1,1}_{\mathbb{C}}=\{(u,\varphi,v)\mid m_{(u,\varphi,v)} \text{ is an equivalence in } \mathrm{Fr}^{1,1}\mathbb{C}\},$$

$$\mathscr{M}^{1,1}_{\mathbb{C}}=\{(u,\varphi,v)\mid e_{(u,\varphi,v)} \text{ is an equivalence in } \mathrm{Fr}^{1,1}\mathbb{C}\}.$$

Proposition 4.2. *The factorization system* $(\mathscr{E}^{1,1}_{\mathbb{C}},\mathscr{M}^{1,1}_{\mathbb{C}})$ *in* $\mathrm{Fr}^{1,1}\mathbb{C}$ *is* (1,1)-*proper.*

Proof. We have to prove that, if $(u,\varphi,v):f\to g$ is an arrow in $\mathrm{Fr}^{1,1}\mathbb{C}$, then $e_{(u,\varphi,v)}$ is cofaithful and $m_{(u,\varphi,v)}$ is faithful.

Let h be an object of $\mathrm{Fr}^{1,1}\mathbb{C}$. Let $[\alpha,\beta],[\alpha',\beta']:(w,\psi,x)\Rightarrow(w',\psi',x'):gu\to h$ be 2-cells of $\mathrm{Fr}^{1,1}\mathbb{C}$ such that

$$[\alpha,\beta]*e_{(u,\varphi,v)}=[\alpha',\beta']*e_{(u,\varphi,v)}.$$

Since $e_{(u,\varphi,v)} = (1_C, \varphi, v) : f \to gu$ (cf. diagram 2), this equation becomes

$$[\alpha, \beta * v] = [\alpha', \beta' * v],$$

which, by definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, is equivalent to

$$h * \alpha = h * \alpha'. \tag{3}$$

This implies that $[\alpha,\beta]=[\alpha',\beta']$, since this last equation is also equivalent, by definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, to Eq. (3). So $e_{(u,\varphi,v)}$ is cofaithful. The proof that $m_{(u,\varphi,v)}$ is faithful is similar. □

Consider the quotient 2-functor $P_{\mathbb{C}}^{1,1} : \mathbb{C}^2 \to \mathrm{Fr}^{1,1}\mathbb{C}$, which is the identity on objects and arrows and maps a 2-cell (α,β) to its equivalence class $[\alpha,\beta]$. We can define the 2-functor $E_{\mathbb{C}}^{1,1} = P_{\mathbb{C}}^{1,2} \circ E_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^2 \to \mathrm{Fr}^{1,1}\mathbb{C}$. Its universal property is stated in the following proposition.

Proposition 4.3. *For any 2-category $\mathbb{C}$ and for any 2-category $(\mathbb{D},(\mathscr{E},\mathscr{M}))$ with (1,1)-proper factorization system, the 2-functor*

$$-\circ E_{\mathbb{C}}^{1,1} : \ \mathrm{PS}_{\mathrm{fs}}(\mathrm{Fr}^{1,1}\mathbb{C}, \mathbb{D}) \to \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. Since $E_{\mathbb{C}}^{1,1} = P_{\mathbb{C}}^{1,1} \circ E_{\mathbb{C}}$ and since Proposition 2.1 tells us that $-\circ E_{\mathbb{C}}$ is a biequivalence, it remains to prove that

$$-\circ P_{\mathbb{C}}^{1,1} : \ \mathrm{PS}_{\mathrm{fs}}(\mathrm{Fr}^{1,1}\mathbb{C}, \mathbb{D}) \to \mathrm{PS}_{\mathrm{fs}}(\mathbb{C}^2, \mathbb{D})$$

is a biequivalence (it is well defined because $P_{\mathbb{C}}^{1,1}$ preserves the factorization system).

It is straightforward to prove that $-\circ P_{\mathbb{C}}^{1,1}$ is locally an equivalence. As far as its surjectivity up to equivalence is concerned, let $G : \mathbb{C}^2 \to \mathbb{D}$ be a pseudo-functor preserving the factorization system. We have to find a pseudo-functor $F : \mathrm{Fr}^{1,1}\mathbb{C} \to \mathbb{D}$ preserving the factorization system, such that $FP_{\mathbb{C}}^{1,1}$ is equivalent to G.

On objects and arrows, we take $F = G$. If $[\alpha,\beta] : (u,\varphi,v) \Rightarrow (w,\psi,x)$ is a 2-cell in $\mathrm{Fr}^{1,1}\mathbb{C}$, we take $F([\alpha,\beta]) = G(\alpha,\beta)$. Then $FP_{\mathbb{C}}^{1,1} = G$ and it remains to prove that F is well defined, i.e. if $[\alpha,\beta] = [\gamma,\delta] : (u,\varphi,v) \Rightarrow (w,\psi,x) : f \to g$, then $G(\alpha,\beta) = G(\gamma,\delta)$.

By definition of $\mathrm{Fr}^{1,1}\mathbb{C}$, $g * \alpha = g * \gamma$ and $\beta * f = \delta * f$. So

$$G(g * \alpha, \beta * f) = G(g * \gamma, \delta * f). \tag{4}$$

But, up to invertible 2-cells, Eq. (4) becomes

$$G(g, 1_g, 1_{D'}) * G(\alpha,\beta) * G(1_C, 1_f, f) = G(g, 1_g, 1_{D'}) * G(\gamma,\delta) * G(1_C, 1_f, f). \tag{5}$$

Since $(g, 1_g, 1_{D'}) \in \mathscr{M}_{\mathbb{C}}$ and G preserves the factorization system, $G(g, 1_g, 1_{D'}) \in \mathscr{M}$. Since $(\mathscr{E},\mathscr{M})$ is (1,1)-proper, $G(g, 1_g, 1_{D'})$ is faithful. Thus Eq. (5) is equivalent to

$$G(\alpha,\beta) * G(1_C, 1_f, f) = G(\gamma,\delta) * G(1_C, 1_f, f).$$

Similarly, $G(1_C, 1_f, f)$ is cofaithful, and we can conclude that $G(\alpha,\beta) = G(\gamma,\delta)$. □

5. (2,1)-proper and (1,2)-proper factorization systems

In this section, we describe the free 2-category with (2,1)-proper factorization system on a given 2-category $\mathbb{C}$

$$E_{\mathbb{C}}^{2,1} : \mathbb{C} \to \mathrm{Fr}^{2,1}\mathbb{C}.$$

Definition 5.1. Let $\mathbb{C}$ be a 2-category. The 2-category $\mathrm{Fr}^{2,1}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell between (u,φ,v) and $(w,\psi,x) : f \to g$ is an equivalence class of 2-cells $\alpha : u \Rightarrow w$ for the equivalence relation

$$\alpha \sim \alpha' \quad \text{iff } g * \alpha = g * \alpha'.$$

Let $[\alpha]$ stand for the class of α. The compositions of 2-cells are easily defined: $[\alpha'] \circ [\alpha] = [\alpha' \circ \alpha]$ and $[\gamma] * [\alpha] = [\gamma * \alpha]$.

The 2-category $\mathrm{Fr}^{2,1}\mathbb{C}$ is equipped with a factorization system $(\mathscr{E}_{\mathbb{C}}^{2,1}, \mathscr{M}_{\mathbb{C}}^{2,1})$, which factorizes the arrows as in diagram (2).

Proposition 5.2. *The factorization system $(\mathscr{E}_{\mathbb{C}}^{2,1}, \mathscr{M}_{\mathbb{C}}^{2,1})$ in the 2-category $\mathrm{Fr}^{2,1}\mathbb{C}$ is (2,1)-proper.*

The 2-functor $P_{\mathbb{C}}^{2,1} : \mathbb{C}^2 \to \mathrm{Fr}^{2,1}\mathbb{C}$ maps (α,β) to $[\alpha]$. We define $E_{\mathbb{C}}^{2,1} = P_{\mathbb{C}}^{2,1} \circ E_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^2 \to \mathrm{Fr}^{2,1}\mathbb{C}$.

Proposition 5.3. *For any 2-category $\mathbb{C}$ and for any 2-category with a (2,1)-proper factorization system $(\mathbb{D},(\mathscr{E},\mathscr{M}))$, the 2-functor*

$$- \circ E_{\mathbb{C}}^{2,1} : \ \mathrm{PS_{fs}}(\mathrm{Fr}^{2,1}\mathbb{C},\mathbb{D}) \to \mathrm{PS}(\mathbb{C},\mathbb{D})$$

is a biequivalence.

Proof. As for Proposition 4.3, we have to prove that

$$- \circ P_{\mathbb{C}}^{2,1} : \ \mathrm{PS_{fs}}(\mathrm{Fr}^{2,1}\mathbb{C},\mathbb{D}) \to \mathrm{PS_{fs}}(\mathbb{C}^2,\mathbb{D})$$

is a biequivalence. The interesting part is, given a pseudo-functor $G : \mathbb{C}^2 \to \mathbb{D}$ which preserves the factorization system, to construct a pseudo-functor $F : \mathrm{Fr}^{2,1}\mathbb{C} \to \mathbb{D}$ which preserves the factorization system and such that $FP_{\mathbb{C}}^{2,1} \cong G$.

We take $F = G$ on objects and arrows of $\mathrm{Fr}^{2,1}\mathbb{C}$. Consider now a 2-cell $[\alpha] : (u,\varphi,v) \Rightarrow (w,\psi,x) : f \to g$ in $\mathrm{Fr}^{2,1}\mathbb{C}$. Define $\nu = \psi^{-1}(g * \alpha)\varphi : vf \Rightarrow xf$ (it is well defined because we only use $g * \alpha$.) We get now a 2-cell ξ in the following way:

$$\begin{tikzcd}
G(u,\varphi,v) \circ G(1_C,1_f,f) \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\xi"'] & G(u,\varphi,vf) \arrow[d, Rightarrow, "G(\alpha,\nu)"] \\
G(w,\psi,x) \circ G(1_C,1_f,f) & G(w,\psi,xf) \arrow[l, Rightarrow, "\cong"']
\end{tikzcd}$$

Since $(1_C,1_f,f) \in \mathscr{E}_{\mathbb{C}}$ and G preserves the factorization system, $G(1_C,1_f,f) \in \mathscr{E}$. Since $(\mathscr{E},\mathscr{M})$ is (2,1)-proper, $G(1_C,1_f,f)$ is fully cofaithful. This implies that there is a unique 2-cell $F([\alpha]) : G(u,\varphi,v) \Rightarrow G(w,\psi,x)$ such that

$$F([\alpha]) * G(1_C, 1_f, f) = \xi. \tag{6}$$

The argument to prove that F is well defined is similar to that in the proof of Proposition 4.3.

Finally, if $(\alpha,\beta) : (u,\varphi,v) \Rightarrow (w,\psi,x) : f \to g$ is a 2-cell in $\mathbb{C}^2$, then $F([\alpha]) = G(\alpha,\beta)$. For this, it is enough to check Eq. (6) for $G(\alpha,\beta)$. This follows from the fact that $v = \beta * f$. $\square$

We can do exactly the same with (1,2)-proper factorization systems, and we get the free 2-category $E^{1,2}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fr}^{1,2}\mathbb{C}$. The difference is that, if $\mathbb{C}$ is a 2-category, the 2-cells of the 2-category $\mathrm{Fr}^{1,2}\mathbb{C}$ from (u,φ,v) to $(w,\psi,x) : f \to g$ are the equivalence classes of 2-cells $\beta : v \Rightarrow x$ for the equivalence relation $\beta \sim \beta'$ iff $\beta * f = \beta' * f$.

6. (2,2)-proper factorization systems

The construction of $\mathrm{Fr}^{2,2}\mathbb{C}$, the free 2-category with a (2,2)-proper factorization system on a given 2-category $\mathbb{C}$, can be done if and only if the 2-category $\mathbb{C}$ is pre-full, in the sense of the following definition.

Definition 6.1. Let $\mathbb{C}$ be a 2-category, and let $f : C \to C'$ be an arrow in $\mathbb{C}$. We say that f is *pre-full* if for each $g, g' : X \to C$, for each $h, h' : C' \to Y$, for each $\alpha : fg \Rightarrow fg'$ and for each $\beta : hf \Rightarrow h'f$, one has

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[r, "f"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\beta"] & C' \arrow[d, "h"] \\ C \arrow[r, "f"'] & C' \arrow[r, "h'"'] & Y \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, phantom, "\Downarrow\alpha"] & C' \arrow[r, "h"] \arrow[dr, phantom, "\Downarrow\beta"] & Y \\ X \arrow[r, "g'"'] \arrow[u, "g"] & C \arrow[r, "f"'] \arrow[u, "f"] & C' \arrow[u, "h'"'] \end{tikzcd} \tag{7}$$

We say that $\mathbb{C}$ is *pre-full* if each arrow in $\mathbb{C}$ is pre-full.

The fact that any 2-category with a (2,2)-proper factorization system is pre-full follows immediately from the next lemma.

Lemma 6.2. *Let $f : C \to C'$ be an arrow in a 2-category $\mathbb{C}$ and consider an invertible 2-cell $\varphi : me \Rightarrow f$. If e and m are such that $- \circ e$ and $m \circ -$ are full functors, then f is pre-full.*

Proof. Let us consider the situation of Definition 6.1. Let $\beta' = (h' * \varphi^{-1})\beta(h * \varphi) : hme \Rightarrow h'me$. Since $- \circ e$ is full, there exists $\delta : hm \Rightarrow h'm$ such that $\delta * e = \beta'$. In the same way, if $\alpha' = (\varphi^{-1} * g')\alpha(\varphi * g) : meg \Rightarrow meg'$, there exists $\gamma : eg \Rightarrow eg'$ such that $m * \gamma = \alpha'$, since $m \circ -$ is full. Then the 2 members of (7) are equal to the 2-cell

$$\begin{tikzcd}
 & C \arrow[rr, "f"] \arrow[dr, "e"] \arrow[rr, phantom, shift right=4ex, "\Downarrow\varphi^{-1}"] & & C' \arrow[dr, "h"] & \\
X \arrow[ur, "g"] \arrow[dr, "g'"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & I \arrow[ur, "m"] \arrow[dr, "m"] \arrow[rr, phantom, "\Downarrow\delta"] & & Y \\
 & C \arrow[rr, "f"'] \arrow[ur, "e"] \arrow[rr, phantom, shift left=4ex, "\Downarrow\varphi"] & & C' \arrow[ur, "h'"'] &
\end{tikzcd}$$

□

Let us explain now the reason why we can define $\mathrm{Fr}^{2,2}\mathbb{C}$ if and only if $\mathbb{C}$ is pre-full. We will define a 2-functor $E^{2,2}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fr}^{2,2}\mathbb{C}$ which is locally faithful. It is easy to see that this fact, together with the pre-fullness of $\mathrm{Fr}^{2,2}\mathbb{C}$ (which comes from its (2,2)-proper factorization system), implies that $\mathbb{C}$ is pre-full.

We arrive at the definition of $\mathrm{Fr}^{2,2}\mathbb{C}$.

Definition 6.3. Let $\mathbb{C}$ be a pre-full 2-category. The 2-category $\mathrm{Fr}^{2,2}\mathbb{C}$ has the same objects and arrows as $\mathbb{C}^2$, but a 2-cell from (u,φ,v) to $(w,\psi,x) : f \to g$ is a 2-cell $\mu : gu \Rightarrow gw$. This is equivalent to give a 2-cell $\breve{\mu} : vf \Rightarrow xf$ related to μ by the equation $\breve{\mu} = \psi^{-1}\mu\varphi$. The vertical composition of $\mu : (u,\varphi,v) \Rightarrow (u',\varphi',v')$ (i.e. $\mu : gu \Rightarrow gu'$) and $\mu' : (u',\varphi',v') \Rightarrow (u'',\varphi'',v'')$ (i.e. $\mu' : gu' \Rightarrow gu''$) is simply $\mu' \circ \mu : gu \Rightarrow gu''$.

The horizontal composition is more problematic. Let $\mu : (u,\varphi,v) \Rightarrow (u',\varphi',v') : f \to g$ and $\nu : (w,\psi,x) \Rightarrow (w',\psi',x') : g \to h$. We define $\nu * \mu = (\psi' * u') \circ \tau_{\mu,\nu} \circ (\psi^{-1} * u) : hwu \Rightarrow hw'u'$, where $\tau_{\mu,\nu}$ is given by the following pasting:

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "u'"'] \arrow[dr, phantom, "\Downarrow\mu"] & D \arrow[r, "g"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\breve{\nu}"] & D' \arrow[d, "x"] \\
D \arrow[r, "g"'] & D' \arrow[r, "x'"'] & E'.
\end{tikzcd}$$

One can check now that $\mathrm{Fr}^{2,2}\mathbb{C}$ is a 2-category: the pre-fullness of $\mathbb{C}$ is needed to prove the interchange law.

The 2-category $\mathrm{Fr}^{2,2}\mathbb{C}$ is equipped with a factorization system $(\mathscr{E}^{2,2}_{\mathbb{C}}, \mathscr{M}^{2,2}_{\mathbb{C}})$, which factorizes the arrows as in diagram (2).

Proposition 6.4. *The factorization system* $(\mathscr{E}^{2,2}_{\mathbb{C}}, \mathscr{M}^{2,2}_{\mathbb{C}})$ *in the 2-category* $\mathrm{Fr}^{2,2}\mathbb{C}$ *is* (2,2)*-proper.*

As in the previous sections, there is a 2-functor $P^{2,2}_{\mathbb{C}} : \mathbb{C}^2 \to \mathrm{Fr}^{2,2}\mathbb{C}$ which is the identity on objects and 1-cells and maps $(\alpha,\beta) : (u,\varphi,v) \Rightarrow (u',\varphi',v') : f \to g$ to $g * \alpha$.

We define the 2-functor

$$E_{\mathbb{C}}^{2,2} : \mathbb{C} \to \mathrm{Fr}^{2,2}\mathbb{C}$$

as the composite $P_{\mathbb{C}}^{2,2} \circ E_{\mathbb{C}}$. The next statement, which gives the universal property of $E_{\mathbb{C}}^{2,2}$, makes sense because a 2-category with a (2,2)-proper factorization system is pre-full.

Proposition 6.5. *For each pre-full* 2*-category* $\mathbb{C}$ *and for each* 2*-category with* (2,2)*-proper factorization system* $(\mathbb{D}, (\mathscr{E}, \mathscr{M}))$*, the* 2*-functor*

$$- \circ E_{\mathbb{C}}^{2,2} : \ \mathrm{PS}_{\mathrm{fs}}(\mathbb{C}^2, \mathbb{D}) \to \mathrm{PS}(\mathbb{C}, \mathbb{D})$$

is a biequivalence.

Proof. Similar to that of Proposition 5.3. □

Remark. If the 2-category $\mathbb{C}$ is locally discrete, then it is pre-full and $\mathrm{Fr}^{2,2}\mathbb{C} = \mathrm{Fr}\mathbb{C}$ is the free category with proper factorization system studied in [15].

7. Examples and an open problem

7.1. Symmetric categorical groups

In [18], two examples of factorization systems are described in the 2-category SCG of symmetric categorical groups, monoidal functors preserving the symmetry and monoidal natural transformations. Let us set some notation. If $F : \mathscr{G} \to \mathscr{H}$ is a morphism in SCG, we write

$$\begin{tikzcd}
\mathrm{Ker}\,F \arrow[r, "e"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, shift right=5ex, "\Downarrow\varepsilon"] & \mathscr{G} \arrow[r, "F"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\pi"] & \mathscr{H} \arrow[r, "p"'] & \mathrm{Coker}\,F
\end{tikzcd}$$

for its kernel and its cokernel; we refer to [18] for their universal properties as bi-limits. If $\mathscr{G}$ is a symmetric cat-group, we write $\pi_0(\mathscr{G})$ for the abelian group of its connected components, and $\pi_1(\mathscr{G})$ for the abelian group $\mathscr{G}(I, I)$, where I is the unit object. If G is an abelian group, we write $D(G)$ for the discrete symmetric cat-group on G, and $G!$ for the symmetric cat-group with a unique object I, and such that $G!(I, I) = G$. These constructions have obvious extensions to morphisms.

In [18], it is proved that, by taking the kernel of the cokernel of an arrow in SCG, we get a factorization system $(\mathscr{E}_1, \mathscr{M}_1)$, where $\mathscr{E}_1$ is the class of full and essentially surjective functors, whereas $\mathscr{M}_1$ is the class of faithful functors. The second factorization system $(\mathscr{E}_2, \mathscr{M}_2)$ on SCG is obtained by taking the cokernel of the kernel of an arrow. In this case $\mathscr{E}_2$ is the class of essentially surjective functors and $\mathscr{M}_2$ is the class of fully faithful functors.

Proposition 7.1. *Let $F:\mathscr{G}\to\mathscr{H}$ be an arrow in* SCG.
1. *F is faithful as an arrow in* SCG *if and only if F is faithful as a functor.*
2. *F is fully faithful as an arrow in* SCG *if and only if F is fully faithful as a functor.*
3. *F is cofaithful if and only if F is essentially surjective.*
4. *F is fully cofaithful if and only if F is full and essentially surjective.*

Proof. Only the necessary condition of 3 was not established in [18]. To prove this condition, let us recall that a functor F in SCG is essentially surjective if and only if $\pi_0 F$ is surjective.

Consider a cofaithful arrow $F:\mathscr{G}\to\mathscr{H}$ in SCG. We have to prove that $\pi_0(F)$ is an epimorphism in the category Ab of abelian groups, i.e. for any $G\in\mathrm{Ab}$ the mapping

$$-\circ\pi_0(F):\ \mathrm{Ab}(\pi_0(\mathscr{H}),G)\to\mathrm{Ab}(\pi_0(\mathscr{G}),G)$$

is surjective. Let us consider the one-object symmetric cat-group $G!$. There is a bijection

$$\varphi_{\mathscr{H}}:\ \mathrm{SCG}(\mathscr{H},G!)(0,0)\to\mathrm{Ab}(\pi_0(\mathscr{H}),G)$$

which maps a monoidal natural transformation $\alpha:0\Rightarrow 0$ onto the group homomorphism $\varphi_{\mathscr{H}}(\alpha):\pi_0(\mathscr{H})\to G:[X]\mapsto\alpha_X$. This map is well defined because α is natural, and it is a group homomorphism because α is monoidal. The inverse of $\varphi_{\mathscr{H}}$ maps a morphism $f:\pi_0(\mathscr{H})\to G$ onto the natural transformation $\varphi_{\mathscr{H}}^{-1}(f)$ such that $(\varphi_{\mathscr{H}}^{-1}(f))_X=f([X])$. In the same way, there is a bijection $\varphi_{\mathscr{G}}:\ \mathrm{SCG}(\mathscr{G},G!)(0,0)\to\mathrm{Ab}(\pi_0(\mathscr{G}),G)$. The announced result is immediate from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{SCG}(\mathscr{H},G!)(0,0) \arrow[r, "-\circ F"] \arrow[d, "\varphi_{\mathscr{H}}"'] & \mathrm{SCG}(\mathscr{G},G!)(0,0) \arrow[d, "\varphi_{\mathscr{G}}"] \\
\mathrm{Ab}(\pi_0(\mathscr{H}),G) \arrow[r, "-\circ\pi_0(F)"'] & \mathrm{Ab}(\pi_0(\mathscr{G}),G)
\end{tikzcd}$$

Indeed, the cofaithfulness of F implies that the top arrow is injective. Since the vertical arrows are bijective, this implies that the bottom arrow is injective. □

As a consequence, we have

1. $(\mathscr{E}_1,\mathscr{M}_1)$ is a (2,1)-proper factorization system;
2. $(\mathscr{E}_2,\mathscr{M}_2)$ is a (1,2)-proper factorization system;
3. let SCG^f be the sub-2-category of SCG whose arrows are the full functors; it is pre-full. Moreover, in SCG^f the systems $(\mathscr{E}_1,\mathscr{M}_1)$ and $(\mathscr{E}_2,\mathscr{M}_2)$ coincide and are (2,2)-proper.

From [18], we know that a morphism $F:\mathscr{G}\to\mathscr{H}$ in SCG is essentially surjective iff it is the cokernel of its kernel $e:\ \mathrm{Ker}\,F\to\mathscr{G}$. Moreover, there is a canonical morphism $c:\pi_1(\mathrm{Ker}\,F)!\to\mathrm{Ker}\,F$, and F is full and essentially surjective iff it is the cokernel of the composite $e \circ c$. Therefore, we obtain the first factorization system taking the cokernel of $e \circ c$. Dually, F is faithful iff it is the kernel of its cokernel $p : \mathscr{H} \to \operatorname{Coker} F$. There is a canonical arrow $d : \operatorname{Coker} F \to D(\pi_0(\operatorname{Coker} F))$, and F is full and faithful iff it is the kernel of the composite $d \circ p$. Therefore, the second system can be obtained by taking the kernel of $d \circ p$.

We want now to describe the systems $(\mathscr{E}_1, \mathscr{M}_1)$ and $(\mathscr{E}_2, \mathscr{M}_2)$ using a different kind of bi-limits. We define the bi-limits we need in an arbitrary pointed 2-category.

Definition 7.2. Let $\mathbb{C}$ be a 2-category with a zero object 0 (that is, for any object $C \in \mathbb{C}$, the categories $\mathbb{C}(C, 0)$ and $\mathbb{C}(0, C)$ are equivalent to the one-arrow category).

1. Consider an arrow $f : C \to C'$ in $\mathbb{C}$. The *pip* of f is given by an object $\operatorname{Pip} f$ and a 2-cell σ as in the following diagram,

$$\begin{tikzcd} \operatorname{Pip} f \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\sigma"] & C \arrow[r, "f"] & C' \end{tikzcd}$$

   such that $f * \sigma = f0$, and such that for any other

$$\begin{tikzcd} X \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\chi"] & C \arrow[r, "f"] & C' \end{tikzcd}$$

   with $f * \chi = f0$, there is an arrow $t : X \to \operatorname{Pip} f$, unique up to a unique invertible 2-cell, such that $\sigma * t = \chi$.
2. Consider a 2-cell

$$\begin{tikzcd} C \arrow[r, bend left, "0"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Downarrow\alpha"] & C' \end{tikzcd}$$

   in $\mathbb{C}$. The *root* of α is an object $\operatorname{Root} \alpha$ and an arrow $r : \operatorname{Root} \alpha \to C$ such that $\alpha * r = 0r$, and such that for any other $x : X \to C$ with $\alpha * x = 0x$, there is $x' : X \to \operatorname{Root} \alpha$ and an invertible 2-cell $\varphi : rx' \Rightarrow x$, the pair (x', φ) being unique up to a unique invertible 2-cell. (The root is a special case of *identifier*.)
3. The *copip* of f and the *coroot* of α are defined by the dual universal property.

We need an explicit description for the pip and the copip of a morphism in SCG. Let $F : \mathscr{G} \to \mathscr{H}$ be an arrow in SCG.

1. The pip of F is given by $\operatorname{Pip} F = D(\operatorname{Ker} \pi_1(F))$ together with the monoidal natural transformation $\sigma : 0 \Rightarrow 0 : \operatorname{Pip} F \to \mathscr{G}$ whose component at $\lambda \in \operatorname{Pip} F$ is λ.
2. The copip of F is given by $\operatorname{Copip} F = (\operatorname{Coker} \pi_0(F))!$ and by $\varrho : 0 \Rightarrow 0 : \mathscr{H} \to \operatorname{Copip} F$, whose component at $X \in \mathscr{H}$ is $\varrho_X = [X]$, the equivalence class of X in $\operatorname{Coker} \pi_0(F)$, that is the isomorphism class of X in $\operatorname{Coker} F$.

Proposition 7.3. *Let $F:\mathcal{G}\to\mathcal{H}$ be a morphism in* SCG.
1. *If F is fully cofaithful, then F is the coroot of its pip.*
2. *If F is fully faithful, then F is the root of its copip.*

Lemma 7.4. *Let $\mathbb{C}$ be a pointed* 2*-category with pips and copips. Let $f:C\to C'$ be an arrow in $\mathbb{C}$.*
1. *If $h:C'\to Y$ is a faithful arrow, then* $\mathrm{Pip}\, f=\mathrm{Pip}\, hf$.
2. *If $g:X\to C$ is a cofaithful arrow, then* $\mathrm{Copip}\, f=\mathrm{Copip}\, fg$.

Proposition 7.5. 1. *By taking the coroot of the pip of an arrow, we get the factorization system $(\mathscr{E}_1,\mathscr{M}_1)$.*

2. *By taking the root of the copip of an arrow, we get the factorization system $(\mathscr{E}_2,\mathscr{M}_2)$.*

Proof. Let $F:\mathcal{G}\to\mathcal{H}$ be a morphism of symmetric cat-groups. Let $M_F\circ E_F$ be the $(\mathscr{E}_1,\mathscr{M}_1)$-factorization of F. Since E_F is fully cofaithful, E_F is the coroot of its pip, by Proposition 7.3. By Lemma 7.4, it is also the coroot of the pip of $F\cong M_F\circ E_F$, since M_F is faithful. So taking the coroot of the pip of F gives exactly its $(\mathscr{E}_1,\mathscr{M}_1)$-factorization. The proof of part 2 is similar. □

7.2. Categories

We discuss now some example in Cat, the 2-category of categories. Let us start with a point of terminology.

Definition 7.6. Let $F:\mathscr{C}\to\mathscr{D}$ be a functor.
1. F is *nearly surjective* (see [20]) if any $D\in\mathscr{D}$ is a retract of FC for some $C\in\mathscr{C}$.
2. F is *retract-stable* if for any $D\in\mathscr{D}$ which is a retract of FC for some $C\in\mathscr{C}$, there exists $C'\in\mathscr{C}$ such that $FC'\cong D$.

Clearly, a functor is essentially surjective on objects if and only if it is nearly surjective and retract-stable.

Example 7.7. The inclusion functor of a reflective subcategory is fully faithful and retract-stable.

Proposition 7.8. *Let $F:\mathscr{C}\to\mathscr{D}$ be a functor.*
1. *F is faithful in the sense of Definition* 3.1 *if and only if F is faithful in the usual sense.*
2. *F is fully faithful in the sense of Definition* 3.1 *if and only if F is fully faithful in the usual sense.*
3. *F is fully faithful and each $F\circ -$ is retract-stable if and only if F fully faithful and retract-stable*
4. *F is cofaithful if and only if F is nearly surjective.*

5. *If F is full and nearly surjective, F is fully cofaithful.*
6. *If F is full and essentially surjective, then F is fully cofaithful and each $-\circ F$ is retract-stable.*
7. *If F is full, then F is pre-full in the sense of Definition* 6.1.

Proof. Points 1–3 are obvious. Point 4 is proved in [1]. Point 5 is proved in [18] in the 2-category SCG for full and essentially surjective functors; the proof for full and nearly surjective functors in Cat is an easy translation.

Let us prove point 6. If F is full and essentially surjective, by point 5, it is fully cofaithful. It remains to prove that each $-\circ F$ is retract-stable. For this, consider $G:\mathscr{D}\to\mathscr{Y}$, $H:\mathscr{C}\to\mathscr{Y}$, $\rho:GF\Rightarrow H$ and $\mu:H\Rightarrow GF$ such that $\rho\circ\mu=1_H$. We define a functor $G':\mathscr{D}\to\mathscr{Y}$ in the following way. Given an object $D\in\mathscr{D}$, since F is essentially surjective there is $C_D\in\mathscr{C}$ and an invertible $\sigma_D:FC_D\to D$. We put $G'D=HC_D$. If $f:D\to D'$, consider the morphism

$$FC_D\xrightarrow{\sigma_D}D\xrightarrow{f}D'\xrightarrow{\sigma_{D'}^{-1}}FC_{D'}. \tag{8}$$

Since F is full, there exists $g_f:C_D\to C_{D'}$ such that Fg_f is equal to the morphism (8). We put $G'f=Hg_f$.

The component at $C\in\mathscr{C}$ of the isomorphism $\omega:G'F\Rightarrow H$ is

$$G'FC=HC_{FC}\xrightarrow{\mu_{C_{FC}}}GFC_{FC}\xrightarrow{G\sigma_{FC}}GFC\xrightarrow{\rho_C}HC.$$

Its inverse is $\omega_C^{-1}=$

$$HC\xrightarrow{\mu_C}GFC\xrightarrow{G\sigma_{FC}^{-1}}GFC_{FC}\xrightarrow{\rho_{C_{FC}}}HC_{FC}=G'FC.$$

Finally, let us prove point 7. Consider two categories $\mathscr{X},\mathscr{Y}$, four functors $G,G':\mathscr{X}\to\mathscr{C}$, $H,H':\mathscr{D}\to\mathscr{Y}$, and two natural transformations $\alpha:FG\Rightarrow FG'$ and $\beta:HF\Rightarrow H'F$. We have to prove that, for each $X\in\mathscr{X}$,

$$H'\alpha_X\circ\beta_{GX}=\beta_{G'X}\circ H\alpha_X. \tag{9}$$

Since F is full, there exists $f:GX\to G'X$ such that $Ff=\alpha_X$. Eq. (9) becomes now $H'Ff\circ\beta_{GX}=\beta_{G'X}\circ HFf$, which holds by naturality of β. $\square$

Let us also recall that fully cofaithful functors are characterized in two different ways in [1].

Example 7.9. 1. The first factorization system $\mathscr{S}_1$ is given by

$$\mathscr{E}_1=\{\text{full and essentially surjective functors}\},$$

$$\mathscr{M}_1=\{\text{faithful functors}\}.$$

A functor $F:\mathscr{C}\to\mathscr{D}$ factors through $\mathrm{Im}_1 F$, which has the same objects as $\mathscr{C}$ and, if $C,C'\in\mathscr{C}$,

$$\mathrm{Im}_1 F(C,C')=F_{C,C'}(\mathscr{C}(C,C')).$$

The composition is that of $\mathscr{D}$. By Proposition 7.8, this factorization system is (2,1)-proper.

2. The second factorization system $\mathscr{S}_2$ is given by

$$\mathscr{E}_2 = \{\text{essentially surjective functors}\},$$

$$\mathscr{M}_2 = \{\text{fully faithful functors}\}.$$

A functor $F:\mathscr{C} \to \mathscr{D}$ factors through $\mathrm{Im}_2 F$, which has the same objects as $\mathscr{C}$ and, if $C, C' \in \mathscr{C}$,

$$\mathrm{Im}_2 F(C, C') = \mathscr{D}(FC, FC').$$

The composition is that of $\mathscr{D}$. By Proposition 7.8, this factorization system is (1,2)-proper.

3. The third factorization system $\mathscr{S}_3$ is given by

$$\mathscr{E}_3 = \{\text{nearly surjective functors}\},$$

$$\mathscr{M}_3 = \{\text{retract-stable fully faithful functors}\}.$$

A functor $F:\mathscr{C} \to \mathscr{D}$ factors through $\mathrm{Im}_3 F$, which is a full subcategory of $\mathscr{D}$. An object is in $\mathrm{Im}_3 F$ if it is a retract of FC for some $C \in \mathscr{C}$. By Proposition 7.8, this factorization system is (1,2)-proper.

4. Here is a simple example of factorization system which is not (1,1)-proper. We write $\emptyset$ for the empty category.

$$\mathscr{E}_4 = \{\text{the identity on } \emptyset \text{ and functors with non-empty domain}\},$$

$$\mathscr{M}_4 = \{\text{equivalences and functors with empty domain}\}.$$

The image of a functor $F:\mathscr{C} \to \mathscr{D}$ is $\mathscr{C}$ if $\mathscr{C} = \emptyset$, and $\mathscr{D}$ if $\mathscr{C} \neq \emptyset$.

5. As for SCG, let Cat^f be the sub-2-category of Cat of full functors. It is pre-full and $\mathscr{S}_1$ restricted to Cat^f is (2,2)-proper.

7.3. An open problem

Let us note that the first factorization system of Example 7.9 is not only (2,1)-proper but also “(3,1)-proper”, in the sense that for any $E \in \mathscr{E}_1$, every composition functor $- \circ E$ is fully faithful and retract-stable. In the same way, the third factorization system is “(1,3)-proper”, i.e. for any $M \in \mathscr{M}_3$, every composition functor $M \circ -$ is fully faithful and retract-stable. This suggests a more general definition of proper factorization system in a 2-category.

Definition 7.10. Let $\mathscr{S}_e = (\mathscr{E}_e, \mathscr{M}_e)$ and $\mathscr{S}_m = (\mathscr{E}_m, \mathscr{M}_m)$ be two factorization systems on the 2-category Cat. A factorization system $(\mathscr{E}, \mathscr{M})$ on a 2-category $\mathbb{C}$ is *$(\mathscr{S}_e, \mathscr{S}_m)$-proper* if

1. for any $e \in \mathscr{E}$, each composition functor $- \circ e$ belongs to $\mathscr{M}_e$;
2. for any $m \in \mathscr{M}$, each composition functor $m \circ -$ belongs to $\mathscr{M}_m$.

Following the notations of Section 7.2, we can reformulate Definition 3.2 in the following way:

A factorization system on a 2-category $\mathbb{C}$ is (i,j)-proper exactly when it is $(\mathscr{S}_i,\mathscr{S}_j)$-proper, for $i,j\in\{1,2\}$ (as well as for $(i,j)=(3,1)$ and $(i,j)=(1,3)$).

(Note that, if we put $\mathscr{S}_0=$ (equivalences, all arrows), every factorization system is $(\mathscr{S}_0,\mathscr{S}_0)$-proper.)

Observe that the free 2-category with (i,j)-proper factorization system $\mathrm{Fr}^{i,j}\mathbb{C}$ on a 2-category $\mathbb{C}$, for $i,j\in\{1,2\}$ (Sections 4–6), can be described in the following way.

Let $f: C\to C'$ and $g: D\to D'$ be in $\mathbb{C}$; consider the $\mathscr{S}_i$-factorization of the functor $-\circ f:\mathbb{C}(C',D')\to\mathbb{C}(C,D')$ and the $\mathscr{S}_j$-factorization of the functor $g\circ -:\mathbb{C}(C,D)\to\mathbb{C}(C,D')$. Then the hom-category $\mathrm{Fr}^{i,j}\mathbb{C}(f,g)$ is given by the following bi-pullback in Cat:

$$\begin{tikzcd}
 & & \mathbb{C}(C,D) \arrow[d, "\mathscr{E}_j\ni"'] \arrow[dd, bend left=60, "g\circ -"] \\
 & \mathrm{Fr}^{i,j}\mathbb{C}(f,g) \arrow[r] \arrow[d] & I_g \arrow[d, "\mathscr{M}_j\ni"'] \\
\mathbb{C}(C',D') \arrow[r, "\in\mathscr{E}_i"'] \arrow[rr, bend right, "-\circ f"'] & I_f \arrow[r, "\in\mathscr{M}_i"'] & \mathbb{C}(C,D')
\end{tikzcd}$$

(This is the case also for $(i,j)=(0,0)$, where $\mathrm{Fr}^{0,0}\mathbb{C}$ is simply the 2-category $\mathbb{C}^2$ of Section 2.)

The problem arising from this remark is if it is possible to generalize the previous construction to get the free 2-category with $(\mathscr{S}_e,\mathscr{S}_m)$-proper factorization system on $\mathbb{C}$. To define the composition functor on the hom-categories, further assumptions on $\mathbb{C}$ are needed (as the example $\mathrm{Fr}^{2,2}\mathbb{C}$ shows), but we are not able to state them explicitly.

8. A glance at the homotopy category

Recall that a weak factorization system in a category $\mathscr{C}$ consists of two classes of morphisms $(\mathscr{E},\mathscr{M})$ satisfying the following conditions:

(1) Given three arrows $A\xrightarrow{f}B\xrightarrow{i}X\xrightarrow{p}B$, if $i\circ f\in\mathscr{E}$ and $p\circ i=1_B$, then $f\in\mathscr{E}$;

(2) Given three arrows $A\xrightarrow{j}X\xrightarrow{q}A\xrightarrow{f}B$, if $f\circ q\in\mathscr{M}$ and $q\circ j=1_A$, then $f\in\mathscr{M}$;

(3) Each arrow has a $(\mathscr{E},\mathscr{M})$-factorization;

(4) Given a commutative square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dotted, "w"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

if $e \in \mathscr{E}$ and $m \in \mathscr{M}$, then there is a (not necessarily unique) arrow $w : B \to C$ such that $w \circ e = u$ and $m \circ w = v$.

The aim of this section is to show that a factorization system in a 2-category $\mathbb{C}$ induces a weak factorization system in the homotopy category $H(\mathbb{C})$ of $\mathbb{C}$ (the category $H(\mathbb{C})$ has the same objects as $\mathbb{C}$, and 2-isomorphism classes of 1-cells as arrows). The main fact is stated in the following proposition.

Proposition 8.1. *Let $\mathbb{C}$ be a 2-category with a factorization system $(\mathscr{E}, \mathscr{M})$.*
(1) *Consider the following diagram*:

$$\begin{tikzcd} & & X \arrow[dr, "p"] & \\ A \arrow[r, "f"] & B \arrow[ur, "i"] \arrow[rr, "1_B"'] \arrow[rr, phantom, shift left=4, "\lambda\Downarrow"] & & B \end{tikzcd}$$

if λ is invertible and $i \circ f \in \mathscr{E}$, then $f \in \mathscr{E}$;
(2) *Consider the following diagram*:

$$\begin{tikzcd} & X \arrow[dr, "q"] & & \\ A \arrow[ur, "j"] \arrow[rr, "1_A"'] \arrow[rr, phantom, shift left=4, "\lambda\Downarrow"] & & A \arrow[r, "f"] & B \end{tikzcd}$$

if λ is invertible and $f \circ q \in \mathscr{M}$, then $f \in \mathscr{M}$.

Proof. We prove the first part, the second one is similar. We have to show that $f \in \mathscr{M}^{\uparrow}$. For this, we check the first condition in Proposition 1.4 and we leave the second one to the reader. Consider the following diagram in $\mathbb{C}$, with $m \in \mathscr{M}$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi \swarrow"] & B \arrow[d, "v"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

We get an arrow $(u, \varphi \circ (v * \lambda * f), vp) : if \to m$ with a universal fill-in (α, w, β) (because $if \in \mathscr{E}$ and $m \in \mathscr{M}$). This fill-in gives rise to a fill-in (α, wi, γ) for $(u, \varphi, v) : f \to m$, where $\gamma = (v * \lambda) \circ (\beta * i)$, and we have to prove that (α, wi, γ) is universal. Let (α', w', β') be another fill-in for (u, φ, v). We get a second fill-in $(\alpha' \circ (w' * \lambda * f), w'p, \beta' * p)$ for $(u, \varphi \circ (v * \lambda * f), vp)$, so that there is a unique comparison $\psi : w \Rightarrow w'p$. This gives us a comparison $\mu = (w' * \lambda) \circ (\psi * i) : wi \Rightarrow w'$ between the two fill-in for (u, φ, v), and we have to prove that such a comparison is unique. Let $\bar{\mu} : wi \Rightarrow w'$ be another comparison between the two fill-in for (u, φ, v). Observe that $(\alpha \circ (wi * \lambda * f), wip, \gamma * p)$ is a third fill-in for $(u, \varphi \circ (v * \lambda * f), vp)$, so that there is a unique comparison $v : wip \Rightarrow w'p$ between $(\alpha \circ (wi * \lambda * f), wip, \gamma * p)$ and $(\alpha' \circ (w' * \lambda * f), w'p, \beta' * p)$ (because, by Lemma 1.5, each fill-in is universal). A diagram chasing shows that both $v = \mu * p$

and $v = \bar{\mu} * p$ work, so that $\mu * p = \bar{\mu} * p$. Finally, observe that, since $\lambda : pi \Rightarrow 1_B$ is an invertible 2-cell, p is cofaithful (that is, the hom-functor

$$\mathbb{C}(p, C) : \mathbb{C}(B, C) \to \mathbb{C}(X, C)$$

is faithful). Now $\mu * p = \bar{\mu} * p$ implies $\mu = \bar{\mu}$. $\square$

In the next corollary, we write $[f]$ for the 2-isomorphism class of an arrow f.

Corollary 8.2. *Let $\mathbb{C}$ be a 2-category with a factorization system $(\mathscr{E}, \mathscr{M})$ and let $H(\mathbb{C})$ be the homotopy category of $\mathbb{C}$. Then $(H(\mathscr{E}), H(\mathscr{M}))$ is a weak factorization system in $H(\mathbb{C})$, where $H(\mathscr{E}) = \{[e] \mid e \in \mathscr{E}\}$ and $H(\mathscr{M}) = \{[m] \mid m \in \mathscr{M}\}$.*

References

[1] J. Adamek, R. El Bashir, M. Sobral, J. Velebil, On functors which are lax epimorphisms, Theory Appl. Categories 8 (2001) 509–521 (http://www.tac.mta.ca/tac/).

[2] J. Adamek, H. Herrlich, J. Rosicky, W. Tholen, Weak factorization systems and topological functors, Appl. Categorical Struct. 10 (2002) 237–249.

[3] M.A. Bednarczyk, A.M. Borzyszkowski, W. Pawlowski, Generalized congruences—epimorphisms in Cat, Theory Appl. Categories 5 (1999) 266–280 (http://www.tac.mta.ca/tac/).

[4] T. Beke, Sheafifiable homotopy model categories, Math. Proc. Cambridge Philos. Soc. 129 (2000) 447–475.

[5] A. Beligiannis, On the Freyd categories of an additive category, Homology Homotopy Appl. 2 (2000) 147–185 (http://www.rmi.acnet.ge/hha/).

[6] R. Betti, D. Schumacher, R. Street, Factorizations in bicategories, preprint.

[7] F. Borceux, Handbook of Categorical Algebra, Cambridge University Press, Cambridge, 1994.

[8] F. Borceux, G. Janelidze, Galois Theories, Cambridge University Press, Cambridge, 2001.

[9] A. Carboni, S. Johnson, R. Street, D. Verity, Modulated bicategories, J. Pure Appl. Algebra 94 (1994) 229–282.

[10] J.M. Cohen, Stable Homotopy, in: Lecture Notes in Mathematics, Vol. 194, Springer, Berlin, 1970.

[11] D. Dikranjan, W. Tholen, Categorical Structure of Closure Operators. With Applications to Topology, Algebra and Discrete Mathematics, Kluwer Academic Publishers, Dordrecht, 1995.

[12] M. Dupont, Systèmes de factorisation dans les 2-catégories, Mémoire de Licence, Université catholique de Louvain, 2001.

[13] P. Freyd, Stable homotopy, Proc. Conf. Categor. Algebra, La Jolla 1965, 1966, pp. 121–172.

[14] M. Grandis, Weak subobjects and the epi-monic completion of a category, J. Pure Appl. Algebra 154 (2000) 193–212.

[15] M. Grandis, On the monad of proper factorization systems in categories, J. Pure Appl. Algebra 171 (2002) 17–26.

[16] M. Hovey, Model Categories, American Mathematical Society, Providence, RJ, 1999.

[17] P.T. Johnstone, Factorization theorems for geometric morphisms, I, Cahiers Topologie G.D. 22 (1981) 1–15.

[18] S. Kasangian, E.M. Vitale, Factorization systems for symmetric cat-groups, Theory Appl. Categories 7 (2000) 47–70 (http://www.tac.mta.ca/tac/).

[19] M. Korostenski, W. Tholen, Factorization systems as Eilenberg–Moore algebras, J. Pure Appl. Algebra 85 (1993) 57–72.

[20] S. Lack, E.M. Vitale, When do completion processes give rise to extensive categories? J. Pure Appl. Algebra 159 (2001) 203–230.

[21] S. Milius, Factorization systems in 2-categories, 2001, preprint.
[22] A. Neeman, Triangulated Categories, Princeton University Press, Princeton, NJ, 2001.
[23] J. Rosicky, W. Tholen, Lax factorization algebras, J. Pure Appl. Algebra 175 (2002) 355–382.
[24] J. Rosicky, E.M. Vitale, Exact completion and representations in abelian categories, Homology Homotopy Appl. 3 (2001) 453–466 (http://www.rmi.acnet.ge/hha/).
[25] R. Street, Two-dimensional sheaf theory, J. Pure Appl. Algebra 23 (1982) 251–270.